Integrating Factors :-

An integrating factor (I.F) is a multiplying factor by which the differential equation can be made exact. ③

OR

A function $\mu(x,y)$ is said to an I.F of the diff. equation $M\,dx + N\,dy = 0$ if it is possible to obtain a function $u(x,y)$ such that -
$\mu(M\,dx + N\,dy) = du$.

Theorem: The number of I.F.s of an eqn $M\,dx + N\,dy = 0$ is infinite.

Proof: Let $\mu(x,y)$ be an I.F of the diff. eqn

$$M\,dx + N\,dy = 0 \quad \text{——} ①$$

Then $\mu(M\,dx + N\,dy) = du$

$$, \quad du = 0 \qquad [\text{by } ①]$$

Int. $\int du = c$

$$, \quad u = C. \text{ is the general sol}^n.$$

Let $f(u)$ be any function of u.

then $\mu f(u)(M\,dx + N\,dy) = f(u)\,du$

$$= d\{F(u)\}, \quad \text{where } F'(u) = f(u).$$

Thus R.H.S is an exact and hence $\mu f(u)$ is also an IF of ①.

Since, $f(u)$ is an arbitrary function of u, so, the number of IFs is infinite.

Ex – II(B)

③ $(1+xy)\,y\,dx + (1-xy)\,x\,dy = 0$

$x\,dy + y\,dx + xy\,(y\,dx - x\,dy) = 0$

or, $d(xy) + xy\,(y\,dx - x\,dy) = 0$

or, ~~Dividing by~~ Multiplying by $\frac{1}{x^2y^2}$, we get

$$\frac{d(xy)}{x^2y^2} + \frac{y\,dx - x\,dy}{xy} = 0$$

$$\frac{d(xy)}{(xy)^2} + \frac{dx}{x} - \frac{dy}{y} = 0$$

Int $\int \frac{d(xy)}{(xy)^2} + \int \frac{dx}{x} - \int \frac{dy}{y} = c$

$-\frac{1}{xy} + \log x - \log y = c$, where c is a.c.

⑤ $(x+y)(dx - dy) = dx + dy$

$(x+y)\,d(x-y) = d(x+y)$

Int $\int d(x-y) = \int \frac{d(x+y)}{x+y}$

$x - y = \log(x+y) + c.$

⑩ $\left\{x + y\cos\left[\frac{y}{x}\right]\right\} dx = x\cos\left[\frac{y}{x}\right] dy$

$x\,dx = \cos\left[\frac{y}{x}\right](x\,dy - y\,dx)$

Multiplying by $\frac{1}{x^2}$, we get

$$\frac{dx}{x} = \cos\left[\frac{y}{x}\right]\frac{x\,dy - y\,dx}{x^2}$$

Int. $\int \frac{dx}{x} = \int \cos\left[\frac{y}{x}\right] \cdot d\left(\frac{y}{x}\right)$

$\log x = \sin\left[\frac{y}{x}\right] + c.$

(12) $x\frac{dy}{dx} - y = \frac{x}{a}\sqrt{x^2+y^2}$

or, $x\,dy - y\,dx = \frac{x}{a}\sqrt{x^2+y^2}\,dx$.

or, $\frac{x\,dy - y\,dx}{x^2} = \frac{1}{a}\sqrt{1+\left(\frac{y}{x}\right)^2}\,dx$

Int. $\int \frac{d\left(\frac{y}{x}\right)}{\sqrt{1+\left(\frac{y}{x}\right)^2}} = \int \frac{1}{a}\,dx$

$\log\left|\frac{y}{x} + \sqrt{1+\left(\frac{y}{x}\right)^2}\right| = \frac{x}{a} + c.$

(17) $x\frac{dy}{dx} - y = x\sqrt{x^2+y^2}$

or, $x\,dy - y\,dx = x\sqrt{x^2+y^2}\,dx$.

, $\frac{x\,dy - y\,dx}{x^2} = \sqrt{1+\left(\frac{y}{x}\right)^2}\,dx$

Int. $\int \frac{d\left(\frac{y}{x}\right)}{\sqrt{1+\left(\frac{y}{x}\right)^2}} = \int dx$

, $\log\left|\frac{y}{x} + \sqrt{1+\left(\frac{y}{x}\right)^2}\right| = x + c.$

(19) Show that $\frac{1}{x^2}$ is an integrating factor of $x\,dy - y\,dx = 0$.

