# GENERAL EQUATION OF THE SECOND DEGREE

**Every general equation of 2nd degree in $x$ and $y$ represents a conic.**

The general equation of 2nd degree in $x$ and $y$ is

$$ax^2+2hxy+by^2+2gx+2fy+c=0. \quad \text{——— (1)}$$

We have to show that this equation represents a conic.
Let us transform the equation by rotation of axes such that the terms containing $xy$ vanishes. We rotate the co-ordinate axes through an angle $\theta$, so that

$$\left.\begin{aligned} x &= x'\cos\theta - y'\sin\theta \\ y &= x'\sin\theta + y'\cos\theta \end{aligned}\right\} \quad \text{——— (2)}$$

Substituting these values in (1), we have

$$x'^2(a\cos^2\theta + h\sin 2\theta + b\sin^2\theta) + x'y'\{2h\cos 2\theta - (a-b)\sin 2\theta\}$$
$$+ y'^2(a\sin^2\theta - h\sin 2\theta + b\cos^2\theta) + 2x'(g\cos\theta + f\sin\theta)$$
$$+ 2y'(f\cos\theta - g\sin\theta) + c = 0. \quad \text{——— (3)}$$

We choose $\theta$ in such a way such that the coeffn. of $x'y'$ in (3) vanishes. i.e,

$$2h\cos 2\theta - (a-b)\sin 2\theta = 0$$

$$\tan 2\theta = \frac{2h}{a-b}$$

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2h}{a-b}\right). \quad \text{——— (4)}$$

Whatever be the values of $a$, $b$ and $h$, there is always a value of $\theta$, which satisfies the equation (4). For the above value of $\theta$, the eqn (3) reduces to the form

$$a'x'^2 + b'y'^2 + 2g'x' + 2f'y' + c' = 0, \quad \text{——— (5)}$$

By the property of invariance,

$$a+b = a'+b', \quad ab-h^2 = a'b' \quad \text{and} \quad g^2+f^2 = g'^2+f'^2.$$

Now, the following cases may arise:—

**CASE-I:** If $ab = h^2$, then $a'b' = 0$
$\Rightarrow$ either $a'=0$ or $b'=0$.

Say, $a'=0$, then eqn (5) becomes -

$$b'y'^2 + 2g'x' + 2f'y' + c' = 0,$$

$$\therefore b'\left(y'^2 + \frac{2f'}{b'}y'\right) = -2g'x' - c'$$

or, $\left(y'+\frac{f'}{b'}\right)^2 = -\frac{2g'}{b'}\left(x'+\frac{b'c'-f'^2}{2b'g'}\right).$

Shifting the origin to the point $\left(\frac{f'^2-b'c'}{2b'g'}, -\frac{f'}{b'}\right)$ without changing the direction of the axes, we have

$$Y^2 = -\frac{2g'}{b'}X \quad \text{where } Y = y'+\frac{f'}{b'}, \quad X = x'+\frac{b'c'-f'^2}{2b'g'}. \qquad \text{—— (7)}$$

The above eqn (7) represents a parabola whose axis is the X-axis.

CASE-II :- Let both $a'\neq 0$ and $b'\neq 0$.

The eqn (5) can then be written as

$$a'x'^2+2g'x'+b'y'^2+2f'y' = -c'$$

or, $$a'\left(x'+\frac{g'}{a'}\right)^2+b'\left(y'+\frac{f'}{b'}\right)^2 = r \qquad \text{—— (8)}$$

where $r = \frac{g'^2}{a'^2}+\frac{f'^2}{b'^2}-c'$. —— (9)

Now, shifting the origin to the point $\left(-\frac{g'}{a'}, -\frac{f'}{b'}\right)$, without changing the direction of the axes, we have

$$a'X^2+b'Y^2 = r, \quad \text{where } X = x'+\frac{g'}{a'}, \quad Y = y'+\frac{f'}{b'} \qquad \text{—— (10)}$$

(a) If $r=0$, then eqn (10) becomes $a'X^2+b'Y^2=0$.

which represents (i) a pair of st. lines if $a'$ and $b'$ are of opposite signs.

