SEMESTER-III

HONOURS

CORE COURSE---C 5T

UNIT-II (MARKS-14)

UNIT-II

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Syllabus for Unit-II: Differentiability of a function at a point and in an interval, Caratheodory's theorem, algebra of differentiable functions, Relative extrema, interior extremum theorem, Rolle's theorem, Mean value theorem, intermediate value property of derivatives, Darboux's theorem, application of mean value theorem to inequalities and approximation of polynomials.

DERIVABILITY

Let $y = f(x)$ be a function where x is independent variable and y is dependent variable.
Let Δx be the increment of x and let Δy be the corresponding increment of y. Therefore, Change in x is Δx and Change in y is Δy. Therefore, the rate of change in y with respect to the change in x is $\frac{\Delta y}{\Delta x}$. **Then the limit $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ (if the limit exists) is called derivative of the function $y = f(x)$ at the point x and it is denoted by $\frac{d}{dx}(y)$ or by $\frac{dy}{dx}$ or by $f'(x)$.**

So, $f'(x) = \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$.

$$\boxed{f'(x) = \frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

At the point $x = c$, $f'(c) = \left(\frac{dy}{dx}\right)_{x=c} = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$ Now let $c + h = x$. Then as $h \to 0$, $x \to c$. Then $f'(c) = \left(\frac{dy}{dx}\right)_{x=c} = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$.

$$\boxed{f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}}$$

DEFINITION : A real valued function $y = f(x)$ defined on an interval $[a,b]$ is said to be derivable at $x = c$ when $a < c <$ b if $\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ exists. This limit, if exists, is called derivative or differential coefficient of the function $y = f(x)$ at the point $x = c$.

DEFINITION (Left-hand derivative): If $\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c}$ or $\lim_{h \to 0^-} \frac{f(c-h) - f(c)}{-h}$ exists, is called left-hand derivative of the function $y = f(x)$ at $x = c$ and it is denoted by $f'(c-0)$ or $f'(c^-)$ or $Lf'(c)$.

DEFINITION (Right-hand derivative): If $\lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}$ or $\lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h}$ exists, is called right-hand derivative of the function $y = f(x)$ at $x = c$ and it is denoted by $f'(c+0)$ or $f'(c^+)$ or $Rf'(c)$.

DEFINITION (Derivablity in an interval): If a function f defined on an interval $[a,b]$ is derivable at all points including the end points a and b then f is called derivable on $[a,b]$.

ALGEBRA OF DERIVATIVES

THEOREM 2.1 : *If* f and g be two functions which are defined on $[a,b]$ and derivable at any point c of $[a,b]$ then

(i) $f + g$ **is also derivable at $x = c$ and $(f+g)' = f'(c) + g'(c)$**

(ii) fg **is also derivable at $x = c$ and $(fg)' = f(c)g'(c) + g(c)f'(c)$**

(iii) $\left(\frac{f}{g}\right)$ **is also derivable at $x = c$ and $\left(\frac{f}{g}\right)'(c) = \frac{g(c)f'(c) - f(c)g'(c)}{\{g(c)\}^2}$ provided $g(c) \neq 0$**

(iv) $\frac{1}{f}$ **is also derivable at $x = c$ and $\left(\frac{1}{f}\right)'(c) = -\frac{f'(c)}{\{f(c)\}^2}$ provided $f(c) \neq 0$**

(v) (kf) **is also derivable at $x = c$ and $(kf)'(c) = kf'(c)$, where k is a real number.**

THEOREM 2.2 : If f be defined on an interval continuous and one to one function and f be derivable at $x = c$ with $f'(c) \neq 0$. Then the invers of the function, i.e., f^- is derivable at $f(c)$ and its derivative at $f(c)$ is $\frac{1}{f'(c)}$.

THEOREM 2.3 ***A function which is derivable at a point is necessarily continuous at that point but converse is not true.***

Proof : Let a function $y = f(x)$ **be derivavle at** $x = c$**. That is,** $f'(c)$ **exists. That is,** $f'(c) = \lim_{x\to c}\frac{f(x)-f(c)}{x-c}$ exists. Now $f(x)-f(c) = \frac{f(x)-f(c)}{(x-c)}(x-c)$ $(x \neq c)$. Therefore,

$$\lim_{x\to c}\{f(x)-f(c)\} = \lim_{x\to c}\left\{\frac{f(x)-f(c)}{x-c}\right\}\lim_{x\to c}(x-c) = f'(c)\times 0 = 0 \qquad \Rightarrow \lim_{x\to c} f(x) - \lim_{x\to c} f(c) = 0$$

$\Rightarrow \lim_{x\to c} f(x) = \lim_{x\to c} f(c) = f(c) \Rightarrow \lim_{x\to c} f(x) = f(c)$. Hence $y = f(x)$ is continuous at $x = c$.

Converse is not true.

Let us consider the function $y = f(x) = |x|$.

According to the definition of modulus function $y = f(x) = |x| = x$ when $x > 0$

$= 0$ when $x = 0$

$= -x$ when $x < 0$.

Now $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(-x) = 0$ and $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(x) = 0$. Therefore, $\lim_{x\to 0} f(x) = 0$. Also $f(0) = 0$. Hence $y = f(x) = |x|$ is continuous at $x = 0$.

Now, $\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-}\frac{|x|-0}{x} = \lim_{x\to 0^-}\frac{-x}{x} = -1$.

