CHAPTER XI

POLAR EQUATIONS

11.1. Polar co-ordinates.

Let O be a fixed point called the *origin* or the *pole* and OX be a fixed straight line called the *initial line* or the *polar axis*.

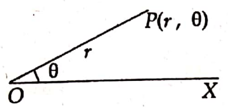


Fig. 38

Let P be any point in the plane ; OP is drawn. Let it be of length r and the angle $\angle XOP$ be θ. The length r is called the *radius vector* and the angle θ is called the *vectorial angle* of the point P. If these two elements be given, the position of the point is determined. These are called the *polar co-ordinates* of the point and are represented as $P(r, \theta)$. The radius vector is positive, if it be measured from the pole along the line bounding the vectorial angle; it is negative, if measured in the opposite direction. Vectorial angle is generally assumed positive, if measured in the anti-clock-wise direction.

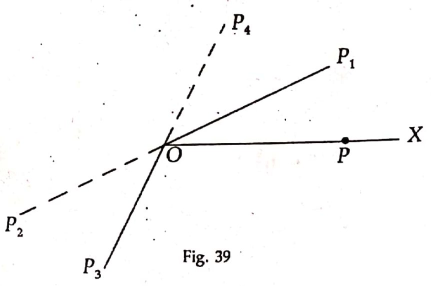


Fig. 39

Let the line segment OP of length 2 units revolve from the initial line OX through an angle of 30° in the anti-clock-wise direction to its new position OP_1. Then the polar co-ordinates of P_1 will be (2, 30°). The point P_2, situated on P_1O produced such that the lengths of

OP_1 and OP_2 are equal, will have its co-ordinates (– 2, 30°). In the same way let OP_3 be the new position of OP after revolving from OX through an angle of 120° in the clock-wise direction. Then the polar co-ordinates of P_3 will be (2, – 120°). The point P_4 situated on P_3O produced such that the lengths of OP_3 and OP_4 are equal, will have its co-ordinates (– 2, – 120°).

It can easily be seen that the same point P_4 may be denoted by each of the following four sets of polar co-ordinates :

$$(2,\ 60°),\ (2,\ -300°),\ (-2,\ 240°),\ (-2,\ -120°).$$

In general, the polar co-ordinates of the same point may be expressed by each of

$$(r,\ \theta),\ \{r,\ -(360° - \theta)\},\ (-r,\ 180° + \theta) \text{ and } \{-r,\ -(180° - \theta)\}.$$

11.2. Change from cartesian to polar system of co-ordinates and vice-versa.

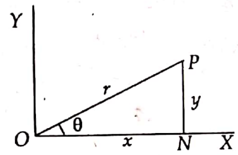


Fig. 40

Let P be any point whose cartesian co-ordinates referred to rectangular axes OXY are (x, y) and whose polar co-ordinates referred to O as pole and OX as initial line are (r, θ). Draw PN perpendicular to OX so that we have

$$ON = x,\ NP = y,$$
$$\angle NOP = \theta \text{ and } OP = r.$$

From the triangle NOP, we have

$$x = ON = OP \cos \angle NOP = r \cos \theta, \quad \ldots (1)$$

$$y = NP = OP \sin \angle NOP = r \sin \theta, \quad \ldots (2)$$

$$r = OP = \sqrt{ON^2 + NP^2} = \sqrt{x^2 + y^2} \quad \ldots (3)$$

and

$$\tan \theta = \frac{NP}{ON} = \frac{y}{x}. \quad \ldots (4)$$

Equations (1) and (2) express the cartesian co-ordinates of P in terms of its polar co-ordinates and equations (3) and (4) express the polar co-ordinates in terms of the cartesian co-ordinates.

11.3. Distance between two points.

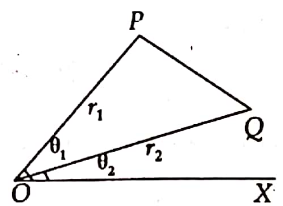


Fig. 41

Let the polar co-ordinates of the two points P and Q be (r_1, θ_1) and (r_2, θ_2) so that $OP = r_1$, $OQ = r_2$, $\angle XOP = \theta_1$, $\angle XOQ = \theta_2$. Then, by Trigonometry, we have in the triangle OPQ,

$$PQ^2 = OP^2 + OQ^2 - 2.OP.OQ \cos \angle QOP$$
$$= r_1^2 + r_2^2 - 2r_1 r_2 \cos (\theta_1 - \theta_2).$$

Therefore $PQ = \{r_1^2 + r_2^2 - 2r_1r_2 \cos (\theta_1 - \theta_2)\}^{\frac{1}{2}}$.

11.4. Area of a triangle.

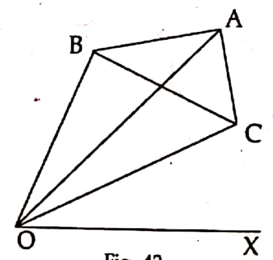


Fig. 42

Let (r_1, θ_1), (r_2, θ_2) and (r_3, θ_3) be the polar co-ordinates of the vertices A, B, C of the triangle ABC. Then

$$\Delta ABC = \Delta AOB + \Delta AOC - \Delta BOC.$$

Now $\Delta AOB = \frac{1}{2} OA . OB \sin \angle BOA$

$= \frac{1}{2} r_1 r_2 \sin (\theta_2 - \theta_1)$;

$\Delta AOC = \frac{1}{2} OA . OC \sin \angle AOC$

$= \frac{1}{2} r_1 r_3 \sin (\theta_1 - \theta_3)$

and $\Delta BOC = \frac{1}{2} OB . OC \sin \angle BOC$

$= \frac{1}{2} r_2 r_3 \sin (\theta_2 - \theta_3)$

$= - \frac{1}{2} r_2 r_3 \sin (\theta_3 - \theta_2)$.

Hence $\Delta ABC = \Delta AOB + \Delta AOC - \Delta BOC$

$= \frac{1}{2} [r_2 r_1 \sin (\theta_2 - \theta_1) + r_1 r_3 \sin (\theta_1 - \theta_3) + r_3 r_2 \sin (\theta_3 - \theta_2)]$.

Cor. The area of a polygon $A_1 A_2 \ldots\ldots A_n$, the polar co-ordinates of whose vertices are $(r_1, \theta_1), (r_2, \theta_2) \ldots\ldots, (r_n, \theta_n)$ is

$$\frac{1}{2} r_1 r_2 \sin (\theta_2 - \theta_1) + \frac{1}{2} r_2 r_3 \sin (\theta_3 - \theta_2) + \ldots\ldots$$
$$\ldots\ldots + \frac{1}{2} r_n r_1 \sin (\theta_1 - \theta_n).$$

Note. If the points A, B, C be collinear, then

$$r_1 r_2 \sin (\theta_2 - \theta_1) + r_2 r_3 \sin (\theta_3 - \theta_2) + r_3 r_1 \sin (\theta_1 - \theta_3) = 0.$$

11.5. Polar equation of a straight line.

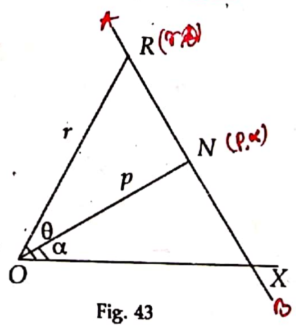


Fig. 43

Let $ON = p$ be the perpendicular on the straight line from O, the pole and α be the vectorial angle of the point N with reference to OX as the initial line. Thus N is the point (p, α).

Let $R\ (r, \theta)$ be any point on the straight line.

Then $\angle XOR = \theta$, so that $\angle NOR = \theta - \alpha$.

Now we have $\frac{ON}{OR} = \cos \angle NOR$

or, $\frac{p}{r} = \cos(\theta - \alpha)$, that is, $r \cos(\theta - \alpha) = p$.

This, being a relation between the polar co-ordinates of any point on the line, is the polar equation of the straight line.

Note 1. This equation may be obtained from the cartesian equation $x \cos\alpha + y \sin\alpha = p$ by puting $x = r\cos\theta$ and $y = r \sin\theta$.

Note 2. If $\alpha = 0$, the straight line becomes $p = r\cos\theta$ and is perpendicular to the polar axis; if $\alpha = \frac{1}{2}\pi$, the equation of the straight line is $p = r \sin\theta$ and is parallel to the polar axis. If $p = 0$, the straight line passes through the pole and in this case $\theta - \alpha = \frac{1}{2}\pi$, that is, $\theta =$ constant is the equation of the straight line.