(ii) a point ellipse, if $a'$ and $b'$ are of same sign.

(b) If $r\neq 0$, then eqn (10) can be written as -

$$\frac{X^2}{r/a'}+\frac{Y^2}{r/b'} = 1.$$

(i) If both $\frac{r}{a'}$ and $\frac{r}{b'}$ are +ve, then it will represents an ellipse

(ii) If $a'=b'$, then it represents a circle.

(iii) If one of $\frac{r}{a'}$ and $\frac{r}{b'}$ be negative and other is +ve, it will represents a hyperbola.

In addition to this if $a'+b'=0$, it will represents a rectangular hyperbola.

(iv) If $r$ is +ve and if both $\frac{r}{a'}$ and $\frac{r}{b'}$ are -ve, then it will represents an imaginary ellipse.

## Nature of Conic :-

The general equation of 2nd degree in $x$ and $y$ is $ax^2+2hxy+by^2+2gx+2fy+c=0$ —— (1)

Now, we introduce the following notations :-

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc+2fgh-af^2-bg^2-ch^2$$

and $D = \begin{vmatrix} a & h \\ h & b \end{vmatrix} = ab-h^2$.

Where $\Delta$ is called the discriminant of (1) and is invariant under translation and rotation of axes, $D$ is also invariant under translation and rotation of axes.

NOTE :-

1. A second degree equation is —
   (i) Elliptic type if $D>0$
   (ii) Hyperbolic type if $D<0$
   (iii) parabolic type if $D=0$.

2. (i) A conic is central conic if $\Delta\neq0$ and $D\neq0$. In this case we first apply translation and then rotation formula.

   (ii) A conic is non-central if $\Delta\neq0$, and $D=0$, In this case we first apply rotation and then translation formula.

## ⊛ Nature of the Conic in Tabular form :-

| D | $\Delta$ | Nature | Canonical form |
|---|---|---|---|
| $D>0$ | $\Delta<0$ | Ellipse | $\frac{x^2}{A^2}+\frac{y^2}{B^2}=1$ |
| | $\Delta>0$ | Imaginary ellipse | $\frac{x^2}{A^2}+\frac{y^2}{B^2}=-1$ |
| | $\Delta\neq0$ | Point ellipse | $Ax^2+By^2=0$ |
| $D<0$ | $\Delta<0$ | Conjugate Hyperbola | $\frac{x^2}{A^2}-\frac{y^2}{B^2}=-1$ |
| | $\Delta>0$ | Hyperbola | $\frac{x^2}{A^2}-\frac{y^2}{B^2}=1$ |
| | $\Delta\neq0$, $a+b=0$ | Rectangular Hyperbola | $x^2-y^2=c^2$ or $xy=c^2$ |
| $D=0$ | $\Delta\neq0$ | parabola | $y^2=4ax$ or $x^2=4by$ |
| | $\Delta=0$ | Parallel st. lines | $y^2=m^2$ or $x^2=n^2$, $(m,n\neq0)$ |
| $D\neq0$ | $\Delta=0$ | Intersecting St. lines | $y^2-k^2x^2=0$ |

**Ex. ①:** Reduce the equation $3x^2+2xy+3y^2-16x+20=0$ to its canonical form and determine the nature of the conic.

**Sol$^n$:** Let, the given equation is of the form -

$$ax^2+2hxy+by^2+2gx+2fy+c=0,$$

where, $a=3$, $h=1$, $b=3$, $g=-8$, $f=0$ and $c=20$.

Therefore. $\Delta = abc+2fgh-af^2-bg^2-ch^2$
$= -32 \ (\neq 0)$

and $D = ab-h^2 = 8 \ (\neq 0)$

Hence it is a central conic. Let the centre of the conic be $(\alpha,\beta)$. Transforming the origin to the point $(\alpha,\beta)$ without changing the directions of the axes, the equation becomes with $(x',y')$ as new co-ordinates

$$3(x'+\alpha)^2+2(x'+\alpha)(y'+\beta)+3(y'+\beta)^2-16(x'+\alpha)+20=0$$

or, $3x'^2+2x'y'+3y'^2+2x'(3\alpha+\beta-8)+2y'(\alpha+3\beta)+(3\alpha^2+2\alpha\beta+3\beta^2-16\alpha+20)=0$ —— (i)

Since the origin being at the centre, the coeff$^n$'s of $x'$ and $y'$ should be separately vanish,

$\therefore$ $3\alpha+\beta-8=0$ and $\alpha+3\beta=0$.