Also $\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+}\frac{|x|-0}{x} = \lim_{x\to 0^+}\frac{x}{x} = 1$. So, $\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0} \neq \lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0}$

$\Rightarrow \lim_{x\to 0}\frac{f(x)-f(0)}{x-0}$ does not exists.

$\Rightarrow f'(0)$ does not exists. Hence $y = f(x)$ is not differentiable at $x = 0$.

MEANING OF THE SIGN OF DERIVATIVE

Let c be an interior point of the domain of definition of f. Let $f'(c)$ exists. Let $f'(c) > 0$. Therefore,

$f'(c) = \lim_{x\to c}\frac{f(x)-f(c)}{x-c} > 0$.

$\Rightarrow \left|\frac{f(x)-f(c)}{x-c} - f'(c)\right| < \varepsilon$ when $|x-c| < \delta$.

$\Rightarrow f'(c) - \varepsilon < \frac{f(x)-f(c)}{x-c} < f'(c) + \varepsilon$, $\forall x \in (c-\delta, c+\delta)$, $x \neq c$.

If $\varepsilon < f'(c)$ then $\frac{f(x)-f(c)}{x-c} > 0$ $\forall x \in (c-\delta, c+\delta)$, $x \neq c$

then $f(x) - f(c) > 0 \Rightarrow f(x) > f(c)$ when $c < x < c+\delta$

$f(x) - f(c) < 0 \Rightarrow f(x) < f(c)$ when $c-\delta < x < c$.

If $f'(c) > 0$ then there exists a neighbourhood $(c-\delta, c+\delta)$ of c such that

$f(x) > f(c) \ \ \forall x \in (c, c+\delta)$

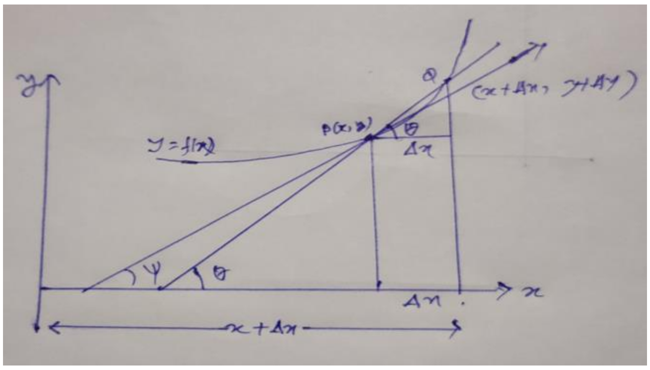
& $f(x) < f(c) \ \ \forall x \in (c-\delta, c)$ **we say** f **is increasing at** c **.**

Similarly, If $f'(c) < 0$ then there exists a neighbourhood $(c-\delta, c+\delta)$ of c such that

$f(x) < f(c) \ \ \forall x \in (c, c+\delta)$

& $f(x) > f(c) \ \ \forall x \in (c-\delta, c)$ **we say** f **is decreasing at** c **.**

GEOMETRICAL INTERPRETATION OF DERIVATIVES



Let $y = f(x)$ be a function. Let $P(x, y)$ be any point on the curve $y = f(x)$. Let $Q(x+\Delta x, y+\Delta y)$ be any neighbouring point taken either Sides of the point $P(x, y)$. Let the chord $\overrightarrow{PQ}$ makes an angle θ with the Positive direction of x-axis . Then $\tan\theta = \dfrac{\Delta y}{\Delta x}$. Let the point Q tends to P along the curve indefinitely so that $\Delta x \to 0$ and $\theta \to \psi$.

Therefore, $\displaystyle \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \tan\theta = \lim_{\theta \to \psi} \tan\theta = \tan\psi$

$$\Rightarrow \lim_{\Delta x \to 0} \frac{y + \Delta y - y}{\Delta x} = \tan\psi$$

$$\Rightarrow \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \tan\psi$$

$$\Rightarrow \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \tan\psi$$

$\Rightarrow \dfrac{dy}{dx} = \tan\psi$. Now the equation of the tangent to the curve $y = f(x)$ at the point $P(x, y)$ is $y - y_1 = m(x - x_1)$. That is, $y - y_1 = \tan\psi(x - x_1)$ That is, $y - y_1 = \left(\dfrac{dy}{dx}\right)_{(x,y)} (x - x_1)$. Therefore, $\dfrac{dy}{dx}$ geometrically represents the slope of the tangent to the curve $y = f(x)$ at the point (x, y).

DARBOUX'S THEOREM

THEOREM 2.4 ***If a function f is derivable In a closed interval $[a,b]$ and $f'(a)$, $f'(b)$ are of opposite signs, then there exists at least one point c of the open interval (a,b) such that $f'(c)=0$.***

Proof : For the sake of definiteness let us suppose that $f'(a)>0$ and $f'(b)<0$. There exist intervals $(a,a+h]$ and $[b-h,b)$, $(h>0)$, such that
$x\in(a,a+h]\Rightarrow f(x)>f(a)$ ....(1)
$x\in[b-h,b)\Rightarrow f(x)>f(b)$ ....(2). Again, since f is derivable in $[a,b]$, f is continuous in $[a,b]$. Therefore, it is bounded and attains its bounds. Thus if M be the least upper bound(*sup*) of f in $[a,b]$ there exists $c\in[a,b]$ such that $f(c)=M$. From (1) & (2), we see that the least upper bound Is not attained at the end points a and b so that c Is interior point of $[a,b]$.
If $f'(c)$ be positive, then there exists an interval $[c,c+\eta]$ $(\eta>0)$ such that for every point x of this interval $f(x)>f(c)=M$ and this is a contradiction.
If $f'(c)$ be negative, then there exists an interval $[c-\eta,c]$ $(\eta>0)$ such that for every point x of this interval $f(x)>f(c)=M$ and this is, again, a contradiction. Hence $f'(c)=0$.