Sol^n: We have, $\frac{1}{x^2}(x\,dy - y\,dx) = \frac{x\,dy - y\,dx}{x^2}$

$= d\left(\frac{y}{x}\right).$

This shows that $\frac{1}{x^2}$ is an I.F. of $x\,dy - y\,dx = 0$.

Ex! Solve!:-

(i) $x\cos\left(\frac{y}{x}\right)(y\,dx + x\,dy) = y\sin\left(\frac{y}{x}\right)(x\,dy - y\,dx)$.

(ii) $x\,dx + y\,dy + \frac{x\,dy - y\,dx}{x^2+y^2} = 0$

(iii) $x\,dx + y\,dy = m(x\,dy - y\,dx)$

(iv) $(xy^2 - e^{1/x^3})\,dx - x^2y\,dy = 0$

Equations solvable by separation of variables

A special case of an exact equation occurs when, in the equation $M\,dx + N\,dy = 0$, M is a function of x-alone [$= f_1(x)$, say] and N is a function of y-alone [$= f_2(y)$, say]. The equation $f_1(x)\,dx + f_2(y)\,dy = 0$ is then said to have separated variables. Its general soln is-

$$\int f_1(x)\,dx + \int f_2(y)\,dy = c.$$

Ex-II(C)

Ex! If $\frac{dy}{dx} = f(ax+by+c)$, show that the substitution $ax+by+c = v$ will change it to a separable equation.

Soln:

$$\frac{dy}{dx} = f(ax+by+c)$$

put $ax+by+c = v$

$$a + b\frac{dy}{dx} = \frac{dv}{dx}$$

$$\frac{dy}{dx} = \frac{1}{b}\left(\frac{dv}{dx} - a\right)$$

$$\frac{1}{b}\left(\frac{dv}{dx} - a\right) = f(v)$$

$$\therefore \frac{dv}{dx} = b f(v) + a$$

$$\therefore \frac{dv}{b f(v) + a} = dx.$$

$$\therefore \frac{1}{bf(v)+a}\cdot dv - dx = 0$$ is a separable equation.

Ex) Find $f(x)$ for which $f'(x) = \log x$ and $f(1) = -5$.

Soln: Given that $f'(x) = \log x$.

$$\therefore f(x) = \int \log x + c.$$

$$= x(\log x - 1) + c. \quad \text{——} (1)$$

Since, $f(1) = -5$

$$\therefore 1.(\log 1 - 1) + c = -5$$

$$\therefore c - 1 = -5$$

$$c = -4$$

From (1), $f(x) = x(\log x - 1) - 4$.

Ans

Homogeneous Equations

Def^n:- A function $f(x,y)$ is said to be homogeneous of degree n if it can be expressed in the form $x^n \phi(\frac{y}{x})$, i.e., $f(x,y) = x^n \phi(\frac{y}{x})$.

Ex:
$$f(x,y) = x^3 + 2x^2y - y^3$$
$$= x^3\left\{1 + 2\left(\frac{y}{x}\right) - \left(\frac{y}{x}\right)^3\right\}$$
$$= x^3 \phi\left(\frac{y}{x}\right), \text{ where } \phi\left(\frac{y}{x}\right) = 1 + 2\left(\frac{y}{x}\right) - \left(\frac{y}{x}\right)^3$$

Thus $f(x,y)$ is a homogeneous function of degree 3

Def^n: The diff. eqn^n $M\,dx + N\,dy = 0$ is said to be homogeneous diff. equation of degree n, if ~~both~~ M and N both are homogeneous functions of degree n.

Ex: $(x^2 - xy)\,dx + 2y^2\,dy$ is a homogeneous diff. equation of degree 2.

Note: To solve homogeneous diff. equation we put $y = vx$ on it, where v may be considered a new dependent variable.

$\boxed{\text{Ex-II(D)}}$

Ex: ① Solve: $x\,dy - y\,dx = \sqrt{(x^2+y^2)}\,dx$

Sol^n:
$$x\,dy - y\,dx = \sqrt{x^2+y^2}\,dx$$

put, $y = vx$
$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\frac{dy}{dx} = \frac{y + \sqrt{x^2+y^2}}{x}$$

$$\cancel{v} + x\frac{dv}{dx} = \frac{vx + \sqrt{x^2+v^2x^2}}{x}$$
$$= \frac{(\cancel{v} + \sqrt{v^2+1})\cancel{x}}{\cancel{x}}$$

$$x\frac{dv}{dx} = \sqrt{v^2+1}$$

Int.
$$\int \frac{dv}{\sqrt{v^2+1}} = \int \frac{dx}{x}$$

$$\log\left|v + \sqrt{v^2+1}\right| = \log x + \log c, \text{ where } c \text{ is Int. Const.}$$
$$v + \sqrt{v^2+1} = xc$$
$$y + \sqrt{x^2+y^2} = cx^2$$