Solving, we get $\alpha=3, \beta=-1$.

Thus the centre of the conic is $(3,-1)$.

Put $\alpha=3, \beta=-1$ in eq$^n$ ①, we have

$$3x'^2+2x'y'+3y'^2-4=0 \quad \text{—— ②}$$

To remove the $x'y'$ term, we rotate the axes through an angle $\theta$ with the same origin.

put, $x'=X\cos\theta-Y\sin\theta$ , where $(X,Y)$ is the new co-ordinate.

$y'=X\sin\theta+Y\cos\theta$

Then eq$^n$ ② becomes –

$$3(X\cos\theta-Y\sin\theta)^2+2(X\cos\theta-Y\sin\theta)(X\sin\theta+Y\cos\theta) + 3(X\sin\theta+Y\cos\theta)^2-4=0$$

or, $$X^2(3+\sin 2\theta)+2XY\cos 2\theta+Y^2(3-\sin 2\theta)=4. \quad \text{—— ③}$$

We choose $\theta$ in such a way that the coeff$^n$ of $XY$ is equal to zero.

$\therefore$ $\cos 2\theta=0$

$\therefore$ $\theta=\pi/4$.

The eq$^n$ ③ becomes – $X^2(3+1)+Y^2(3-1)=4$

$$\Rightarrow \frac{X^2}{1}+\frac{Y^2}{2}=1.$$

This is the required canonical form of the given equation and it will represent an ellipse.

Ex:-② :- Reduce the following equations to their canonical forms and determine the nature of the conics represented by them:-

(i) $6x^2-5xy-6y^2+14x+5y+4=0$

(ii) $x^2+4xy+4y^2-20x+10y-50=0$

Solⁿ: (i) Let the given equation is of the form

$$ax^2+2hxy+by^2+2gx+2fy+c=0.$$

where $a=6,\ b=-6,\ h=-\frac{5}{2},\ g=7,\ f=\frac{5}{2},\ c=4.$

Therefore, $\Delta = abc+2fgh-af^2-bg^2-ch^2$

$= 0$

and $D = ab-h^2 = -\frac{169}{4}\ (\neq 0).$

Thus the given equation represents a pair of intersecting st. lines. Let the st. lines intersect at $(\alpha,\beta)$.

We put $x = x'+\alpha$ and $y = y'+\beta$ in the given eqⁿ.

Then, $6(x'+\alpha)^2-5(x'+\alpha)(y'+\beta)-6(y'+\beta)^2+14(x'+\alpha)+5(y'+\beta)+4=0$

or, $6x'^2-5x'y'-6y'^2+x'(12\alpha-5\beta+14)+y'(-5\alpha-12\beta+5)+6\alpha^2-5\alpha\beta-6\beta^2+14\alpha+5\beta+4=0 \quad \text{—— (1)}$

Since, the origin being at the centre, the coefficient of $x'$ and $y'$ should separately vanish.

$\therefore\ 12\alpha-5\beta+14=0$

and $-5\alpha-12\beta+5=0.$

Solving, we get $\alpha = -\frac{11}{13},\ \beta=\frac{10}{13}$

$\therefore$ The intersecting point is at $\left(-\frac{11}{13}, \frac{10}{13}\right)$.

put $\alpha=-\frac{11}{13}$ and $\beta=\frac{10}{13}$ in eqⁿ (1), we have

$$6x'^2-5x'y'-6y'^2=0. \quad \text{—— (2)}$$

To remove $x'y'$-term, we rotate the axes through an angle $\theta$ with the same origin, we put