EXAMPLE 17 : If $f(x)=x^2\sin\left(\frac{1}{x}\right)$ when $x\neq 0$

$=0$ when $x=0$, show that f is derivable for every value x of but derivative is not continuous for $x=0$.

<u>SOLUTION</u> (FIRST PART) :

$$Rf'(0)=\underset{h\to 0^+}{Lim}\frac{f(0+h)-f(0)}{h}=\underset{h\to 0^+}{Lim}\frac{h^2\sin\left(\frac{1}{h}\right)-0}{h}=\underset{h\to 0^+}{Lim}\,h\sin\left(\frac{1}{h}\right)=0\times k=0,$$ where $-1\le k\le 1$.

$$Lf'(0)=\underset{h\to 0^-}{Lim}\frac{f(0-h)-f(0)}{-h}=\underset{h\to 0^-}{Lim}\frac{h^2\sin\left(-\frac{1}{h}\right)-0}{-h}=\underset{h\to 0^-}{Lim}\,h\sin\left(\frac{1}{h}\right)=0\times k=0,$$

where $-1\le k\le 1$.
Therefore, as $Rf'(0)=Lf'(0)=0$, f is derivable at $x=0$ and $f'(0)=0$.

(SECOND PART) : Here $f'(x)=2x\sin\left(\frac{1}{x}\right)-\cos\left(\frac{1}{x}\right)$ at $x\neq 0$.

$= 0$ at $x=0$.

Right-hand limit

$$=\underset{h\to 0^+}{Lim}\,f'(x)=\underset{h\to 0^+}{Lim}\,f'(0+h)=\underset{h\to 0^+}{Lim}\,f'(h)=\underset{h\to 0^+}{Lim}\left(2h\sin\left(\frac{1}{h}\right)-\cos\left(\frac{1}{h}\right)\right)$$

$= 2\times 0\times k - \underset{h\to 0^+}{Lim}\cos\left(\frac{1}{h}\right) = -\underset{h\to 0^+}{Lim}\cos\left(\frac{1}{h}\right)$, when $-1\le k\le 1$. As $h\to 0$, $\cos\left(\frac{1}{h}\right)$ oscillates between -1 and 1 and hence $\cos\left(\frac{1}{h}\right)$ does not tend to a fixed and definite limit. Hence Right-hand limit does not exist. Similarly, it can be shown that Left-hand limit also does not exist. Hence $f'(x)$ is not continuous at $x=0$.

EXAMPLE 18 : Show that the function defined by $f(x)=x^2$ is derivable on $[0,1]$.

SOLUTION : Let $c\in(0,1)$ be any point, then

$f'(c) = \underset{x\to c}{Lim}\dfrac{f(x)-f(c)}{x-c} = \underset{x\to c}{Lim}\dfrac{x^2-c^2}{x-c} = \underset{x\to c}{Lim}(x+c) = 2c$. At end point $x=0$

$$f'(0) = \underset{x\to 0^+}{Lim}\frac{f(x)-f(0)}{x-0} = \underset{x\to 0^+}{Lim}\frac{x^2-0^2}{x-0} = \underset{x\to 0^+}{Lim}\,x = 0$$

$f'(1) = \underset{x\to 1^-}{Lim}\dfrac{f(x)-f(1)}{x-1} = \underset{x\to 1^-}{Lim}\dfrac{x^2-1^2}{x-1} = \underset{x\to 1^-}{Lim}(x+1) = 2$. Thus $f'(0)$ and $f'(1)$ both exist. Hence the given function is derivable in the closed interval $[0,1]$.

EXAMPLE 19 : A function f is defind on R by $f(x)=x$ if $0\le x\le 1$
$=1$ if $x\ge 1$

SOLUTION : At $x=1$,

$$Lf'(1) = \underset{x\to 1^-}{Lim}\frac{f(x)-f(1)}{x-1} = \underset{x\to 1^-}{Lim}\frac{x-1}{x-1} = 1$$

$Rf'(1) = \underset{x\to 1^+}{Lim}\dfrac{f(x)-f(1)}{x-1} = \underset{x\to 1^+}{Lim}\dfrac{1-1}{x-1} = 0$. Therefore, $Lf'(1)\ne Rf'(1)$

Thus f is not derivable at $x=1$.

EXAMPLE 20: A function f is defind on R by $f(x)=|x|=x$ if $x>0$
$=0$ if $x=0$
$=-x$ if $x<0$.
Check the derivability of f at $(0,0)$.

SOLUTION : At $(0,0)$,

$$Lf'(0) = \underset{x\to 0^-}{Lim}\frac{f(x)-f(0)}{x-0} = \underset{x\to 0^-}{Lim}\frac{-x-0}{x-0} = -1$$

$Rf'(0) = \underset{x\to 0^+}{Lim}\dfrac{f(x)-f(0)}{x-0} = \underset{h\to 0^-}{Lim}\dfrac{x-0}{x-0} = 1$. Therefore, $Lf'(0)\ne Rf'(0)$

Thus f is not derivable at $x=0$.