Ex! ② $x^3 \frac{dy}{dx} = y^3 + y^2\sqrt{(y^2-x^2)}$ put $y = vx$

$\frac{dy}{dx} = v + x\frac{dv}{dx}$

or $\frac{dy}{dx} = \left(\frac{y}{x}\right)^3 + \left(\frac{y}{x}\right)^2 \sqrt{\left(\frac{y}{x}\right)^2 - 1}$

$v + x\frac{dv}{dx} = v^3 + v^2\sqrt{v^2-1}$

$x\frac{dv}{dx} = (v^3 - v) + v^2\sqrt{v^2-1}$

$= v(v^2-1) + v^2\sqrt{v^2-1}$

$= v\sqrt{v^2-1}\left(v + \sqrt{v^2-1}\right)$

or, $\frac{dv}{v\sqrt{v^2-1}(v+\sqrt{v^2-1})} = \frac{dx}{x}$

$\frac{(v-\sqrt{v^2-1})\,dv}{v\sqrt{v^2-1}\{v^2-(v^2-1)\}} = \frac{dx}{x}$

$\frac{(v-\sqrt{v^2-1})\,dv}{v\sqrt{v^2-1}} = \frac{dx}{x}$

Int $\int \frac{dv}{\sqrt{v^2-1}} - \int \frac{dv}{v} = \int \frac{dx}{x}$

$\log|v + \sqrt{v^2-1}| - \log v = \log x + \log c$

$\frac{v + \sqrt{v^2-1}}{v} = cx$

$y + \sqrt{y^2-x^2} = cxy$

Ex ③ $(1+e^{x/y})\,dx + e^{x/y}\left(1-\frac{x}{y}\right)dy = 0$

or, $\frac{dx}{dy} = \frac{e^{x/y}\left(\frac{x}{y}-1\right)}{(e^{x/y}+1)}$, put $x = vy$

$\frac{dx}{dy} = v + y\frac{dv}{dy}$

$v + y\frac{dv}{dy} = \frac{e^v(v-1)}{(e^v+1)}$

$y\frac{dv}{dy} = \frac{ve^v - e^v}{e^v+1} - v$

$$y\frac{dv}{dy} = \frac{\cancel{ve^v} - e^v - \cancel{ve^v} - v}{e^v+1}$$

or

$$\int \frac{(e^v+1)\,dv}{e^v+v} + \int \frac{dy}{y} = \log c$$

$$\int \frac{d(e^v+v)}{(e^v+v)} + \int \frac{dy}{y} = \log c$$

$$\log(e^v+v) + \log y = \log c$$

$$(e^v+v)\,y = c$$

$$\left(e^{x/y} + \frac{x}{y}\right) y = c$$

$$y e^{x/y} + x = c$$

Ex! What curve through $(1,1)$ has at every point $\frac{dy}{dx} = \frac{x-y}{x+y}$?

Soln! $\frac{dy}{dx} = \frac{x-y}{x+y}$

$$(x+y)\,dy = (x-y)\,dx.$$

$$x\,dy + y\,dx = x\,dx - y\,dy$$

Int. $\int d(xy) = \int x\,dx - \int y\,dy$

$$xy = \frac{x^2}{2} - \frac{y^2}{2} + \frac{c}{2}$$

$$2xy = x^2 - y^2 + c \quad \text{--------- (1)}$$

If it passes through $(1,1)$, we have

$$2 = 1 - 1 + c$$

$$c = 2$$

Then the curve becomes $2xy = x^2 - y^2 + 2$.

Ex! Solve! (i) $(x^2+y^2)\,dx - 2xy\,dy = 0$

(ii) $\left(x + y\cos\frac{y}{x}\right) dx = x\cos\frac{y}{x}\,dy$

(iii) $x\sin\frac{y}{x}\,dy = \left(y\sin\frac{y}{x} - x\right) dx.$

(iv) $(x^3 - y^3)\,dx + xy^2\,dy = 0$

(v) $y^2 + x^2\frac{dy}{dx} = xy\frac{dy}{dx}.$

(vi) $x^2\,dy + (xy + y^2)\,dx = 0.$

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