Note 3. The equations of two parallel straight lines are of the form $r\cos(\theta - \alpha) = p$ and $r\cos(\theta - \alpha) = p'$. The equations of two mutually perpendicular straight lines are of the form

$r\cos(\theta - \alpha) = p$ and $r\cos(\theta - \alpha') = p'$, where $\alpha' \sim \alpha = \frac{1}{2}\pi$.

Note 4. The equation $r\cos(\theta - \alpha) = p$ may be written as

$$\cos\theta \cos\alpha + \sin\theta \sin\alpha = \frac{p}{r}$$

or, $$\frac{1}{r} = \frac{\cos\alpha}{p}\cos\theta + \frac{\sin\alpha}{p}\sin\theta.$$

Thus another form of the polar equation of a straight line is

$$\frac{1}{r} = A\cos\theta + B\sin\theta,$$

where A and B are constants.

This is the general form of the polar equation of a straight line.

11.6 Polar equation of a circle.

Let $C(R, \alpha)$ be the centre of the circle with O as the pole and OX as the initial line. Let a be the radius of the circle, $OC = R$ and $\angle XOC = \alpha$.

Let any line through O making an angle θ with the initial line meet the circle at P and Q and let $OP = r$ and $\angle XOP = \theta$. Thus the polar co-ordinates of P are (r, θ).

Then, in the triangle CPO,

$$CP^2 = OC^2 + OP^2 - 2\,OC.OP \cos \angle COP$$

or, $$a^2 = R^2 + r^2 - 2Rr\cos(\theta - \alpha)$$

or, $$r^2 - 2Rr\cos(\theta - \alpha) + R^2 - a^2 = 0. \quad \ldots \quad (1)$$

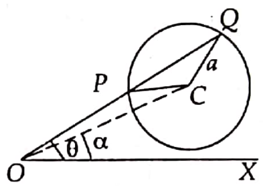
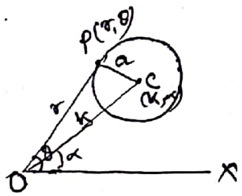


Fig. 44



(1), being a relation between the polar co-ordinates of any point on the circle, is the required polar equation of a circle of radius a.

Cor. If the initial line passes through the centre of the circle, then $\alpha = 0$ and the equation becomes $r^2 - 2Rr\cos\theta + R^2 - a^2 = 0$.

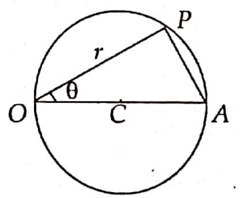


Fig. 45

R=a

If the pole be taken on the circle, then $R = a$ and the general equation reduces to $r = 2a\cos(\theta - \alpha)$.

If $R = a$ and $\alpha = \frac{1}{2}\pi$, then the polar axis becomes the tangent to the circle at the pole and the equation of the circle becomes

$$r = 2a \sin\theta.$$

11.7. Polar equation of a conic referred to a focus as pole.

Let S be the focus, XM_1M be the directrix of the conic having e for its eccentricity. Let us take the pole of the polar system at S and SX, the axis of the conic, as the initial line. Referred to the pole S and the initial line SX, let the co-ordinates of any point P on the conic be (r, θ), so that $SP = r$ and $\angle XSP = \theta$.

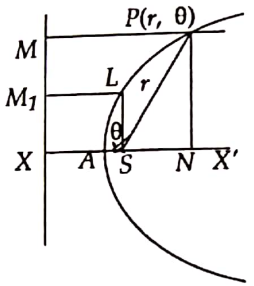


Fig. 46

PN is drawn perpendicular to the axis and SL is the semi-latus rectum.

Therefore $SL = e.LM_1 = e.SX = l$ (say).

PM is drawn perpendicular to the directrix from P.

Thus $r = SP = e.PM = e.NX = e(SX + SN)$

$= e.SX + e.SN = l + er \cos \angle PSN$

$= l + er \cos(\pi - \theta) = l - er \cos\theta$

or, $l = r + er\cos\theta$, whence $\dfrac{l}{r} = 1 + e\cos\theta$.

This, being a relation between the polar co-ordinates of any point on the conic, is the polar equation of the conic.

Note 1. If SX' be taken as the initial line so that $\angle PSN = \theta$, then the equation of the conic becomes $\dfrac{l}{r} = 1 - e\cos\theta$.

Note 2. From the equations it is evident that the curve is symmetrical about the initial line ; for, changing θ to $(-\theta)$ or to $(2\pi - \theta)$ the nature of the equation remains unaltered.

Note 3. If the axis of the conic makes an angle α with the initial line SX and $P(r, \theta)$ be any point on the conic, then

$\angle XSP = \theta - \alpha$, so that $\angle PSN = \pi - \theta + \alpha$.

Therefore the equation of the conic becomes

$$\frac{l}{r} = 1 + e\cos(\theta - \alpha).$$

If, in this case, the initial line be SX', then the equation of the conic is

$$\frac{l}{r} = 1 - e\cos(\theta - \alpha).$$

11.8. Nature of the conic $l/r = 1 + e\cos\theta$.

Case I. *If $e = 1$, then the curve is a parabola whose equation becomes*

$\dfrac{l}{r} = 1 + \cos\theta$, *whence* $r = \dfrac{l}{1 + \cos\theta} = \dfrac{l}{2\cos^2\frac{1}{2}\theta}$.

At A, the vertex, $\theta = 0$, $r = \frac{1}{2}l$.

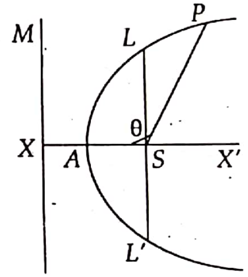


Fig. 47

When $\theta = \frac{1}{2}\pi$, $\cos\theta = 0$ and $r = l$, giving the end L of the latus rectum.

$(1 + \cos\theta)$ decreases with increase in θ, therefore r increases beyond limit as θ tends to π, since l is constant. Thus the curve extends up to infinity.

$(1 + \cos\theta)$ again will be increasing continuously as θ increases beyond π until when $\theta = \frac{3}{2}\pi$ when r becomes equal to l again, giving the other end L' of the latus rectum. Again, when $\theta = 2\pi$, $r = \frac{1}{2}l$. Thus the curve described is a parabola as shown in the figure.

Note. If SX' be taken as the initial line, the equation of the parabola becomes $$r = \frac{l}{1 - \cos\theta} = \frac{l}{2\sin^2\frac{1}{2}\theta} = \frac{l}{2}\operatorname{cosec}^2\frac{\theta}{2}.$$

Case II. *If $e < 1$, then the curve is an ellipse.*

At A, $\theta = 0$ and $\frac{l}{r} = 1 + e$.

The equation of the curve being $\frac{l}{r} = 1 + e \cos \theta$, as $\cos \theta$ decreases with increase in θ, r also increases until θ reaches the value π at A' when $\frac{l}{r} = 1 - e$ and e being less than 1, r is positive.

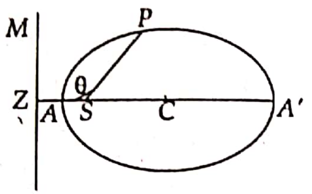


Fig. 48

As θ increases beyond π and until it reaches 2π, $\cos \theta$ goes on increasing from (-1) to 1.

Hence r decreases continuously from $\frac{l}{1-e}$ to $\frac{l}{1+e}$. So the maximum and minimum values of r are $\frac{l}{1-e}$ at A' and $\frac{l}{1+e}$ at A respectively.

Again, for any value of θ, $\cos \theta = \cos (2\pi - \theta)$, showing that the curve is symmetrical about the axis, which is the initial line.

Thus, for $e < 1$, the equation gives a closed curve symmetrical about the axis, as shown in the diagram.

Note. The lengths of the semi-axes of the ellipse are

$$\frac{l}{1-e^2} \text{ and } \frac{l}{\sqrt{1-e^2}}.$$

Case III. *If $e > 1$, then the curve is a hyperbola.*

At A, $\theta = 0$ and $\frac{l}{r} = 1 + e$.

When $\theta = \pi/2$, $\cos \theta = 0$ and $l = r$.