EXPANSION OF FUNCTIONS

ROLLE'S THEOREM

If a function f defined on $[a,b]$ is

(i) Continuous on $[a,b]$

(ii) Derivable in (a,b), i.e., $f'(x)$ exists in (a,b)

(iii) $f(a)=f(b)$

Then there exists at least value of x (say c) between a and b , such that $f'(c)=0$.

Proof : As f is continuous in $[a,b]$, f is bounded. Let l.u.b of f be M and g.l.b of f be m. Let c, $d \in [a,b]$ such that $f(c)=M$, $f(d)=m$. There are two possibilities, either $m=M$ or, $m \neq M$.

Case-I : If $m=M$ then $f(x)=M \quad \forall x \in [a,b]$

$$\Rightarrow f'(x)=0 \ \forall x \in [a,b]$$

$$\Rightarrow f'(c)=0, \ c \in [a,b]$$

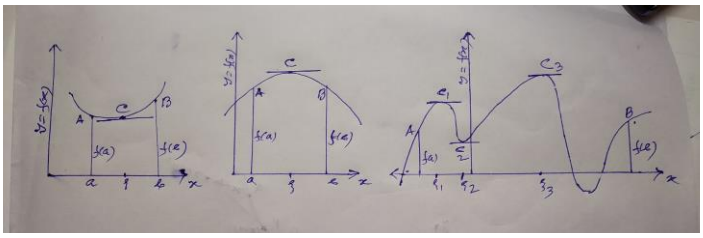
Case-II : Let $m \neq M$. As $f(a)=f(b)$ and $m \neq M$ then atleast one of the numbers M and m be different from $f(a)$ and $f(b)$. Let $M \neq f(a)$, $M \neq f(b)$. Then $f(c) \neq f(a)$, $f(c) \neq f(b)$ which implies $c \neq a$, $c \neq b$. Thus $a<c<b$. As the function f is derivable in (a,b) at c, $f'(c)$ exists .

If $f'(c)>0$ then there exists a neighbourhood $(c-\delta, c+\delta)$ of c such that $f(x)>f(c)(=M)$ when $c<x<c+\delta$ (That is, $x \in (c, c+\delta)$) which contradicts the fact that M is the l.u.b of f. Hence our assumption i.e, $f'(c)>0$, is not true.

Again if, $f'(c)<0$ then there exists a neighbourhood $(c-\delta, c+\delta)$ of c such that $f(x)>f(c)(=M)$ when $c-\delta<x<c$ (That is, $x \in (c-\delta, c)$) which contradicts the fact that M is the l.u.b of f. Hence our assumption i.e, $f'(c)<0$, is not true. So, there remains the only possibilities, i.e., $f'(c)=0$.

GEOMETRICAL INTERPRETATION

Let f be a continuous function defined on $[a,b]$ and derivable on (a,b). Let the graph (curve) be drawn.



Rolle's theorem simply states that between two points A and B with equal ordinate on the graph of the function f , there exists at least one point where the tangent is parallel to $x-$axis.

ALGEBRAIC INTERPRETATION

Between two roots a, b of $f(x)=0$ there exists at least one root ξ of $f'(x)=0$.

LAGRANGE'S MEAN VALUE THEOREM

(FIRST MEAN VALUE THEOREM OF DIFFRENTIAL CALCULUS)

If a function f be defined on $[a,b]$, is

- **i) Continuous on $[a,b]$ and**
- **ii) derivable on (a,b)**

then there exists at least one real number ξ between a and b such that $f(b)-f(a)=(b-a)f'(\xi)$.

Proof : Let us consider the function $\phi(x)=f(x)+Ax$, where A is a constant to be determined such that $\phi(a)=\phi(b)$. Therefore, $A=\dfrac{-f(b)-f(a)}{b-a}$. Now the function $\phi(x)$ is the sum of two continuous and derivable functions $f(x)$ and $A(x)$. Therefore, the function $\phi(x)$ is

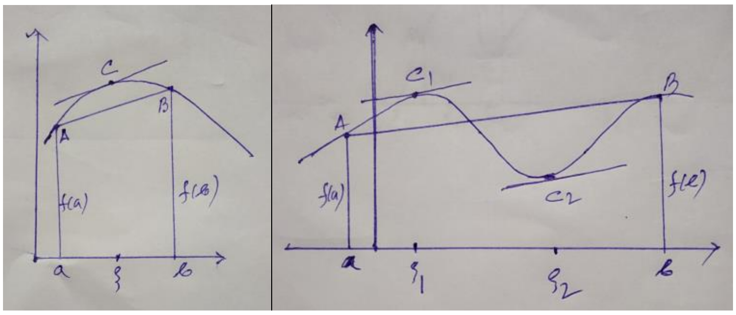
- i) continuous on $[a,b]$
- ii) derivable on (a,b)
- iii) $\phi(a)=\phi(b)$.

Therefore, by Rolle's theorem, there exists a real number $\xi\in[a,b]$ suct that $\phi'(\xi)=0$. As $\phi'(x)=f'(x)+A$, we have $0=\phi'(\xi)=f'(\xi)+A$. That is, $f'(\xi)=-A=\dfrac{f(b)-f(a)}{(b-a)}$. Hence $f(b)-f(a)=(b-a)f'(\xi)$.

