Semester-III

Core Course 5T

Chapter-5

Partial Differential Equations

Class Note 4 (1 hour)

(# Solution of radial part of Laplace equation in Spherical Polar Coordinates
Some Problems involving application of solution of Laplace equation in Polar coordinates)

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Class Note-4

6. **Continued from Class Note-3 Laplace's Equation in spherical polar coordinate systems: Solution of Radial Equation and Some Problems involving application of solution of Laplace equation in Polar coordinates.**

Radial Equation

Radial equation obtained from Laplace equation through separation of variables is:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR(r)}{dr}\right)-\frac{\lambda}{r^2}R(r)=0\ \ldots\ldots\ldots(6.4)\ [\textbf{Radial or } \boldsymbol{r} \textbf{ eqn}.]$$

$$\Rightarrow\quad r^2\frac{d^2R(r)}{dr^2}+2r\frac{dR(r)}{dr}-\lambda R(r)=0\ \ldots\ldots\ldots(6.24)$$

Let

$$R(r)=r^{\alpha}, \text{where } \alpha \text{ is a constant} \ldots\ldots.(6.25)$$

is a trial solution. Then:

$$r^2\alpha(\alpha-1)r^{\alpha-2}+2r\alpha r^{\alpha-1}-\lambda r^{\alpha}=0$$

$$\Rightarrow\quad [\alpha(\alpha-1)+2\alpha-\lambda]r^{\alpha}=0$$

Since r is not always zero, therefore:

$$\alpha(\alpha-1)+2\alpha-\lambda=0$$

$$\alpha^2+\alpha-\lambda=0\ \ldots\ldots\ldots(6.26)$$

If the roots of this quadratic equation are α_1 and α_2, then:

$$\alpha_1=\frac{-1+\sqrt{1+4\lambda}}{2}\quad and\quad \alpha_2=\frac{-1-\sqrt{1+4\lambda}}{2}$$

i.e. both of α_1 and α_2 are real.

Also from (6.26) we have:

$$\lambda=\alpha^2+\alpha=\alpha(\alpha+1)=\alpha_1(\alpha_1+1)=\alpha_2(\alpha_2+1)$$

Starting from (6.4) we obtain that λ should have the form $\alpha(\alpha+1)$, where α is a real number. We have obtained similar condition during the solution of θ equation, where we have seen that for physically meaningful solution we require the extra restriction: $\alpha = l =$ zero or positive integer i.e. $l = 0,1,2 \ldots$.

Thus equation (6.24) should be written as:

$$r^2 \frac{d^2R(r)}{dr^2} + 2r\frac{dR(r)}{dr} - l(l+1)R(r) = 0 \ldots\ldots\ldots (6.24A)$$

And equation (6.26) will become:

$$\alpha^2 + \alpha - l(l+1) = 0$$

$$\alpha^2 + \alpha - l^2 - l = 0$$

$$(\alpha + l)(\alpha - l) + \alpha - l = 0$$

$$(\alpha + l + 1)(\alpha - l) = 0$$

$$\boldsymbol{\alpha = l \ \ or\ \alpha = -(l+1)}$$

Therefore the solution of the radial equation will be:

$$\boldsymbol{R_l(r) = A_l r^l + B_l r^{-(l+1)}}, \qquad l = 0,1,2 \ldots\ldots\ldots\ldots (\mathbf{6.25})$$

Where A_l and B_l are constants to be determined from boundary conditions.

Now we can write the complete solution of Laplace equation in spherical polar coordinates as:

$$u_{lm}(r,\theta,\varphi) = R_l(r)Y_{lm}(\theta,\varphi) = R_l(r)\Theta_{lm}(\theta)\Phi_{lm}(\varphi) \ldots\ldots\ldots (6.26)$$

And the general solution will be:

$$u(r,\theta,\varphi) = \sum_{l,m} u_{lm}(r,\theta,\varphi) = \sum_{l,m} R_l(r)\Theta_{lm}(\theta)\Phi_{lm}(\varphi)$$

$$= \sum_{l,m}\left(\boldsymbol{A_l r^l + B_l r^{-(l+1)}}\right)\boldsymbol{P_l^m(\cos\theta)e^{im\varphi}} \ldots (6.27)$$

Problem: 6.1

An uncharged conducting sphere of radius a is placed at the origin in an initially uniform electrostatic field E. Obtain the expression of potential at an external point and show that it behaves as an electric dipole.