As θ increases, $\cos \theta$ decreases and hence r increases, but remains finite till $1 + e \cos \theta = 0$.

When $1 + e \cos \theta = 0$, that is, $\cos \theta = -\frac{1}{e}$, r is infinite.

If $e > 1$, then the conic has two infinite radius vectors SL and SL' corresponding to the two values of $\cos \theta = -\frac{1}{e}$.

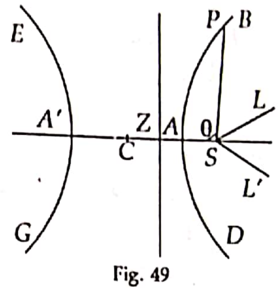


Fig. 49

Let the two values of θ be α and β which are $\angle ASL$ and $\angle ASL'$, the inclinations of SL and SL' to the initial line SA. SL and SL' are equally inclined to the initial line, that is, to the transverse axis at $\cos^{-1}\left(\frac{1}{e}\right)$ and are also parallel to the asymptotes.

If the vectorial angle θ lies between α and β, then the radius vector is negative and we get the other branch of the hyperbola. Hence a hyperbola has two branches tending to infinity, one, if θ lies outside α and β, and the other, if θ lies between α and β.

Note. The lengths of the semi-axes of the hyperbola are

$$\frac{l}{e^2-1} \text{ and } \frac{l}{\sqrt{e^2-1}}.$$

11.9. Equations of the directrices.

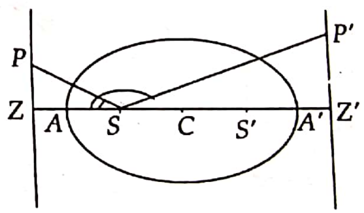


Fig. 50

Let the equation of the conic be $\frac{l}{r} = 1 + e \cos \theta$.

If $e < 1$, the conic is an ellipse. Let the focus S be the pole and $P(r, \theta)$ be a point on the directrix ZP which is nearer to the pole.

Now, $SZ = \frac{l}{e}$ and $SZ' = 2CZ - SZ = \frac{2l}{e(1-e^2)} - \frac{l}{e} = \frac{l}{e} \cdot \frac{1+e^2}{1-e^2}$,

since $b^2 = a^2(1-e^2)$, $l = \frac{b^2}{a}$ and $CZ = SZ + ae$.

Now, for the directrix PZ, we have

$$r \cos\theta = SZ = \frac{l}{e}$$

or, $$\frac{l}{r} = e \cos\theta. \qquad \ldots \quad (1)$$

For the other directrix $P'Z'$, if P' be the point (r, θ), we have

$$r \cos(\pi - \theta) = SZ' = \frac{l}{e} \cdot \frac{1+e^2}{1-e^2}$$

or, $$-r\cos\theta = \frac{l}{e} \cdot \frac{1+e^2}{1-e^2}$$

or, $$\frac{l}{r} = -\frac{1-e^2}{1+e^2} e \cos\theta. \qquad \ldots \quad (2)$$

(1) and (2) are the equations of the directrices.

If $e > 1$, then the conic is a hyperbola.

In this case $SZ = \frac{l}{e}$ and $SZ' = SZ + 2CZ$.

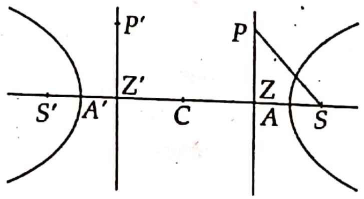


Fig. 51

Hence $SZ' = \frac{l}{e} + \frac{2l}{e(e^2-1)} = \frac{l}{e} \cdot \frac{e^2+1}{e^2-1}$,

since $b^2 = a^2(e^2-1)$ and $l = \frac{b^2}{a}$.

For the directrix PZ, we have, if the point P be (r, θ), then

$$r\cos\theta = \frac{l}{e}$$

or, $$\frac{l}{r} = e\cos\theta. \qquad \ldots \quad (3)$$

For the other directrix $P'Z'$, if P' be the point (r, θ), then

$$r\cos\theta = SZ'$$

or, $$r\cos\theta = \frac{l}{e} \cdot \frac{e^2+1}{e^2-1}$$

or, $$\frac{l}{r} = \frac{e^2-1}{e^2+1} e \cos\theta. \qquad \ldots \quad (4)$$

(3) and (4) are the equations of the directrices.

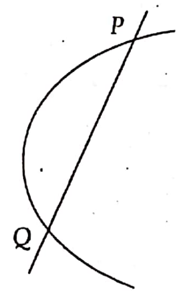
For a parabola $e = 1$ and the equation of the directrix is

$$\frac{l}{r} = \cos\theta.$$

11.10. Equation of the chord of a conic $\frac{l}{r} = 1 + e\cos\theta$, joining two points whose vectorial angles are $(\alpha-\beta)$ and $(\alpha+\beta)$.

Let the equation of the conic be $\frac{l}{r} = 1 + e\cos\theta$. $\qquad \ldots \quad (1)$

Let P and Q be two points on the conic such that the radius vectors of P and Q are r_1 and r_2 and their respective vectorial angles are $(\alpha - \beta)$ and $(\alpha + \beta)$.



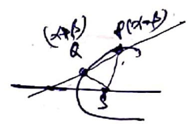


Fig. 52

Let the equation of the straight line (chord) PQ be

$$\frac{l}{r} = A\cos\theta + B\sin\theta. \qquad \ldots \quad (2)$$

This equation contains two independent constants A and B which can be obtained from the condition that it passes through the two given points P and Q.

Now, $P(r_1, \alpha - \beta)$ is a common point on the conic (1) and the straight line (2).

Therefore $\dfrac{l}{r_1} = 1 + e\cos(\alpha - \beta)$

and $\dfrac{l}{r_1} = A\cos(\alpha - \beta) + B\sin(\alpha - \beta)$.

Therefore $A\cos(\alpha - \beta) + B\sin(\alpha - \beta) = 1 + e\cos(\alpha - \beta)$

or, $$(A - e)\cos(\alpha - \beta) + B\sin(\alpha - \beta) = 1. \qquad \ldots (3)$$

Similarly for the point $Q(r_2, \alpha + \beta)$, we have

$$(A - e)\cos(\alpha + \beta) + B\sin(\alpha + \beta) = 1. \qquad \ldots (4)$$

Solving from (3) and (4), we get

$A = e + \cos\alpha\sec\beta$ and $B = \sin\alpha\sec\beta$.

Thus, from (2), the required equation of the chord is

$$\frac{l}{r} = e\cos\theta + \sec\beta\cos(\theta - \alpha). \qquad \ldots (5)$$

Cor. If the equation of the conic be $\dfrac{l}{r} = 1 - e\cos\theta$, then the equation of the chord PQ will be

$$\frac{l}{r} = \sec\beta\cos(\theta - \alpha) - e\cos\theta.$$

Note 1. Equation of the chord joining the two points whose vectorial angles are α and β is

$$\frac{l}{r} = \sec\frac{\beta - \alpha}{2}\cos\left(\theta - \frac{\alpha + \beta}{2}\right) + e\cos\theta.$$

Note 2. If the equation of the conic be $\dfrac{l}{r} = 1 + e\cos(\theta - \gamma)$, then the chord joining the points whose vectorial angles are $(\alpha - \beta)$ and $(\alpha + \beta)$ is

$$\frac{l}{r} = e\cos(\theta - \gamma) + \sec\beta\cos(\theta - \alpha).$$

11.11. Equation of the tangent to a conic.

To find the equation of the tangent at a point whose vectorial angle is α, we are to put $\beta = 0$ in the equation of the chord joining the two points whose vectorial angles are $(\alpha - \beta)$ and $(\alpha + \beta)$ of the conic and we have the equation of the tangent to the conic $\dfrac{l}{r} = 1 + e\cos\theta$ as

$$\frac{l}{r} = e\cos\theta + \cos(\theta - \alpha).$$

Cor. If the equation of the conic be $\dfrac{l}{r} = 1 - e\cos\theta$, then the equation of the tangent at the point α is $\dfrac{l}{r} = \cos(\theta - \alpha) - e\cos\theta$.

Note. The equation of the tangent to the conic

$\dfrac{l}{r} = 1 + e\cos(\theta - \gamma)$ at α is $l/r = e\cos(\theta - \gamma) + \cos(\theta - \alpha)$.