$x'=x\cos\theta-y\sin\theta$

$y'=x\sin\theta+y\cos\theta$, in (2),

$\therefore\ 6(x\cos\theta-y\sin\theta)^2-5(x\cos\theta-y\sin\theta)(x\sin\theta+y\cos\theta)-6(x\sin\theta+y\cos\theta)^2=0$

or, $\left(6\cos2\theta-\frac{5}{2}\sin2\theta\right)x^2-(5\cos2\theta+12\sin2\theta)xy-6\left(\cos2\theta-\frac{5}{2}\sin2\theta\right)y^2=0 \quad \text{—— (3)}$

We choose $\theta$ in such a way such that the coeff$^n$ of $xy$ is equal to zero.

$\therefore \quad 5\cos 2\theta + 12\sin 2\theta = 0$

$\Rightarrow \quad \dfrac{\sin 2\theta}{-5} = \dfrac{\cos 2\theta}{12} = \dfrac{1}{13}$

$\therefore \quad \sin 2\theta = -\dfrac{5}{13}, \quad \cos 2\theta = \dfrac{12}{13}.$

Put these values in eqn $(3)$, we have

$$x^2 - y^2 = 0$$

or, $\quad x - y = 0, \quad x + y = 0.$

This is the required canonical form of the conic and it will represents an intersecting st. lines.

) Comparing given equation with the general equation of second degree $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

we have, $\quad a = 1 \qquad h = 2$
$\qquad\qquad b = 4 \qquad g = -10$
$\qquad\qquad c = -50 \qquad f = 5$

$\therefore \Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = -625 < 0$

$D = ab - h^2 = 0$

Since $\Delta \neq 0, D = 0$, it is a non-central conic.

To remove $xy$-term, we rotate the axes through an angle $\theta$ with the same origin.

put $\quad x = x\cos\theta - y\sin\theta$
$\qquad\quad y = x\sin\theta + y\cos\theta$, we have

$(x\cos\theta - y\sin\theta)^2 + 4(x\cos\theta - y\sin\theta)(x\sin\theta + y\cos\theta)$
$+ 4(x\sin\theta + y\cos\theta)^2 - 20(x\cos\theta - y\sin\theta) + 10(x\sin\theta + y\cos\theta) - 50 = 0$

$\Rightarrow (\cos^2\theta + 4\cos\theta\sin\theta + 4\sin^2\theta)x^2 + (\sin^2\theta - 4\cos\theta\sin\theta + 4\cos^2\theta)y^2$
$+ (-2\sin\theta\cos\theta + 4\cos^2\theta - 4\sin^2\theta + 8\sin\theta\cos\theta)xy + (-20\cos\theta + 10\sin\theta)x + (20\sin\theta + 10\cos\theta)y - 50 = 0 \quad \text{——} (1)$

We choose $\theta$, in such a way, such that the coeff$^n$ of $xy$ is equal to zero.

$\therefore \; 4\cos^2\theta - 4\sin^2\theta + 6\sin\theta\cos\theta = 0$

$\Rightarrow \; 2\tan^2\theta - 3\tan\theta - 2 = 0$

$\Rightarrow \; (2\tan\theta + 1)(\tan\theta - 2) = 0$

$\therefore \; \tan\theta = -\frac{1}{2}$ or $\tan\theta = 2.$

we take $\tan\theta = 2$.

$$\frac{\sin\theta}{2} = \frac{\cos\theta}{1} = \frac{1}{\sqrt{5}}$$

$$\therefore \quad \sin\theta = \frac{2}{\sqrt{5}}, \quad \cos\theta = \frac{1}{\sqrt{5}}.$$

put these values in ①, we have.

$$5x^2 + 10\sqrt{5}\,y - 50 = 0$$

$$\Rightarrow x^2 + 2\sqrt{5}\,y - 10 = 0$$

$$\Rightarrow x^2 = -2\sqrt{5}\,(y - \sqrt{5})$$

$$\Rightarrow x'^2 = -2\sqrt{5}\,y', \quad \text{where } x' = x - 0$$

$$y' = y - \sqrt{5}.$$

which is the required canonical form of the given equation and it will represent parabola.

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