ANOTHER STATEMENT OF LAGRANGE'S MEAN-VALUE THEOREM

In the statement of Lagrange's Mean-Value theorem, let b is replaced by $a+h$, then the number ξ between a and b may be written as $a+\theta h$ where $0<a<1$. Thus $f(a+h)-f(a)=hf'(a+\theta h)$, $0<a<1$. That is, $f(a+h)=f(a)+hf'(a+\theta h)$ where $0<a<1$.

GEOMETRICAL INTERPRETATION



The Lagrange's M.V.T states that between two points A and B of the graph of the function f there exists at least one point where the tangent is parallel to the chord $\overline{AB}$.

APPLICATIONS OF MEAN-VALUE THEOREM

EXAMPLE 21 : Using Lagrange's M.V.T, show that $\frac{x}{1+x} < \log(1+x) < x$, $x > 0$

SOLUTION : Let $f(x) = \log(1+x)$ in $[0, x]$. Therefore, $f'(x) = \frac{1}{1+x}$ ....(1). This implies that f is continuous in $[0, x]$. Since f is continuous in $[0, x]$ and derivable in $(0, x)$, so by Lagrange's M.V.T, there exists some θ, $0 < \theta < 1$ such that $\frac{f(x) - f(0)}{x - 0} = f'(\theta x)$ That is, $\log(1+x) = \frac{x}{1+\theta x}$ (using (1))..............(2). As $0 < \theta < 1$ and $x > 0$, we have $\theta x < x$. That is, $1 + \theta x < 1 + x$ $\Rightarrow \frac{1}{1+\theta x} > \frac{1}{1+x}$ $\Rightarrow \frac{x}{1+\theta x} > \frac{x}{1+x}$ ....(3). Again, as $0 < \theta < 1$ and $x > 0$, we have $1 < 1 + \theta x$ $\frac{1}{1+\theta x} < 1$ $\Rightarrow \frac{x}{1+\theta x} < x$...................(4). From (3) & (4) we have, $\frac{x}{1+x} < \frac{x}{1+\theta x} < x$ ....(5). From (2) & (5) we obtain, $\frac{x}{1+x} < \log(1+x) < x$, $x > 0$.

EXAMPLE 22 : Applying Lagrange's M.V.T, prove that $\frac{x}{1+x^2} < \tan^{-1} x < x$, $x > 0$.

SOLUTION : Let $f(x) = \tan^{-1} x$ in $[0, x]$. Therefore, $f'(x) = \frac{1}{1+x^2}$. Clearly, $f'(x)$ exists in $[0, x]$. As f is continuous in $[0, x]$ and derivable in $(0, x)$, by Lagrange's M.V.T, there exists some θ, $0 < \theta < 1$ such that $\frac{f(x) - f(0)}{x - 0} = f'(\theta x)$ That is, $\tan^{-1} x = \frac{x}{(1+\theta^2 x^2)}$ ....(1). As $0 < \theta < 1$ and $x > 0$, we have $\theta x < x$ $\Rightarrow \theta^2 x^2 < x^2$ $\Rightarrow 1 + \theta^2 x^2 < 1 + x^2$ $\Rightarrow \frac{x}{1+\theta^2 x^2} > \frac{x}{1+x^2}$ ....(2). Again, as $0 < \theta < 1$ and $x > 0$, we have $1 < 1 + \theta^2 x^2$ $\Rightarrow \frac{1}{1+\theta^2 x^2} < 1 \Rightarrow \frac{x}{1+\theta^2 x^2} < x$ ....(3). From (2) & (3), we obtain $\frac{x}{1+x^2} < \frac{x}{1+\theta^2 x^2} < x$, $x > 0$ ....(4). From (1) & (4), we have $\frac{x}{1+x^2} < \tan^{-1} x < x$, $x > 0$.

EXAMPLE 23 : Show that $\frac{v-u}{1+v^2}<\tan^{-1}v-\tan^{-1}u<\frac{v-u}{1+u^2}$ if $0<u<v$ and deduce that $\frac{\pi}{4}+\frac{3}{25}<\tan^{-1}\frac{4}{3}<\frac{\pi}{4}+\frac{1}{6}$.

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SOLUTION : Applying L.M.V.T to the function $f(x)=\tan^{-1}x$ in $[u,v]$, we obtain $\frac{f(v)-f(u)}{v-u}=f'(c)$ for some $c\in(u,v)$. That is, $\frac{\tan^{-1}v-\tan^{-1}u}{v-u}=\frac{1}{1+c^2}$ for $u<c<v$....(1) Now

$c>u\Rightarrow 1+c^2>1+u^2\Rightarrow\frac{1}{1+c^2}<\frac{1}{1+u^2}$ ....(2)

Again, $c<v\Rightarrow 1+c^2<1+v^2\Rightarrow\frac{1}{1+c^2}>\frac{1}{1+v^2}$ ....(3). From (1), (2) & (3), we get $\frac{v-u}{1+v^2}<\tan^{-1}v-\tan^{-1}u<\frac{v-u}{1+u^2}$. (since $u<v\Rightarrow v-u>0$).................(4)

SECOND PART : Let $u=1$ and $v=\frac{4}{3}$. Then by (4), we get $\frac{3}{25}<\tan^{-1}\frac{4}{3}-\tan^{-1}1<\frac{1}{6}$

$\Rightarrow\tan^{-1}1+\frac{3}{25}<\tan^{-1}\frac{4}{3}<\tan^{-1}1+\frac{1}{6}$

$\Rightarrow\frac{\pi}{4}+\frac{3}{25}<\tan^{-1}\frac{4}{3}<\frac{\pi}{4}+\frac{1}{6}$.