11.12. Equation of the normal to a conic.

Let the equation of the conic be $$\frac{l}{r} = 1 + e\cos\theta \qquad \ldots (1)$$

so that the equation of the tangent at a point whose vectorial angle is α is $\dfrac{l}{r} = e\cos\theta + \cos(\theta - \alpha)$.

The equation of a straight line perpendicular to this line is

$$\frac{k}{r} = e\cos(\theta + \tfrac{1}{2}\pi) + \cos(\theta + \tfrac{1}{2}\pi - \alpha)$$

or, $$\frac{k}{r} = -e\sin\theta - \sin(\theta - \alpha). \qquad \ldots (2)$$

Now k is so chosen that this perpendicular is the normal, that is, it passes through the point $\left(\dfrac{l}{1 + e\cos\alpha}, \alpha\right)$, which is the point of contact. Putting these in (2), we get

$$k\frac{1 + e\cos\alpha}{l} = -e\sin\alpha, \text{ whence } k = -\frac{le\sin\alpha}{1 + e\cos\alpha}.$$

Hence the equation of the normal is

$$\frac{le\sin\alpha}{1 + e\cos\alpha}\cdot\frac{1}{r} = e\sin\theta + \sin(\theta - \alpha).$$

Cor. If the equation of the conic be $\dfrac{l}{r} = 1 - e\cos\theta$, then the equation of the normal at the point α is

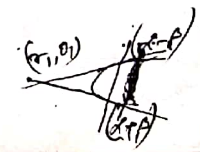
$$\frac{le\sin\alpha}{1 - e\cos\alpha}\cdot\frac{1}{r} = e\sin\theta - \sin(\theta - \alpha).$$

11.13. Equation of the chord of contact of tangents.

Let the equation of the conic be $\dfrac{l}{r} = 1 + e\cos\theta$.

Let $(\alpha - \beta)$ and $(\alpha + \beta)$ be the vectorial angles of the two points of contact of the tangents from a given point (r_1, θ_1) to the conic. Then the equation of the chord is

$$\frac{l}{r} = e\cos\theta + \sec\beta\cos(\theta - \alpha). \qquad (1)$$



Now the equations of the tangents at the points whose vectorial angles are $(\alpha - \beta)$ and $(\alpha + \beta)$ are

$$\frac{l}{r} = e \cos \theta + \cos (\theta - \alpha + \beta)$$

and $$\frac{l}{r} = e \cos \theta + \cos (\theta - \alpha - \beta).$$

Both these tangents pass through the point (r_1, θ_1).

$$\left.\begin{aligned} \text{Therefore } \frac{l}{r_1} &= e \cos \theta_1 + \cos (\theta_1 - \alpha + \beta) \\ \text{and } \frac{l}{r_1} &= e \cos \theta_1 + \cos (\theta_1 - \alpha - \beta). \end{aligned}\right\} \quad \ldots \quad (2)$$

Hence $\cos (\theta_1 - \alpha + \beta) = \cos (\theta_1 - \alpha - \beta)$

or, $$\theta_1 - \alpha + \beta = 2n\pi \pm (\theta_1 - \alpha - \beta).$$

As the upper sign gives a particular value of β, we take the lower sign.

Therefore $\theta_1 - \alpha + \beta = 2n\pi - \theta_1 + \alpha + \beta$, whence $\alpha = \theta_1 - n\pi$.

From (2), $$\frac{l}{r_1} = e \cos \theta_1 + \cos (n\pi - \beta)$$
$$= e \cos \theta_1 + (-1)^n \cos \beta$$

or, $$\frac{l}{r_1} - e \cos \theta_1 = (-1)^n \cos \beta.$$

Hence, from (1), we get the required equation of the chord of contact of tangents from the point (r_1, θ_1) to the conic as

$$\left(\frac{l}{r} - e \cos \theta\right)\left(\frac{l}{r_1} - e \cos \theta_1\right) = (-1)^n \cos (\theta - \theta_1 + n\pi)$$
$$= \cos (\theta - \theta_1).$$

Cor. If the equation of the conic be $\frac{l}{r} = 1 - e \cos \theta$, then the equation of the chord of contact of the tangents to the conic from the point (r_1, θ_1) is $\left(\frac{l}{r} + e \cos \theta\right)\left(\frac{l}{r_1} + e \cos \theta_1\right) = \cos (\theta - \theta_1)$.

11.14. Equation of the polar of a point (r_1, θ_1) with respect to a conic.

Let the equation of the conic be $\frac{l}{r} = 1 + e \cos \theta$.

Let AB be a chord passing through $P(r_1, \theta_1)$. Let the tangents at A and B meet in $T(r', \theta')$. The locus of T is the polar of P with respect to the conic. Now AB is the chord of contact of the tangents drawn from T to the conic. So its equation is

$$\left(\frac{l}{r} - e \cos \theta\right)\left(\frac{l}{r'} - e \cos \theta'\right) = \cos (\theta - \theta').$$

Since it passes through the point $P(r_1, \theta_1)$, we have

$$\left(\frac{l}{r_1} - e \cos \theta_1\right)\left(\frac{l}{r'} - e \cos \theta'\right) = \cos (\theta_1 - \theta').$$

Hence the locus of $T(r', \theta')$ is

$$\left(\frac{l}{r} - e \cos \theta\right)\left(\frac{l}{r_1} - e \cos \theta_1\right) = \cos (\theta - \theta_1).$$

This is the equation of the polar of the point P with respect to the conic.

11.15. Equation of the asymptotes of a conic.

Let the equation of the conic be $\frac{l}{r} = 1 + e \cos \theta$.

The equation of its tangent at a point whose vectorial angle is α is

$$\frac{l}{r} = e \cos \theta + \cos (\theta - \alpha). \quad \ldots \quad (1)$$

This tangent will be an asymptote of the conic, if its point of contact be at infinity.

Then $r \to \infty$ as $\theta \to \alpha$.

Therefore $0 = 1 + e \cos \alpha$.

Hence $\cos \alpha = -\frac{1}{e}$ and $\sin \alpha = \pm \sqrt{1 - \frac{1}{e^2}}$.

Putting these values of $\sin \alpha$ and $\cos \alpha$ in (1), we get the required equation of the asymptotes of the conic as

$$\frac{l}{r} = \left(e - \frac{1}{e}\right) \cos \theta \pm \sqrt{\left(1 - \frac{1}{e^2}\right)} \sin \theta$$

or, $$\left\{\frac{l}{r} - \left(e - \frac{1}{e}\right) \cos \theta\right\}^2 = \left(1 - \frac{1}{e^2}\right) \sin^2 \theta$$

or, $$\left\{\frac{el}{r} + (1 - e^2) \cos \theta\right\}^2 = (e^2 - 1) \sin^2 \theta.$$

11.16. Illustrative examples.

Ex. 1. *Find the nature of the conic* $\frac{8}{r} = 4 - 5\cos\theta$.

The given equation can be written as

$$\frac{2}{r} = 1 - \tfrac{5}{4}\cos\theta.$$

Comparing this equation with the equation

$$\frac{l}{r} = 1 - e\cos\theta,$$

we see that here $e = \frac{5}{4} > 1$.

Hence the given equation represents a hyperbola.

Ex. 2. *Find the points on the conic* $\frac{14}{r} = 3 - 8\cos\theta$ *whose radius vector is* 2.

For the points with radius vector 2 on the given curve, we have

$$\frac{14}{2} = 3 - 8\cos\theta$$

or, $7 = 3 - 8\cos\theta$

or, $\cos\theta = -\frac{4}{8} = -\frac{1}{2}$.

Hence $\theta = \frac{2}{3}\pi, -\frac{2}{3}\pi$.

Thus the required points are $(2, \frac{2}{3}\pi)$ and $(2, -\frac{2}{3}\pi)$.

Ex. 3. *Find the polar equation of the ellipse* $\frac{x^2}{36} + \frac{y^2}{20} = 1$, *if the pole be at its right-hand focus and the positive direction of the x-axis be the positive direction of the polar axis.* [T. H. 1992]

For the ellipse $\frac{x^2}{36} + \frac{y^2}{20} = 1$, the lengths of the semi-major and semi-minor axes (that is, a and b) are given by $a^2 = 36$ and $b^2 = 20$.