Solution:

Let the initially (before introducing the conducting sphere) uniform electrostatic field is directed along the Z-axis and its magnitude is E_0. Then

$$\vec{E} = E_0\hat{k} \dots\dots\dots (P6.1.1)$$

In a charge free region the electrostatic field is conservative and can be derived from electrostatic potential, say u. i.e.:

$$\vec{E} = -\vec{\nabla}u$$

Thus, if $u = u_0$ represent the potential of the initial field, then:

$$-\vec{\nabla}u_0 = E_0\hat{k}$$

$$-\left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial z}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right)u_0 = E_0\hat{k}$$

$$-\frac{\partial u_0}{\partial z} = E_0$$

$$u_0 = -E_0 z + C = -E_0 r\cos\theta + C; \quad (C \text{ is arbitrary till now}) \dots\dots (P6.1.2)$$

In a charge free region, here outside the sphere ($r \geq a$), the electrostatic potential u satisfies Laplace equation:

$$\nabla^2 u = 0$$

The symmetry of the present problem suggests that we should use the solution of Laplace equation in spherical polar coordinates. i.e.:

$$\left.\begin{array}{l} u(r,\theta,\varphi) = \sum_{l,m}\left(A_l r^l + B_l r^{-(l+1)}\right)P_l^m(\cos\theta)e^{im\varphi} \\ with\ l = 0,1,2\dots;\ m = 0, \pm 1 \dots\dots \pm l \end{array}\right\} \dots\dots\dots (6.27)$$

The problem has axial or azimuthal symmetry. i.e. the potential does not depend on φ. Therefore the acceptable value of m is:

$$m = 0$$

Then:

$$u(r,\theta,\varphi) = \sum_{l=0}^{\infty}\left(A_l r^l + B_l r^{-(l+1)}\right)P_l^0(\cos\theta)$$

$$= \sum_{l=0}^{\infty}\left(A_l r^l + B_l r^{-(l+1)}\right)P_l(\cos\theta) \dots\dots (P6.1.3)$$

$$\left[since \quad P_l^0(w) = (1-w^2)^{\frac{|0|}{2}}\left(\frac{d}{dw}\right)^{|0|}[P_l(w)] = P_l(w)\right]$$

Now, after introduction of the conducting sphere in the initially uniform electric field, the field and potential at infinite distance from the sphere should not be affected and should remain as before. i.e.:

$$u(r\to\infty,\theta,\varphi) = u_0(r\to\infty,\theta,\varphi) = -E_0 z + C = -E_0 r\cos\theta + C$$

From (P6.1.2) and (P6.1.3):

$$\lim_{r\to\infty}\sum_{l=0}^{\infty}\left(A_l r^l + B_l r^{-(l+1)}\right)P_l(\cos\theta) = -E_0 r\cos\theta + C$$

$$\lim_{r\to\infty}\sum_{l=0}^{\infty}A_l r^l P_l(\cos\theta) = -E_0 r\cos\theta + C$$

$$A_0P_0(\cos\theta) + A_1 r P_1(\cos\theta) + A_2 r^2 P_2(\cos\theta) + \cdots\dots = -E_0 r\cos\theta + C$$

$$A_0 + A_1 r P_1(\cos\theta) + A_2 r^2 P_2(\cos\theta) + \cdots\dots = -E_0 r P_1(\cos\theta) + C$$

$$[As\ P_0(\cos\theta) = 1\ \&\ P_1(\cos\theta) = \cos\theta]$$

Equating the coefficients of $P_l(\cos\theta)$ from the two sides we can write:

$$A_1 = -E_0, \qquad A_0 = C \quad \text{and} \quad A_l = 0 \text{ for } l \geq 2$$

Then from (P6.1.3):

$$u(r,\theta,\varphi) = C - E_0 r\cos\theta + \sum_{l=0}^{\infty}B_l r^{-(l+1)}P_l(\cos\theta) \dots\dots (P6.1.4)$$

