বিদ্যাসাগর বিশ্ববিদ্যালয়

VIDYASAGAR UNIVERSITY

B.Sc. Honours Examination 2021

(CBCS)

1st Semester

MATHEMATICS

PAPER—C2T

ALGEBRA

Full Marks : 60

Time : 3 Hours

The figures in the right-hand margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Answer any *four* questions. 4×12

1. (a) If $a_1, a_2, \ldots a_n$ be all positive real numbers and

$S = a_1 + a_2 + \ldots + a_n$;

Prove that $\left(\frac{s-a_1}{n-1}\right)\left(\frac{s-a_2}{n-1}\right)\cdots\left(\frac{s-a_n}{n-1}\right)$

$> a_1a_2 \ldots a_n$ unless $a_1 = a_2 = \ldots = a_n$

(b) If α, β, γ, δ are the roots of the equation $t^4 + t^2 + 1 = 0$ and n is a positive integer, prove that $\alpha^{2n+1} + \beta^{2n+1} + \gamma^{2n+1} + \delta^{2n+1} = 0$.

(c) Find the relation among the coefficients of the equation $ax^3 + 3bx^2 + 3cx + d = 0$ if its roots be in arithmetic progression. 4+5+3

2. (a) Let C[0, 1] be the set of all real continuous functions on the closed interval [0, 1] and T be a mapping from c[0,1] to R defined by $T(f) = \int_0^1 f(x)dx, f \in c[0,1]$. Show that T is a linear transformation.

(b) Let v be a real vector space with a basis $\{\vec{\alpha}_1, \vec{\alpha}_2, .., \vec{\alpha}_n\}$,

Examine if $\{\vec{\alpha}_1 + \vec{\alpha}_2, \vec{\alpha}_2 + \vec{\alpha}_3, .., \vec{\alpha}_n + \vec{\alpha}_1\}$ is also a basis of V.

(c) Find $K \in R$ so that the set S = {(1, 2, 1), (k, 3, 1), (2, k, 0)} is linearly dependent in $1R^3$. 4+5+3

3. (a) Prove that 6|n(n + 1) (n + 2), $n \in \mathbb{Z}$.

(b) Use the theory of congruence to find the remainder when the sum $1^5 + 2^5 + 3^5 + \ldots + 100^5$ is divided by 5. 5+5+2

(c) Find the values of a for which the equation $ax^3 - 6x^2 + 9x - 4 = 0$ may have multiple roots. 5+5+2

4. (a) Find x if the rank of the matrix $\begin{pmatrix} 1 & 3 & -3 & x \\ 2 & 2 & x & -4 \\ 1 & 1-x & 2x+1 & -8-3x \end{pmatrix}$ be 2.

(b) Find the value of λ for which the system of equations

$2x_1 - x_2 + x_3 + x_4 = 1$, $x_1 + 2x_2 - x_3 + 4x_4 = 2$, $x_1 + 7x_2 - 4x_3 + 11x_4 = \lambda$ is solvable.

(c) If $\alpha + \beta + \gamma = 0$, Prove that $\frac{\alpha^5+\beta^5+\gamma^5}{5} = \frac{\alpha^3+\beta^3+\gamma^3}{3} . \frac{\alpha^2+\beta^2+\gamma^2}{2}$

4+4+4

5. (a) If α, β, γ be the roots of the equation $x^3 - 2x^2 + 3x - 1 = 0$,

find the equation whose roots are $\frac{\beta\gamma-\alpha^2}{\beta+\gamma-2\alpha}, \frac{\gamma\alpha-\beta^2}{\gamma+\alpha-2\beta}, \frac{\gamma\beta-\gamma^2}{\alpha+\beta-2\gamma}$

(b) Solve : $(1+x)^{2n} + (1-x)^{2n} = 0$

(c) If $S_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$, prove that $S_n > \frac{2n}{n+1}$ if $n > 1$. 4+5+3

6. (a) Show that $(2n + 1)^2 \equiv 1 \pmod 8$ for any natural number n.

(b) Use Cayley Hamiltan theorem, to find A^{50} where $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

(c) Find the dimension of the subspace $S \cap T$ of $\mathbb{R}^4$ where

$S = \{(x, y, z, w) \in \mathbb{R}^4 : x + y + z + w = 0\}$.

$T = \{(x, y, z, w) \in \mathbb{R}^4 : 2x + y - z + w = 0\}$. 3+4+5

7. (a) If the roots of the equation $x^3 + px^2 + qx + r = 0$ are in A. P where p,q,r are real numbers, prove that $p^2 \geq 3q$.

(b) Find all values of $i^{1/7}$.

(c) Prove that for any two integers U and V > 0,. there exist two unique integers m and n such that

$U = mV + n, \quad o \leq n < V$. 4+4+4

8. (a) If $a \equiv b (\bmod\ m)$ and $a \equiv c (\bmod n)$, prove that $b \equiv c (\bmod d)$ where d = gcd(m, n).

(b) Find the basis for the column space of the matrix

$$\begin{pmatrix} 1 & 2 & -1 \\ 2 & 3 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

(c) Determine the conditions for which the system of equations

$x + 2y + z = 1$

$2x + y + 3z = b$

$x + ay + 3z = b + 1$

has unique solution, many solutions and no solution.

Answer any *six* questions. 6×2

9. Find the general values of the equation

$(\cos\theta + i\sin\theta)(\cos 2\theta + i\sin 2\theta) \ldots (\cos n\theta + i\sin n\theta) = -i$, where θ is real.

10. If the equation $x^4 + px^2 + qx + r = 0$ has three equal roots then show that $8p^3 + 27q^2 = 0$.

11. Solve the equations $x + py + p^2z = p^3$, $x + qy + q^2z = q^3$, $x + ry + r^2z = r^3$.

12. Find the equation whose roots are cubes of the roots of the cubic $x^3 + 3x^2 + 2 = 0$.

13. Prove that $n^2 + 2$ is not divisible by 4 for any integer n.

14. Show that the set of all points on the line $y = mx$ forms a sub space of the vector space $\mathbb{R}^2$.

15. Find the number of divisors and their sum of 10800.

16. Find the greatest value of xyz where x, y and z are positive real numbers satisfying $xy + yz + zx = 27$.

17. If A and B be two square invertible matrices, then prove that AB and BA have the same eigen values.

18. Show that eigen values of the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$ are all real.

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