Therefore the semi-latus rectum of the ellipse is

$$l = \frac{b^2}{a} = \frac{20}{6} = \frac{10}{3}$$

and the eccentricity e of the ellipse is given by

$$b^2 = a^2(1 - e^2)$$

or, $20 = 36(1 - e^2)$

or, $1 - e^2 = \frac{20}{36} = \frac{5}{9}$

or, $e^2 = 1 - \frac{5}{9} = \frac{4}{9}$

or, $e = \frac{2}{3}$.

The required polar equation of the ellipse is

$$\frac{l}{r} = 1 + e\cos\theta$$

or, $\frac{\frac{10}{3}}{r} = 1 + \frac{2}{3}\cos\theta$

or, $\frac{10}{r} = 3 + 2\cos\theta$.

Ex. 4. *Show that the straight line* $r\cos(\theta - \alpha) = p$ *touches the conic* $\frac{l}{r} = 1 + e\cos\theta$, *if* $(l\cos\alpha - ep)^2 + l^2\sin^2\alpha = p^2$. [C. H. 1997]

Let the given straight line

$$r\cos(\theta - \alpha) = p$$

that is, $\frac{p}{r} = \cos(\theta - \alpha) = \cos\theta\cos\alpha + \sin\theta\sin\alpha$ ... (1)

touch the conic at the point 'β'.

The equation of the tangent to the given conic at the point 'β' is

$$\frac{l}{r} = e\cos\theta + \cos(\theta - \beta)$$

$= \cos\theta\,(e + \cos\beta) + \sin\theta\sin\beta$. ... (2)

Equations (1) and (2) are identical.

Therefore $\frac{p}{l} = \frac{\cos\alpha}{e + \cos\beta} = \frac{\sin\alpha}{\sin\beta}$.

Hence $\sin\beta = \frac{l\sin\alpha}{p}$ and $\cos\beta = \frac{l\cos\alpha}{p} - e$.

Now $\sin^2\beta + \cos^2\beta = 1$

or, $\frac{l^2\sin^2\alpha}{p^2} + \left(\frac{l\cos\alpha}{p} - e\right)^2 = 1$

or, $(l\cos\alpha - ep)^2 + l^2\sin^2\alpha = p^2$.

Ex. 5. (a) *Show that the semi-latus rectum of a conic is a harmonic mean between the segments of any focal chord.* [B. H. 1991]

(b) *If PSQ and PS'R be two chords of an ellipse through the foci S and S', then prove that* $\left(\frac{SP}{SQ} + \frac{S'P}{S'R}\right)$ *is independent of the position of P.*

(a) Let PSQ be a focal chord. If the vectorial angle of P be θ, then that of Q will be $(\theta + \pi)$.

Let the equation of the conic be $\frac{l}{r} = 1 + e\cos\theta$.

Let $SP = r$ and $SQ = r'$.

Therefore

$$\frac{l}{r} = 1 + e\cos\theta \text{ and } \frac{l}{r'} = 1 + e\cos(\theta + \pi) = 1 - e\cos\theta.$$

Adding these, we get $\frac{l}{SP} + \frac{l}{SQ} = 2$, whence $\frac{1}{SP} + \frac{1}{SQ} = \frac{2}{l}$.

This also shows that in any conic the sum of the reciprocals of the segments of a focal chord is constant.

(*b*) Let PSQ be the focal chord such that the vectorial angles of P and Q are α and $(\alpha + \pi)$ respectively.

Therefore $\frac{l}{SP} = 1 + e\cos\alpha$

and $\frac{l}{SQ} = 1 + e\cos(\alpha + \pi) = 1 - e\cos\alpha.$

Hence $\frac{1}{SP} + \frac{1}{SQ} = \frac{2}{l}$

or, $\frac{SP}{SQ} = \frac{2}{l}\cdot SP - 1.$ ... (1)

Again $PS'R$ is also a focal chord.

Therefore $\frac{1}{S'P} + \frac{1}{S'R} = \frac{2}{l}$

or, $\frac{S'P}{S'R} = \frac{2}{l}\cdot S'P - 1.$ ... (2)

Adding (1) and (2), we get

$$\frac{SP}{SQ} + \frac{S'P}{S'R} = \frac{2}{l}(SP + S'P) - 2 = \frac{4a}{l} - 2$$

where $2a$ is the major axis of the ellipse.

The right-hand side is independent of the position of P.

Ex. 6. *PSP' is a focal chord of the conic. Prove that the angle between the tangents at P and P' is $\tan^{-1}\frac{2e\sin\alpha}{1-e^2}$, where α is the angle between the chord and the major axis.* [C. H. 1989, 1993]

The focal chord PSP' makes an angle α with the major axis of the conic $\frac{l}{r} = 1 + e\cos\theta$. Then the vectorial angles of P and P' are α and $(\alpha + \pi)$ respectively. The tangent at α is

$$\frac{l}{r} = \cos(\theta - \alpha) + e\cos\theta$$

or, $l = r\cos\theta(\cos\alpha + e) + r\sin\alpha\sin\theta = x(\cos\alpha + e) + y\sin\alpha.$

Therefore m = slope of the tangent at $P = -\frac{\cos\alpha + e}{\sin\alpha}$.

Similarly m' = slope of the tangent at P'

$$= -\frac{\cos(\alpha + \pi) + e}{\sin(\alpha + \pi)} = \frac{e - \cos\alpha}{\sin\alpha}.$$

If ϕ be the angle between the tangents, then

$$\tan\phi = \frac{m - m'}{1 + mm'} = \frac{2e\sin\alpha}{\sin^2\alpha - e^2 + \cos^2\alpha} = \frac{2e\sin\alpha}{1 - e^2}$$

or, $\phi = \tan^{-1}\frac{2e\sin\alpha}{1-e^2}$.

Ex. 7. (*a*) *Find the point of intersection of the two tangents at α and β to the conic $\frac{l}{r} = 1 + e\cos\theta$.*

(*b*) *If the tangents at P and Q of a conic meet at a point T, and S be the focus of the conic, then prove that*

$$ST^2 = SP\,.\,SQ, \text{ if the conic be a parabola.}$$ [C. H. 1992]

(*a*) The tangents at α and β are

$$\frac{l}{r} = \cos(\theta - \alpha) + e\cos\theta \quad \ldots (1)$$

and $$\frac{l}{r} = \cos(\theta - \beta) + e\cos\theta. \quad \ldots (2)$$

Subtracting (2) from (1), we have

$$\cos(\theta - \alpha) = \cos(\theta - \beta), \text{ whence } \theta - \alpha = \pm(\theta - \beta).$$

The positive sign is inadmissible ; for, in that case, $\alpha = \beta$.

Therefore $\theta - \alpha = -(\theta - \beta)$, giving $\theta = \frac{1}{2}(\alpha + \beta)$.

Substituting this value of θ in (1), we get

$$\frac{l}{r} = \cos\{\tfrac{1}{2}(\alpha + \beta) - \alpha\} + e\cos\tfrac{1}{2}(\alpha + \beta)$$

or, $\frac{l}{r} = \cos\frac{1}{2}(\beta - \alpha) + e\cos\frac{1}{2}(\alpha + \beta)$.

If the point of intersection of the tangents be (r_1, θ_1), then

$$\theta_1 = \tfrac{1}{2}(\alpha + \beta) \text{ and } \frac{l}{r_1} = \cos\tfrac{1}{2}(\beta - \alpha) + e\cos\tfrac{1}{2}(\alpha + \beta).$$

(*b*) Let the given parabola be $\frac{l}{r} = 1 + \cos\theta$ and let the vectorial angles of P and Q on it be α and β respectively. If S be the focus of the parabola, then

$$SP = \frac{l}{1 + \cos\alpha} = \frac{l}{2}\sec^2\frac{\alpha}{2}$$

and $$SQ = \frac{l}{1 + \cos\beta} = \frac{l}{2}\sec^2\frac{\beta}{2}.$$

Therefore $SP\,.\,SQ = \frac{l^2}{4}\sec^2\frac{\alpha}{2}\sec^2\frac{\beta}{2}.$ ... (1)

As in (a) the co-ordinates (r_1, θ_1) of the point of intersection T of tangents at α and β to the parabola are (since $e = 1$)

$$\theta_1 = \frac{1}{2}(\alpha + \beta) \text{ and } \frac{l}{r_1} = \cos\frac{1}{2}(\beta - \alpha) + \cos\frac{1}{2}(\alpha + \beta) = 2\cos\frac{\alpha}{2}\cos\frac{\beta}{2}.$$

Therefore $\quad \dfrac{l}{ST} = 2\cos\dfrac{\alpha}{2}\cos\dfrac{\beta}{2}$

or, $\quad ST = \dfrac{l}{2}\sec\dfrac{\alpha}{2}\sec\dfrac{\beta}{2}$

or, $\quad ST^2 = \dfrac{l^2}{4}\sec^2\dfrac{\alpha}{2}\sec^2\dfrac{\beta}{2} = SP \cdot SQ$, by (1).