EXAMPLE 24 : Show that $\frac{2}{\pi}<\frac{\sin x}{x}<1$ if $0<x<\frac{\pi}{2}$

SOLUTION : Let $\phi(x)=\frac{\sin x}{x}$ when $x\neq 0$

$=1$ when $x=0$. Cleary, $\phi(x)$ is continuous in $0\le x\le\frac{\pi}{2}$ and derivable in $0<x<\frac{\pi}{2}$. Therefore, $\phi'(x)=\frac{x\cos x-\sin x}{x^2}$. Let $\psi(x)=x\cos x-\sin x$ be defined in $0\le x\le\frac{\pi}{2}$. Therefore, $\psi'(x)=-x\sin x<0$ for $0<x<\frac{\pi}{2}$. Hence $\psi(x)$ is strictly decreasing in $0\le x\le\frac{\pi}{2}$. So $\psi(x)<\psi(0)=0$ for all x, $0\le x\le\frac{\pi}{2}$. This implies $\phi'(x)<0$, for $0\le x\le\frac{\pi}{2}$. Hence $\phi(x)$ is strictly decreasing in $0\le x\le\frac{\pi}{2}$. Therefore, $\phi(0)>\phi(x)>\phi\left(\frac{\pi}{2}\right)$ for $0<x<\frac{\pi}{2}$. That is, $1>\frac{\sin x}{x}>\frac{1}{\frac{\pi}{2}}$

$\Rightarrow\frac{2}{\pi}<\frac{\sin x}{x}<1$ for $0<x<\frac{\pi}{2}$.

EXAMPLE 25 : Find the value of c of Lagrange's M.V.T when $f(x)=2x^2+3x+4$ in $[1,2]$.

SOLUTION : The given polynomial $f(x)=2x^2+3x+4$ is continuous in the closed interval $[1,2]$. Then $f(x)$ is derivable in $(1,2)$. Thus $f(x)$ satisfies conditions of **Lagrange's M.V.T.** So there must exist $c\in(1,2)$ such that $\frac{f(2)-f(1)}{2-1}=f'(c)\Rightarrow\frac{18-9}{2-1}=4c+3\Rightarrow c=\frac{3}{2}$ Since $c\in(1,2)$, the required value of c is $\frac{3}{2}$.

EXAMPLE 26 : Verify Lagrange's M.V.T for the polynomial $f(x)=x(x-1)(x-2)$ in $\left[0,\frac{1}{2}\right]$.

SOLUTION : The given polynomial $f(x)=x(x-1)(x-2)$ is continuous in the closed interval $\left[0,\frac{1}{2}\right]$. Then $f(x)$ is derivable in $(0,\frac{1}{2})$. Thus $f(x)$ satisfies conditions of **Lagrange's M.V.T.** So there must exist $c\in(0,\frac{1}{2})$ such that $\frac{f(\frac{1}{2})-f(0)}{\frac{1}{2}-0}=f'(c)$ ....(1). We see $f'(x)=(x-1)(x-2)+x(x-1)+x(x-2)\Rightarrow f'(x)=3x^2-6x+2$. So from (1), we have $\frac{\frac{3}{8}}{\frac{1}{2}}=3c^2-6c+2\Rightarrow c=\frac{(6\pm\sqrt{21})}{6}$. Out of two values of c only $\left(1-\frac{\sqrt{21}}{6}\right)$ lies in $(0,\frac{1}{2})$. Hence $c=\left(1-\frac{\sqrt{21}}{6}\right)$ and the **Lagrange's M.V.Theorem** verified.

EXAMPLE 27 : Evaluate the value of θ that appears in Lagrange's M.V.Theorem for the polynomial $f(x)=x^2-2x+3$, given that $a=1,b=\frac{1}{2}$.

SOLUTION : Given polynomial is $f(x)=x^2-2x+3$. Therefore, $f'(x)=2x-2$ ....(1). Since the given polynomial satisfies the conditions of **Lagrange's M.V.T.** So there exists θ, where $0<\theta<1$, satisfying $f(a+h)-f(a)=hf'(a+\theta h)$.

That is, $f(1+h)-f(1)=hf'(1+\theta h)$

SEMESTER-III

HONOURS

CORE COURSE---C 5T

UNIT-III (MARKS-11)

UNIT-III **Dr. Pradip Kumar Gain**

Syllabus for Unit-III: **Cauchy's mean value theorem. Taylor's theorem with Lagrange's form of remainder, Taylor's theorem with Cauchy's form of remainder, application of Taylor's theorem to convex functions, relative extrema. Taylor's series and Maclaurin's series expansions of exponential and trigonometric functions, ln (1 + x), 1/(ax + b) and $(x+1)^n$. Application of Taylor's theorem to inequalities.**

CAUCHY'S MEAN-VALUE THEOREM

If two functions f , g defined on $[a,b]$ are

i) Continuous on $[a,b]$

ii) Derivable on (a,b) and

iii) $g'(x) \neq 0$ For any $x \in [a,b]$

then there exists at least one real number ξ between a and b such that $\dfrac{f(b)-f(a)}{g(b)-g(a)} = \dfrac{f'(\xi)}{g'(\xi)}$.