Now, it is well known that the surface of a conductor is an equipotential surface. Then the potential on the surface of the sphere will be constant, **say $\boldsymbol{V}$**. Therefore, if a is the radius of the sphere, then:

$$u(a,\theta,\varphi) = C - E_0 a\cos\theta + \sum_{l=0}^{\infty} B_l a^{-(l+1)} P_l(\cos\theta) = V$$

$$C - E_0 a P_1(\cos\theta) + \frac{B_0}{a} + \frac{B_1}{a^2}P_1(\cos\theta) + \frac{B_2}{a^3}P_2(\cos\theta) + \frac{B_3}{a^4}P_3(\cos\theta) + \cdots = V$$

$$C + \frac{B_0}{a} + \left(\frac{B_1}{a^2} - E_0 a\right)P_1(\cos\theta) + \frac{B_2}{a^3}P_2(\cos\theta) + \frac{B_3}{a^4}P_3(\cos\theta) + \cdots = V$$

Equating the coefficients of $P_l(\cos\theta)$ from the two sides we can write:

$$\frac{B_1}{a^2} - E_0 a = 0 \Rightarrow \frac{B_1}{a^2} = E_0 a \Rightarrow B_1 = E_0 a^3$$

$$B_2 \Rightarrow B_3 = 0\ \dots;\ \ B_l = 0 \text{ for } l \geq 2$$

$$C + \frac{B_0}{a} = V \Rightarrow C = V - \frac{B_0}{a}$$

Then (P6.1.4) reduces to:

$$u(r,\theta,\varphi) = C - E_0 r\cos\theta + \sum_{l=0}^{\infty} B_l r^{-(l+1)} P_l(\cos\theta)$$

$$= V - \frac{B_0}{a} - E_0 r\cos\theta + \frac{B_0}{r} + \frac{E_0 a^3}{r^2}\cos\theta$$

$$= V + B_0\left(\frac{1}{r} - \frac{1}{a}\right) - E_0\left(r - \frac{a^3}{r^2}\right)\cos\theta \qquad \dots\dots\dots (P6.1.5)$$

To determine the constant B_0 we would utilize the fact that the total charge on the surface of the initially uncharged sphere will remain zero even after placing it in the external electric field. Charge density on the surface of the sphere is given by:

$$\sigma(r,\theta) = -\epsilon \left.\frac{\partial u}{\partial r}\right|_{r=a} = -\frac{B_0}{r^2} - E_0\left(1 + \frac{2a^3}{r^3}\right)\cos\theta$$

Charge density at every point on an annular ring like portion of the surface between θ and $\theta + d\theta$ will be same and equal to:

$$\sigma(a,\theta) = -\frac{B_0}{a^2} - E_0\left(1 + \frac{2a^3}{a^3}\right)\cos\theta = -\frac{B_0}{a^2} - 3E_0\cos\theta$$

Total charge on that annular ring-like portion of the surface between θ and $\theta + d\theta$ is zero. Therefore

$$dQ = \left(-\frac{B_0}{a^2} - 3E_0 \cos\theta\right) 2\pi a \sin\theta\; a\; d\theta = 2\pi \left(-\frac{B_0}{a^2} - 3E_0 \cos\theta\right) a^2 \sin\theta\;\; d\theta$$

$$= -2\pi B_0 \sin\theta\;\; d\theta - 6\pi E_0 a^2 \cos\theta \sin\theta\;\; d\theta$$

Total charge on the surface is zero. Therefore

$$0 = Q = \int dQ = -2\pi B_0 \int_{\theta=0}^{\pi} \sin\theta\;\; d\theta - 6\pi E_0 a^2 \int_{\theta=0}^{\pi} \cos\theta \sin\theta\;\; d\theta$$

$$= -4\pi B_0$$

$$\Rightarrow B_0 = 0$$

Then finally we can write:

$$u(r \geq a, \theta, \varphi) = V - E_0 \left(r - \frac{a^3}{r^2}\right) \cos\theta$$

And:

$$\vec{E}(r \geq a, \theta, \varphi) = -\vec{\nabla} u\big|_{r \geq a} = -\frac{\partial u}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial u}{\partial \theta}\hat{\theta} - \frac{1}{r\sin\theta}\frac{\partial u}{\partial \varphi}\hat{\varphi}$$

$$\vec{E}(r \geq a, \theta, \varphi) = E_0 \left(1 + \frac{2a^3}{r^3}\right) \cos\theta\, \hat{r} - E_0 \left(1 - \frac{a^3}{r^3}\right) \sin\theta\, \hat{\theta}$$

Note: Potential everywhere in a conductor is same. Therefore the potential inside the sphere is constant and equal to V. i.e.:

$$u(r \leq a, \theta, \varphi) = V$$

$$\vec{E}(r \leq a, \theta, \varphi) = -\vec{\nabla} u\big|_{r \leq a} = -\vec{\nabla} V = 0$$

i.e. the electric field intensity inside the conducting sphere is zero.