Ex. 8. *A circle of given diameter d passes through the focus of a given conic and cuts it in four points whose distances from the focus are r_1, r_2, r_3, r_4. Prove that*

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4} = \frac{2}{l} \text{ and } r_1 r_2 r_3 r_4 = \frac{d^2 l^2}{e^2},$$

where l is the semi-latus rectum and e is the eccentricity of the conic. [B. H. 1984]

Since the circle passes through the focus (which is the pole) of the conic whose equation is assumed to be

$$\frac{l}{r} = 1 + e\cos\theta, \qquad \ldots \quad (1)$$

its equation will be $r = d\cos(\theta - \alpha)$ $\qquad \ldots \quad (2)$

in which the diameter passing through the focus is inclined at an angle α to the axis.

If we eliminate θ between (1) and (2), we shall get a biquadratic equation in r, whose roots r_1, r_2, r_3, r_4 will give the distances of the points of intersection of (1) and (2) from the focus of the conic.

From (1), we have $\cos\theta = \dfrac{l - r}{er}$ and hence $\sin\theta = \sqrt{1 - \left(\dfrac{l-r}{er}\right)^2}$.

Then, from (2), we have

$$r = d\cos\theta\cos\alpha + d\sin\theta\sin\alpha$$

or, $\quad \{er^2 - d(l - r)\cos\alpha\}^2 = \{e^2 r^2 - (l - r)^2\}\, d^2\sin^2\alpha$

or, $\quad e^2 r^4 + 2edr^3\cos\alpha + r^2(d^2 - 2eld\cos\alpha - e^2 d^2\sin^2\alpha) - 2ld^2 r + d^2 l^2 = 0.$

Therefore $\quad r_1 r_2 r_3 r_4 = \dfrac{d^2 l^2}{e^2},$

and $\quad \dfrac{1}{r_1} + \dfrac{1}{r_2} + \dfrac{1}{r_3} + \dfrac{1}{r_4} = \dfrac{\Sigma r_1 r_2 r_3}{r_1 r_2 r_3 r_4} = \dfrac{\dfrac{2ld^2}{e^2}}{\dfrac{d^2 l^2}{e^2}} = \dfrac{2}{l}.$

Examples XI

1. (a) Find the rectangular cartesian co-ordinates of the points whose polar co-ordinates are

(i) $(4, \frac{2}{3}\pi)$; (ii) $(2, \frac{7}{6}\pi)$.

(b) Find the polar co-ordinates of the points whose cartesian co-ordinates are

(i) $(-1, -1)$; (ii) $(3, -3)$.

(c) Transform the following to cartesian equations :

(i) $r = 2$; (ii) $\theta = \frac{1}{4}\pi$; (iii) $1/r = 1 + \cos\theta$.

(d) Transform the following to polar equations :

(i) $(x^2 + y^2)^2 = ax^2$; (ii) $x^3 = y^2(2a - x)$.

2. (a) Find the distance between the following points :

(i) $(1, 30°)$ and $(3, 90°)$; (ii) $(2, 40°)$ and $(4, 100°)$.

(b) Show that the locus of the point, which is always at a distance of 2 units from the point $(3, \frac{1}{3}\pi)$, is $r^2 - 6r\cos(\theta - \frac{1}{3}\pi) + 5 = 0$.

3. Find the area of the triangle whose vertices are $A(2, -30°)$, $B(3, 120°)$ and $C(1, 210°)$.

Also find the area of the square whose one side is AC.

4. (a) Find the polar equation of the straight line joining the two points $(1, \frac{1}{2}\pi)$ and $(2, \pi)$.

(b) Show that the polar equation of the straight line passing through the points (r_1, θ_1) and (r_2, θ_2) is

$$\frac{1}{r}\sin(\theta_1 - \theta_2) - \frac{1}{r_1}\sin(\theta - \theta_2) + \frac{1}{r_2}\sin(\theta - \theta_1) = 0.$$

Hence find the condition of collinearity of the points (r_1, θ_1), (r_2, θ_2) and (r_3, θ_3).

(c) The vectorial angle of a point P on the straight line joining the points (r_1, θ_1) and (r_2, θ_2) is $\frac{1}{2}(\theta_1 + \theta_2)$. Find the radius vector of P.

(d) Show that the perpendicular distance of the point (r_1, θ_1) from the straight line $r\cos(\theta - \alpha) = p$ is

$$r_1\cos(\theta_1 - \alpha) - p.$$

(e) Show that the locus of a point whose distance from the pole is equal to its distance from the straight line $r \cos\theta + k = 0$ is

$$2r \sin^2 \tfrac{1}{2}\theta = k. \qquad \text{[C. H. 1980]}$$

(f) Find the polar equation of the straight lines bisecting the angles between the straight lines $\theta = \alpha$ and $\theta = \beta$.

5. Find the condition that the three straight lines

$$r \cos(\theta - \alpha) = a,\ r \cos(\theta - \beta) = b \text{ and } r \cos(\theta - \gamma) = c$$

may meet at a point.

6. (a) Find the centre of the circle $r = 3 \sin\theta + 4 \cos\theta$.

(b) Show that the co-ordinates of the centre of the circle $r = a \cos\theta + b \sin\theta$ is $\{\tfrac{1}{2}\sqrt{a^2 + b^2},\ \tan^{-1}\left(\frac{b}{a}\right)\}$.

(c) Find the polar co-ordinates of the centre and the radius of each of the circles

(i) $r = -2 \cos\theta$; (ii) $r = 6 \cos\left(\frac{\pi}{3} - \theta\right)$;

(iii) $r = 8 \sin\left(\theta - \frac{\pi}{3}\right)$.

7. (a) Find the equation of the circle described on the line segment joining (a, α) and (b, β) as diameter.

(b) Show that $r^2 - 2ar \cos\theta - 3a^2 = 0$ is the polar equation of a circle whose centre lies on the polar axis.

(c) Find the polar equation of the circle which passes through the pole and the two points whose polar co-ordinates are $(d, 0)$ and $(2d, \tfrac{1}{3}\pi)$. Find also the radius of the circle. [C. H. 1977]

(d) Prove that the equations

$$r = a \cos(\theta - \alpha) \text{ and } r = b \sin(\theta - \alpha)$$

represent two orthogonal circles.

(e) If d be the diameter of the circle through the pole and the points (r_1, θ_1), (r_2, θ_2), then show that

$$d^2 \sin^2(\theta_1 - \theta_2) = r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2). \quad \text{[C. H. 1978]}$$

8. Show that the triangle formed by the pole and the [illegible] two intersection of the circle $r = 4 \cos\theta$ with the straight line $r \cos$ [illegible] is equilateral. [C. H. 1979]

9. If from a fixed point a straight line be drawn to cut a given circle, then show that the rectangle contained by the segments is constant.

10. (a) Find the nature of the following conics :

(i) $\frac{2}{r} = 3 - 3 \cos\theta$; (ii) $\frac{3}{r} = 2 + 4 \cos\theta$;

(iii) $\frac{17}{r} = \sqrt{5} - 2 \cos\theta$; (iv) $\frac{3}{r} = 1 + 2 \sin\theta$.

(b) Determine the nature of the conic $r = \frac{1}{4 - 5 \cos\theta}$.

Find the eccentricity, the length of the latus rectum and directrices.

(c) Show that the equations $\frac{l}{r} = 1 + e \cos\theta$ and $\frac{l}{r} = -1 + e \cos\theta$ represent the same conic.

(d) Find the lengths of the semi-axes of the conics

(i) $\frac{l}{r} = 1 + e \cos\theta$; (ii) $r = \frac{144}{13 - 5 \cos\theta}$;

(iii) $r = \frac{18}{4 - 5 \cos\theta}$.