Proof : Let a function ϕ be defined by $\phi(x) = f(x) + Ag(x)$, where A is a constant to be determined that $\phi(a) = \phi(b)$. That is, $f(a) + Ag(a) = f(b) + Ag(b)$. Hence $A = -\dfrac{(f(b)-f(a))}{(g(b)-g(a))}$ ....(1) as $(g(b)-g(a)) \neq 0$. If $(g(b)-g(a)) = 0$, then the function g would satisfy all the conditions of Rolle's theorem and derivative of g would, therefore, vanish at least once in (a,b) and the condition (iii) would be violated. The function ϕ is continuous in the closed interval $[a,b]$, derivable in the open

interval (a,b) and $\phi(a)=\phi(b)$. Hence by Rolle's theorem, there exists at least one point $\xi \in (a,b)$ such that $\phi'(\xi)=0$. That is, $\phi'(\xi)=f'(\xi)+Ag'(\xi)=0$. That is, $\dfrac{f'(\xi)}{g'(\xi)}=-A=\dfrac{(f(b)-f(a))}{(g(b)-g(a))}$ [by (1)].

TAYLOR'S THEOREM (Generalized Mean Value Theorem)(FINITE FORM)

If a function f possesses differential co-efficients of the first $(n-1)$ orders for every value of x in the closed interval $[a,b]$ and the n^th derivative of f exists in the open interval (a,b), i.e., if $f^{n-1}(x)$ is continuous in $[a,b]$ and $f^{n}(x)$ exists in (a,b), then

$$f(b)=f(a)+(b-a)f'(a)+\frac{(b-a)^2}{2!}f''(a)+\frac{(b-a)^3}{3!}f'''(a)+....+\frac{(b-a)^{n-1}}{(n-1)!}f^{n-1}(a)+\frac{(b-a)^n}{n!}f^{n}(\xi)$$

where $a<\xi<b$ ....(A)

If $b=a+h$ so that $b-a=h$, then

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\frac{h^3}{3!}f'''(a)+.............................+\frac{h^{n-1}}{(n-1)!}f^{n-1}(a)+\frac{h^n}{n!}f^{n}(a+\theta h)$$

where $0<\theta<1$..................................(B)

If we write x for a, we have

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\frac{h^3}{3!}f'''(a)+.............................+\frac{h^{n-1}}{(n-1)!}f^{n-1}(x)+\frac{h^n}{n!}f^{n}(x+\theta h)$$

where $0<\theta<1$..................................(C)

Proof : Let us consider the function $\psi(x)$ in (a,b) defined by $\psi(x)=\phi(x)-\dfrac{(b-x)^n}{(b-a)^n}\phi(a)$ ....(1) where

$$\phi(x)=f(b)-f(x)-(b-x)f'(x)-\frac{(b-x)^2}{2!}f''(x)-...........-\frac{(b-x)^{n-1}}{(n-1)!}f^{n-1}(x)\text{.................................(2)}$$

Clearly, $\psi(a)=0$ and $\psi(b)=0$.

Again,

$$\phi'(x)=-f'(x)+\{f'(x)-(b-x)f''(x)\}+\left\{(b-x)f''(x)-\frac{(b-x)^2}{2!}f'''(x)\right\}+..............................$$

$$..............................\left\{\frac{(b-x)^{n-2}}{(n-2)!}f^{n-1}(x)-\frac{(b-x)^{n-1}}{(n-1)!}f^{n}(x)\right\}$$

$$= -\frac{(b-x)^{n-1}}{(n-1)!} f^n(x) .$$

Now from (1),

$$\psi'(x) = \phi'(x) + \frac{n(b-x)^{n-1}}{(b-a)^n} \phi(a)$$

$$= -\frac{(b-x)^{n-1}}{(n-1)!} f^n(x) + \frac{n(b-x)^{n-1}}{(b-a)^n} \phi(a) .$$

As $\psi(a) = \psi(b)$ **and** $\psi'(x)$ **exists in** (a,b)**, by Rolle's Theorem there exists at least one value of** x **(say** ξ **) such that** $\psi'(\xi) = 0$ **.**

That is, $= -\frac{(b-\xi)^{n-1}}{(n-1)!} f^n(\xi) + \frac{n(b-\xi)^{n-1}}{(b-a)^n} \phi(a) = 0$

$\Rightarrow \phi(a) = \frac{(b-a)^n}{n!} f^n(\xi)$**. Then from (2), we have**

$$\phi(a) = f(b) - f(a) - (b-a) f'(a) - \frac{(b-a)^2}{2!} f''(a) - \ldots\ldots\ldots. - \frac{(b-a)^{n-1}}{(n-1)!} f^{n-1}(a)$$

or, $\frac{(b-a)^n}{n!} f^n(\xi) = f(b) - f(a) - (b-a) f'(a) - \frac{(b-a)^2}{2!} f''(a) - \ldots\ldots\ldots. - \frac{(b-a)^{n-1}}{(n-1)!} f^{n-1}(a)$

or,

$f(b) = f(a) + (b-a) f'(a) + \frac{(b-a)^2}{2!} f''(a) + \ldots\ldots\ldots. + \frac{(b-a)^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{(b-a)^n}{n!} f^n(\xi)$ **where** $a < \xi < b$ **..................................(A)**

Putting $b = a + h$ **we have**

$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n}{n!} f^n(a+\theta h)$

where $0 < \theta < 1$**..................................(B)**

writing x **for** a**, we have**

$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. + \frac{h^{n-1}}{(n-1)!} f^{n-1}(x) + \frac{h^n}{n!} f^n(x+\theta h)$

where $0 < \theta < 1$**..................................(C)**

REMARK : The series (A) or (B) or (C) is called Taylor's series in finite form with remainder in Lagrange's form.