Problem: 6.2

Calculate the gravitational potential at a general point in space due to a uniform ring of radius a and total mass M.

Solution:

The gravitational potential $u(\vec{r})$ obeys Laplace equation in a region, where mass density is zero. Therefore in the present problem, $u(\vec{r})$ satisfies Laplace equation ($\nabla^2 u = 0$) everywhere except on the ring. The symmetry of the problem suggests that cylindrical or spherical polar coordinates should be used. However, we here use the solution of Laplace equation in spherical polar coordinates:

$$\left.\begin{array}{l} u(r,\theta,\varphi) = \sum_{l,m}\left(A_l r^l + B_l r^{-(l+1)}\right)P_l^m(\cos\theta)e^{im\varphi} \\ \quad\quad\quad\quad with\ l = 0,1,2\ldots;\ m = 0, \pm 1 \ldots\ldots \pm l \end{array}\right\} \ldots\ldots\ldots (6.27)$$

Let the Z axis is along the symmetry axis of the ring. Due to azimuthal symmetry of the present problem, the solution should not depend on φ, and we must have:

$$m = 0$$

And therefore:

$$u(r,\theta,\varphi) = \sum_{l=0}^{\infty}\left(A_l r^l + B_l r^{-(l+1)}\right)P_l^0(\cos\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + B_l r^{-(l+1)}\right)P_l(\cos\theta) \ldots (P6.2.1)$$

The solution is valid everywhere including $r = 0$ and $r \to \infty$, except on the ring i.e. for $r, \theta = a, \frac{\pi}{2}$, where a is the radius of the ring. Therefore we should expect (i) u to be finite at $r = 0$ and (ii) $u \to 0$ for $r \to \infty$. To satisfy the conditions at $r = 0$ and $r \to \infty$ it seems that both of A_l and B_l should be zero. But then $u(r,\theta,\varphi)$ becomes zero everywhere which is not an acceptable result. Since Laplace equation Laplace equation $\nabla^2 u = 0$ is not satisfied on the ring ($r = a$), therefore the solution P6.2.1 is not valid on the ring and the expression of $u(r,\theta,\varphi)$ changes at $r = a$. Therefore let us take two regions:

(i) $r < a$. In this region we must have $B_l = 0$, since otherwise $u(r,\theta,\varphi)$ becomes infinite at $r = 0$ for the term $B_l r^{-(l+1)}$.

Therefore:

$$u(r,\theta,\varphi) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \quad \text{for } r < a \quad \ldots (P6.2.2)$$

(ii) $r > a$. In this region we must have $A_l = 0$, since otherwise $u(r,\theta,\varphi)$ becomes infinite at $r \to \infty$ for the term $A_l r^l$. *i.e.*

$$u(r,\theta,\varphi) = \sum_{l=0}^{\infty} B_l r^{-(l+1)} P_l(\cos\theta) \;\text{ for } r > a \quad \dots (P6.2.3)$$

A_l and B_l are to be determined using known expressions of $u(r,\theta,\varphi)$ for particular values of r, θ and φ. We can easily obtain the expression of $u(r,\theta,\varphi)$ for axial points i.e. at $(z,0)$:

$$u(r,\theta,\varphi) = u(r,0,\varphi) = u(z,0) = -\frac{Gm}{\sqrt{a^2+z^2}} \dots\dots\dots (P6.2.4)$$

From (P6.2.2), at $(z,0)$ with $r < a$,

$$u(r,0,\varphi) = u(z,0) = \sum_l A_l z^l P_l(\cos 0)$$

$$= \sum_{l=0}^{\infty} A_l z^l\,; \;\; [\text{as } P_l(\cos 0) = P_l(1) = 1] \dots\dots\dots (P6.2.2A)$$