(e) Find the directrices of the conic $\frac{12}{r} = 2 - \cos\theta$.

(f) Find the directrices of the conics :

(i) $r(1 - \cos\theta) = 2$; (ii) $r(5 - 2 \cos\theta) = 21$;

(iii) $r(3 - 5 \cos\theta) = 16$.

11. (a) Find the points on the conic $\frac{15}{r} = 1 - 4 \cos\theta$ whose radius vector is 5.

(b) Find the points on the conic $\frac{3}{r} = 1 + 2 \cos\theta$ whose vectorial angle is $\frac{\pi}{3}$.

(c) Find the point on the conic $\frac{l}{r} = 1 - \cos\theta$ which has the smallest radius vector.

12. (a) Show that the point of intersection of the straight lines $r\cos(\theta-\alpha)=p$ and $r\cos(\theta-\beta)=p$ is $\left(p\sec\frac{\alpha-\beta}{2}, \frac{\alpha+\beta}{2}\right)$.

[T. H. 1991 ; B. H. 1995]

(b) Find the points of intersection of the straight line $r\cos\theta = a$ and the circle $r = 2a\cos\theta$.

(c) Find the points of intersection of the two conics $\frac{1}{r} = 1 + \cos\theta$ and $\frac{3}{r} = 1 - \cos\theta$.

13. Show that the locus of the equation

$$r^2 - ar\cos 2\theta\sec\theta - 2a^2 = 0$$

consists of a straight line and a circle. [N. B. H. 1979]

14. (a) Find the polar equation of the ellipse $\frac{x^2}{64} + \frac{y^2}{28} = 1$, if the pole be at its right-hand focus and the positive direction of the x-axis be the positive direction of the polar axis. [C. H. 1996]

(b) Find the polar equation of the left branch of the hyperbola $\frac{x^2}{16} - \frac{y^2}{36} = 1$, if the pole be at its left-hand focus and the positive direction of the x-axis be the direction of the polar axis.

(c) Show that the polar equation of the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$, if the pole be at its centre and the positive direction of the x-axis be the direction of the polar axis, is

$$r^2 = \frac{36}{9 - 5\cos^2\theta}.$$

[Put $x = r\cos\theta$ and $y = r\sin\theta$ in the given cartesian equation of the ellipse.]

15. (a) Find the polar equation of the parabola whose latus rectum is 8.

(b) Show that the polar equation of a parabola may be written in the form $\sqrt{r}\cos\frac{1}{2}\theta = \sqrt{a}$, where $4a$ is the length of the latus rectum.

16. (a) Find the polar equation of the straight line joining two points on the parabola $\frac{2a}{r} = 1 + \cos\theta$ with $(\alpha - \beta)$ and $(\alpha + \beta)$ as their vectorial angles. [B. H. 1992 ; V. H. 1993]

(b) Find the polar equation of the straight line joining two points on the conic $\frac{l}{r} = 1 - e\cos\theta$ whose vectorial angles are α and β.

(c) Find the polar equation of the straight line joining two points on the circle $r = 2d\cos\theta$ with θ_1 and θ_2 as their vectorial angles. Also find the equation of the tangent to the circle at the point θ_1.

17. (a) Find the polar equation of the tangent to the conic $\frac{2}{r} = 1 - \cos\theta$ at $\theta = \frac{1}{2}\pi$.

(b) Show that the equation of the tangent to the conic $\frac{l}{r} = 1 + e\cos\theta$, parallel to the tangent at $\theta = \alpha$, is given by

$$l(e^2 + 2e\cos\alpha + 1) = r(e^2 - 1)\{\cos(\theta - \alpha) + e\cos\theta\}.$$

[V. H. 1987]

18. (a) Show that the condition that the straight line $\frac{1}{r} = a\cos\theta + b\sin\theta$ may touch the circle $r = 2k\cos\theta$ is

$$b^2k^2 + 2ak = 1.$$

(b) Show that the condition that the straight line $\frac{l}{r} = A\cos\theta + B\sin\theta$ may be a tangent to the conic $\frac{l}{r} = 1 + e\cos(\theta - \gamma)$ is

$$A^2 + B^2 - 2e(A\cos\gamma + B\sin\gamma) + e^2 - 1 = 0.$$

(c) Show that the straight line $\frac{l}{r} = A\cos\theta + B\sin\theta$ touches the conic $\frac{l}{r} = 1 + e\cos\theta$, if $(A - e)^2 + B^2 = 1$.

(d) Show that the straight line $\frac{5}{r} = \cos(\theta - \frac{1}{4}\pi) - \cos\theta$ is a tangent to the parabola $\frac{5}{r} = 1 - \cos\theta$.

(e) If the straight line $r\cos(\theta - \alpha) = p$ touches the parabola $\frac{l}{r} = 1 + \cos\theta$, then show that $p = \frac{1}{2}l\sec\alpha$. [C. H. 1991]

(f) Show that the straight line $r\cos\theta = \rho + a$ touches the circle $r^2 - 2r\rho\cos\theta + \rho^2 = a^2$. Find the point of contact.

19. Prove that the two conics

$$\frac{l_1}{r} = 1 - e_1 \cos\theta \text{ and } \frac{l_2}{r} = 1 - e_2 \cos(\theta - \alpha)$$

will touch one another, if

$l_1^2(1 - e_2^2) + l_2^2(1 - e_1^2) = 2l_1 l_2(1 - e_1 e_2 \cos\alpha)$. [N. B. H. 1992]

[The two conics will touch one another, if they have a common tangent. The tangents of the two conics at $\theta = \beta$ (say) are thus identical.]

20. (a) Show that the equation of the circle, which passes through the focus of the parabola $\frac{2a}{r} = 1 + \cos\theta$ and touches it at the point $\theta = \alpha$ is given by $r\cos^3 \frac{1}{2}\alpha = a\cos(\theta - \frac{3}{2}\alpha)$. [V. H. 1987 ; C. H. 1994]

(b) Show that the equation of the circle which passes through the focus of the conic $\frac{l}{r} = 1 - e\cos\theta$ and touches it at the point $\theta = \alpha$ is $r(1 - e\cos\alpha)^2 = l\cos(\theta - \alpha) - el\cos(\theta - 2\alpha)$. [V. H. 1993]

21. If P, Q be variable points on a conic $\frac{l}{r} = 1 - e\cos\theta$ with vectorial angles α and β where $\alpha - \beta = 2\gamma$ = constant, then show that the chord PQ touches the conic $\frac{l}{r}\cos\gamma = 1 - e\cos\gamma\cos\theta$ and that this conic has the same directrix as the original one.

[C. H. 1984, 1992 ; V. H. 1988]

22. (a) If the tangent at any point P of a conic meets the directrix in K, then prove that the angle KSP is a right angle, S being the corresponding focus.

(b) Prove that the exterior angle between any two tangents to a parabola is equal to half the difference of the vectorial angles of their points of contact.

23. (a) Show that the sum of the reciprocals of two perpendicular focal chords of a conic is constant.

(b) If PSP' and QSQ' be two perpendicular focal chords of a conic with focus S, then prove that $\left(\frac{1}{SP \cdot SP'} + \frac{1}{SQ \cdot SQ'}\right)$ is constant. [B. H. 1995]

24. (a) Prove that the chords of a conic subtending a constant angle at the focus touch a fixed conic.

(b) Prove that the tangents at the extremities of a focal chord intersect on the directrix.

25. If r_1 and r_2 be two mutually perpendicular radius vectors of the ellipse $r^2 = \frac{b^2}{1 - e^2\cos^2\theta}$, then show that $\frac{1}{r_1^2} + \frac{1}{r_2^2} = \frac{1}{a^2} + \frac{1}{b^2}$.

26. (a) The latus rectum of a conic is 6 and its eccentricity is $\frac{1}{2}$. Find the length of the focal chord making an angle of 45° with the major axis. [T. H. 1992]

(b) Prove that the length of the focal chord of the conic $\frac{l}{r} = 1 - e\cos\theta$, which is inclined to the initial line at an angle α, is

$$\frac{2l}{1 - e^2\cos^2\alpha}.$$ [V. H. 2000]

27. (a) Prove that the locus of the middle point of any focal chord of the conic $\frac{l}{r} = 1 + e\cos\theta$ is $r(1 - e^2\cos^2\theta) = el\cos\theta$. [B. H. 2002]

(b) Find the equation of the circum-circle of the triangle formed by the tangents at α, β, γ, which lie on the parabola $\frac{l}{r} = 1 + \cos\theta$, to the parabola.

