বিদ্যাসাগর বিশ্ববিদ্যালয়

VIDYASAGAR UNIVERSITY

Question Paper

B.Sc. Honours Examination 2022

(Under CBCS Pattern)

Semester - VI

Subject : MATHEMATICS

Paper : C 14 - T

Full Marks : 60

Time : 3 Hours

Candidates are required to give their answers in their own words as far as practicable.

The figures in the margin indicate full marks.

[RING THEORY AND LINEAR ALGEBRA-II]

1. Answer any ***five*** questions : 2×5=10

(a) Show that $1+\sqrt{-5}$ is irreducible element in $Z\left(\sqrt{-5}\right)$.

(b) Let *a, b* be two nonzero elements of an Euclidean domain *R*. If *b* is not a unit in *R*, then prove that $d\left(a\right)<d\left(ab\right)$.

(c) Give an example (with justification) of a division ring which is not a field.

(d) Determine all the associates of [8] in the ring $\mathbb{Z}_{10}$.

(e) Give an example of a linear operator *T* on a finite dimensional vector space *V* over a field *F* such that *T* is not diagonalizable.

(f) Find the *dual basis* of the basis {(1, 1, 2), (1, 0, 1), (2, 1, 0)} of the vector space $\mathbb{R}^3$.

(g) If a real symmetric matrix is positive definite then show that all its eigen values are positive.

(h) If $T \in A(V)$ and S is regular in $A(V)$, prove that T and STS^{-1} have same minimal polynomial, where $A(V)$ is the annihilator of V.

2. Answer any ***four*** questions : 5×4=20

(a) (i) Prove that 1 and –1 are the only units of the ring $\mathbb{Z}\sqrt{-5}$.

(ii) Show that the integral domain $\mathbb{Z}\sqrt{-5}$ is a factorization domain. 3+2=5

(b) Find *gcd* of $11 + 7i$ and $18 - i$ in $Z + iZ$.

(c) Let $T : V \to V$ be a linear mapping, where V is a Euclidean space. Show that T is orthogonal if and only if T maps an orthogonal basis to an orthonormal basis.

(d) Let V be a finite dimensional vector space over the field F and T be a diagonalizable linear operator on V. Let $\{c_1, c_2, ..., c_k\}$ be the set of all distinct eigen values of T. Then prove that the characteristic polynomial of T is of the form $(x - c_1)^{d_1}(x - c_2)^{d_2}....(x - c_k)^{d_k}$ for some positive integers $d_1, d_2, ..., d_k$.

(e) (i) Let T be a linear operator on a finite dimensional vector space V over F. Define minimal polynomial of T.

(ii) If V is finite dimensional over F, then prove that $T \in A(V)$ is invertible if and only if the constant term of the minimal polynomial of T is not 0, where $A(V)$ is the annihilator of V. 1+4

(f) Find the eigen values and bases for the eigen space of the matrix $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 4 \end{pmatrix}$.

Is A diagonalizable?

3. Answer any ***three*** questions : 10×3=30

(a) (i) Let R be a PID. Prove that p is irreducible in R if and only if the ideal generated by p is a non-zero maximal ideal. Hence show that $\mathbb{Q}[x]/<x^2-2>$ is a field.

(ii) Prove that for any linear operator T on a finite-dimensional inner product space V, there exists a unique linear operator T^* on V such that $<T\alpha,\beta>=<\alpha,T^*\beta>$ for all $\alpha,\beta\in V$. (4+2)+4=10

(b) (i) Let N be a 2 × 2 complex matrix such that $N^2 = 0$. Then prove that either N = 0 or N is similar to the matrix $\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$ over $\mathbb{C}$.

(ii) Use Gram-Schmidt process to obtain an orthonormal basis from the following basis $\mathcal{B}$ = { (1, 2, –2), (2, 0, 1), (1, 1, 0)} of $\mathbb{R}^3$ with the standard inner product. 4+6=10

(c) (i) Show that an element x in a Euclidean domain is a unit if and only if $d(x)=d(1)$. Hence find all units in the ring $Z + iZ$ of Gaussian integers.

(ii) Define unique factorization domain (UFD). Show that $R=\{a+b\sqrt{-5}\mid a,b\in Z\}$ is not UFD. 6+4

(d) (i) Consider the polynomial $f(x)=5x^4+4x^3-6x^2-14x+2$ in $\mathbb{Z}[x]$. Using Eisenstein's criterion show that *f(x)* is irreducible in $\mathbb{Z}$.

(ii) Let $A=\begin{pmatrix}0 & -1 & 1\\ 1 & 2 & -1\\ 1 & 1 & 0\end{pmatrix}$. Find its minimal polynomial over $\mathbb{R}$ and hence check whether A is similar to a diagonal matrix or not.

(iii) Consider the inner product space $\mathbb{C}^2$ over $\mathbb{C}$ with the standard inner product. Let T be a linear operator on $\mathbb{C}^2$ such that the matrix representation of T with respect to the standard ordered basis is $A=\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}$. Show that T is a normal operator. 3+(3+1)+3=10

(e) (i) Let a linear transformation $T:\mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$T(x,y,z)=(2x+y-2z,2x+3y-4z,x+y-z)$. Find all eigen values of T and find a basis of each eigen space.

(ii) The matrix of a linear mapping $f:\mathbb{R}^3 \to \mathbb{R}^3$ relative to the standard basis

is $\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & -1 & 0 \end{vmatrix}$. Find f and its matrix with respect to the basis

$\{(0,1,-1),(1,-1,1),(-1,1,0)\}$. 5+5