From $P6.2.4$:

$$u(z,0) = -\frac{Gm}{\sqrt{a^2+z^2}} = -\frac{Gm}{a}\left(1+\frac{z^2}{a^2}\right)^{-1/2} = -\frac{Gm}{a}\left(1+\frac{z^2}{a^2}\right)^{-1/2}$$

$$= -\frac{Gm}{a}\sum_{l=0}^{\infty} \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)\cdots\left(\left(-\frac{1}{2}\right)-l+1\right)}{l!}\left(\frac{z^2}{a^2}\right)^l$$

$$= -\frac{Gm}{a}\sum_{l=0}^{\infty} \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(\frac{1-2l}{2}\right)}{l!}\left(\frac{z}{a}\right)^{2l} = -Gm\sum_{l=0}^{\infty}(-1)^l \frac{1\cdot 3\cdot 5\cdots(1-2l)}{2^l l!\, a^{2l+1}} z^{2l}$$

$$= -Gm\left[\frac{1}{a} - \frac{1\cdot 3}{2a^3}z^2 + \frac{1\cdot 3\cdot 5}{2^2 2!\, a^{4+1}}z^4 - \cdots\right] \dots\dots\dots (P6.2.5)$$

Comparing $(P6.2.2A)$ and $(P6.2.5)$ we have:

$$\left.\begin{aligned} A_{2l} &= -(-1)^l Gm \frac{1\cdot 3\cdot 5\cdots(1-2l)}{2^l l!\, a^{2l+1}} \\ A_{2l+1} &= 0 \end{aligned}\right\} \dots\dots\dots (P6.2.6)$$

Therefore from ($P6.2.2$) and ($P6.2.6$) we can write:

$$u(r,\theta,\varphi) = \frac{-Gm}{a}\sum_{l=0}^{\infty}(-1)^l \frac{1\cdot 3\cdot 5 \cdots (1-2l)}{2^l l!}\left(\frac{r}{a}\right)^{2l} P_{2l}(\cos\theta) \quad \text{for } r < a$$

From (P6.2.3), at $(z, 0)$ with $r < a$,

$$u(r,0,\varphi) = u(z,0) = \sum_{l=0}^{\infty} B_l z^{-(l+1)} = \frac{1}{z}\sum_{l=0}^{\infty} B_l z^{-l} \ \ldots\ldots\ldots (P6.2.3A)$$

From $P6.2.4$:

$$u(z,0) = -\frac{Gm}{\sqrt{a^2+z^2}} = -\frac{Gm}{z}\left(1+\frac{a^2}{z^2}\right)^{-1/2} = -\frac{Gm}{z}\left(1+\frac{a^2}{z^2}\right)^{-1/2}$$

$$= -\frac{Gm}{z}\sum_{l=0}^{\infty}(-1)^l \frac{1\cdot 3\cdot 5 \cdots (1-2l)a^{2l}}{2^l l!}\frac{1}{z^{2l}} \ \ldots\ldots\ldots (P6.2.7)$$

Comparing ($P6.2.3A$) and ($P6.2.7$) we have:

$$\left.\begin{aligned} B_{2l} &= -(-1)^l Gm \frac{1\cdot 3\cdot 5 \cdots (1-2l)a^{2l}}{2^l l!} \\ B_{2l+1} &= 0 \end{aligned}\right\} \ldots\ldots\ldots (6.2.8)$$

Therefore ($P6.2.3$) and ($P6.2.7$) we can write:

$$u(r,\theta,\varphi) = \frac{-Gm}{r}\sum_{l=0}^{\infty}(-1)^l \frac{1\cdot 3\cdot 5 \cdots (1-2l)}{2^l l!}\left(\frac{a}{r}\right)^{2l} P_{2l}(\cos\theta) \quad \text{for } r > a$$

Therefore finally we have:

$$\left.\begin{aligned} u(r,\theta,\varphi) &= \frac{-Gm}{a}\left[1+\sum_{l=1}^{\infty}(-1)^l \frac{1\cdot 3\cdot 5 \cdots (1-2l)}{2^l l!}\left(\frac{r}{a}\right)^{2l} P_{2l}(\cos\theta)\right] \ \text{for } r < a \\ u(r,\theta,\varphi) &= \frac{-Gm}{r}\left[1+\sum_{l=1}^{\infty}(-1)^l \frac{1\cdot 3\cdot 5 \cdots (1-2l)}{2^l l!}\left(\frac{a}{r}\right)^{2l} P_{2l}(\cos\theta)\right] \ \text{for } r > a \end{aligned}\right\} \ldots (P6.2.8)$$

Any of these two relations are valid at $r = a$ for all values of θ except for $\theta = \pi/2$, since $r = a, \theta = \pi/2$ means the material of the ring, where Laplace equation is not applicable.