28. If PQ be a variable chord of the conic $\frac{l}{r} = 1 - e\cos\theta$ subtending a constant angle 2β at the focus S where S is the pole, then show that the locus of the foot of the perpendicular from S on PQ is the curve

$r^2(e^2 - \sec^2\beta) + 2elr\cos\theta + l^2 = 0$. [N. B. H. 1994]

29. If the normal be drawn at one extremity $(l, \frac{1}{2}\pi)$ of the latus rectum PSP' of the conic $\frac{l}{r} = 1 + e\cos\theta$ where S is the pole, then show that the distance from the focus S of the other point in which the normal meets the conic is $\frac{l(1 + 3e^2 + e^4)}{1 + e^2 - e^4}$. [C. H. 1966 ; V. H. 1988]

30. (a) If the normals at α, β, γ on the parabola $\frac{l}{r} = 1 + \cos\theta$ be concurrent at the point (ρ, ϕ), then show that $2\phi = \alpha + \beta + \gamma$.

(b) If the normals at the points of vectorial angles α, β, γ, δ on the conic $\frac{l}{r} = 1 + e\cos\theta$ meet at a point, then prove that

$$\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}\tan\frac{\delta}{2} + \left(\frac{1+e}{1-e}\right)^2 = 0.$$ [V. H. 1992]

31. (a) Show that the equation of the pair of tangents to the conic $\frac{l}{r} = 1 + e\cos\theta$ from the point (r', θ') is

$$\left\{\left(\frac{l}{r} - e\cos\theta\right)^2 - 1\right\}\left\{\left(\frac{l}{r'} - e\cos\theta'\right)^2 - 1\right\} = \left\{\left(\frac{l}{r} - e\cos\theta\right)\left(\frac{l}{r'} - e\cos\theta'\right) - \cos(\theta - \theta')\right\}^2.$$

[The equation of the tangent to the conic at $\theta = \alpha$ is $l/r = e\cos\theta + \cos(\theta - \alpha)$. If it passes through the point (r', θ'), then $l/r' = e\cos\theta' + \cos(\theta' - \alpha)$. Eliminate α from these two.]

(b) Show that the polar equation of the asymptotes of the hyperbola $r(3 - 5\cos\theta) = 16$ are

$r(3\sin\theta - 4\cos\theta) = 20$ and $r(3\sin\theta + 4\cos\theta) + 20 = 0$.

32. (a) Show that the auxiliary circle of the conic

$\frac{l}{r} = 1 - e\cos\theta$ is $r^2(e^2 - 1) + 2ler\cos\theta + l^2 = 0$. [C.H. 1972]

[It is the locus of the foot of the perpendicular from the pole on a variable tangent of the conic.]

(b) Show that the director circle of the conic

$\frac{l}{r} = 1 + e\cos\theta$ is $r^2(e^2 - 1) - 2ler\cos\theta + 2l^2 = 0$. [C.H. 1970]

[Let the tangents of the conic at the points whose vectorial angles are α and β meet at the point (r_1, θ_1). Then the tangents

$$\frac{l}{r} = e\cos\theta + \cos(\theta - \alpha) \text{ and } \frac{l}{r} = e\cos\theta + \cos(\theta - \beta)$$

pass through (r_1, θ_1).

Therefore $\frac{\alpha + \beta}{2} = \theta_1$ and $\cos\frac{\beta - \alpha}{2} = \frac{l}{r_1} - e\cos\theta_1$.

Now the tangents will be perpendicular to one another, if

$$(\cos\alpha + e)(\cos\beta + e) + \sin\alpha\sin\beta = 0$$

or, $e^2 - \frac{2el}{r_1}\cos\theta_1 + \frac{2l^2}{r_1^2} - 1 = 0$.

The locus of (r_1, θ_1) is the required director circle.]

Answers

1. (a) (i) $(-2, 2\sqrt{3})$. (ii) $(-\sqrt{3}, -1)$.
   (b) (i) $(\sqrt{2}, \frac{5}{4}\pi)$. (ii) $(3\sqrt{2}, -\frac{1}{4}\pi)$.
   (c) (i) $x^2 + y^2 = 4$. (ii) $x = y$. (iii) $y^2 + 2x = 1$.
   (d) (i) $r^2 = a\cos^2\theta$. (ii) $r\cos\theta = 2a\sin^2\theta$.
2. (a) (i) $\sqrt{7}$ units (ii) $2\sqrt{3}$ units.

3. $\frac{1}{2}(6 + \sqrt{3})$ square units ; 7 square units.

4. (a) $2 = r(2\sin\theta - \cos\theta)$.

(b) $\frac{1}{r_1}\sin(\theta_2 - \theta_3) + \frac{1}{r_2}\sin(\theta_3 - \theta_1) + \frac{1}{r_3}\sin(\theta_1 - \theta_2) = 0$.

(c) $\frac{2r_1r_2}{r_1 + r_2}\cos\frac{1}{2}(\theta_1 - \theta_2)$.

(f) $\theta = \frac{1}{2}(\alpha + \beta)$, $\theta = \frac{1}{2}\pi + \frac{1}{2}(\alpha + \beta)$.

5. $a\sin(\beta - \gamma) + b\sin(\gamma - \alpha) + c\sin(\alpha - \beta) = 0$.

6. (a) $(\frac{5}{2}, \tan^{-1}\frac{3}{4})$. (c) (i) $(1, \pi)$; 1. (ii) $\left(3, \frac{\pi}{3}\right)$; 3. (iii) $\left(4, \frac{5}{6}\pi\right)$; 4.

7. (a) $r^2 - r\{a\cos(\theta - \alpha) + b\cos(\theta - \beta)\} + ab\cos(\alpha - \beta) = 0$.
   (c) $r = 2d\cos(\theta - \frac{1}{3}\pi)$; d.

10. (a) (i) Parabola. (ii) Hyperbola. (iii) Ellipse. (iv) Hyperbola.
   (b) Hyperbola ; $\frac{5}{4}, \frac{1}{2}$; $5r\cos\theta + 1 = 0$, $45r\cos\theta + 41 = 0$.
   (d) (i) $\frac{l}{1 - e^2}, \frac{l}{\sqrt{1 - e^2}}$. (ii) 13, 12. (iii) 8, 6.
   (e) $r\cos\theta = -12$, $r\cos\theta = 20$.
   (f) (i) $r\cos\theta + 2 = 0$. (ii) $2r\cos\theta + 21 = 0$, $2r\cos\theta = 29$.
   (iii) $5r\cos\theta + 34 = 0$, $5r\cos\theta + 16 = 0$.

11. (a) $(5, \frac{2}{3}\pi)$, $(5, -\frac{2}{3}\pi)$. (b) $(\frac{3}{2}, \frac{1}{3}\pi)$. (c) $(\frac{1}{2}l, \pi)$.

12. (b) $(a\sqrt{2}, \frac{1}{4}\pi)$, $(-a\sqrt{2}, \frac{3}{4}\pi)$. (c) $(2, \frac{2}{3}\pi)$, $(2, -\frac{2}{3}\pi)$.

14. (a) $\frac{14}{r} = 4 + 3\cos\theta$. (b) $\frac{18}{r} = 2 + \sqrt{13}\cos\theta$.

15. (a) $\frac{4}{r} = 1 + \cos\theta$.

16. (a) $\frac{2a}{r} = \cos\theta + \sec\beta\cos(\theta - \alpha)$.
   (b) $\frac{l}{r} + e\cos\theta = \sec\frac{1}{2}(\alpha - \beta)\cos\{\theta - \frac{1}{2}(\alpha + \beta)\}$.
   (c) $r\cos(\theta - \theta_1 - \theta_2) = 2d\cos\theta_1\cos\theta_2$; $r\cos(\theta - 2\theta_1) = 2d\cos^2\theta_1$.

17. (a) $\frac{2}{r} = \sin\theta - \cos\theta$. 18. (f) $(\rho + a, 0)$. 26. (a) $\frac{48}{7}$ units.

27. (b) $r = \frac{l}{2}\sec\frac{\alpha}{2}\sec\frac{\beta}{2}\sec\frac{\gamma}{2}\cos(\theta - \frac{\alpha + \beta + \gamma}{2})$.