Here remainder term means the term after nth term in the series. Remainder is denoted by R_n **.**

LAGRANGE'S FORM OF REMAIDER (R_n) FOR TAYLOR'S SERIES.

Lagrange's form of remaider for Taylor's series(A) is

$$\boxed{R_n = \frac{(b-a)^n}{n!} f^n(\xi). \quad a < \xi < b}$$

Lagrange's form of remaider for Taylor's series(B) is

$$\boxed{R_n = \frac{h^n}{n!} f^n(a+\theta h),\ 0 < \theta < 1}$$

Lagrange's form of remaider for Taylor's series(C) is

$$\boxed{R_n = \frac{h^n}{n!} f^n(x+\theta h),\ 0 < \theta < 1}$$

CAUCHY'S FORM OF REMAIDER (R_n) FOR TAYLOR'S SERIES.

Cauchy's form of remaider for Taylor's series(B) is

$$\boxed{R_N = \frac{h^n(1-\theta)^{n-1}}{(n-1)!} f^n(a+\theta h),\ 0 < \theta < 1}$$

Cauchy's form of remaider for Taylor's series(C) is

$$R_N = \frac{h^n(1-\theta)^{n-1}}{(n-1)!} f^n(x+\theta h), \; 0<\theta<1$$

MACLAURIN'S SERIES (FINITE FORM)

Putting $x=0$ and $h=x$ in Taylor's series (C) we get,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots\dots\dots\dots + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0) + \frac{x^n}{n!} f^n(\theta x), \quad 0<\theta<1.$$ **This is known as Maclaurin's series in finite form with remainder $R_n = \frac{x^n}{n!} f^n(\theta x)$ in Lagrange's form.**

And

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots\dots\dots + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0) + \frac{x^n(1-\theta)^{n-1}}{(n-1)!} f^n(\theta x), \quad 0<\theta<1.$$

This is known as Maclaurin's series in finite form with remainder $R_n = \frac{x^n(1-\theta)^{n-1}}{(n-1)!} f^n(\theta x)$ in Cauchy's form.

TAYLOR'S INFINITE SERIES

If $f(x)$, . $f'(x)$, . $f''(x)$,..........................., $f^n(x)$ exist finitely, however large n may be in any $\delta-$neighbourhood of x and if R_n tends to zero as n tends to infinity then Taylor's series extended to infinity is valid and we have

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \dots\dots\dots\dots\dots\dots \text{ to } \infty \quad (|h|<\delta).$$

MACLAURIN'S SERIES (EXTENDED TO INFINITY)

If $f(x)$, . $f'(x)$, . $f''(x)$,.............................., $f^n(x)$ exist finitely, however large n may be in any $\delta-$neighbourhood of x and if R_n tends to zero as n tends to infinity then Maclaurin's series extended to infinity is valid and we have

$$\boxed{f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\ldots\ldots\ldots\ldots\ldots\ldots\ \text{to}\ \infty\ \ (|x|<\delta).}$$

EXPANSION OF SOME FUNCTIONS

EX-(1) Expand the function $f(x)=e^x$ in a finite series in powers of x (i.e., in the neighbourhood of $x=0$) with the remainder in Lagrange's form and also in cauchy's form.

Solution : Given function is $f(x)=e^x \to f(0)=1$

$$f'(x)=e^x \to f'(0)=1$$
$$f''(x)=e^x \to f''(0)=1$$
$$f'''(x)=e^x \to f'''(0)=1$$
$$\vdots$$
$$f^{n-1}(x)=e^x \to f^{n-1}(0)=1$$
$$f^n(x)=e^x \to f^n(\theta x)=e^{\theta x}$$

Now,

$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\ldots\ldots\ldots.+\frac{x^{n-1}}{(n-1)!}f^{n-1}(0)+\frac{x^n}{n!}f^n(\theta x)$, where $\frac{x^n}{n!}f^n(\theta x)$ is the remainder in Lagrange's form, $0<\theta<1$.

Hence expansion of the function $f(x)=e^x$ in finite form with remainder in Lagrange's form is

$$\boxed{e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\ldots\ldots+\frac{x^{n-1}}{(n-1)!}+\frac{x^n}{n!}e^{\theta x},\quad 0<\theta<1\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)}$$

Again,

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\ldots\ldots\ldots.+\frac{x^{n-1}}{(n-1)!}f^{n-1}(0)+\frac{x^n(1-\theta)^{n-1}}{(n-1)!}f^n(\theta x),$$

where $\frac{x^n(1-\theta)^{n-1}}{(n-1)!}f^n(\theta x)$ is the remainder in Cauchy's form, $0<\theta<1$.

Hence expansion of the function $f(x)=e^x$ in finite form with remainder in Cauchy's form is

$$\boxed{e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\ldots\ldots+\frac{x^{n-1}}{(n-1)!}+\frac{x^n(1-\theta)^{n-1}}{(n-1)!}e^{\theta x},\ 0<\theta<1\ldots\ldots\ldots\ldots\ldots\ldots\mathbf{(2)}}$$

EX-(2) Expand the function $f(x)=e^x$ in powers of x (i.e., in the neighbourhood of $x=0$) in infinite series and state the condition under which the expansion is valid.

Solution : Given function is $f(x)=e^x \to f(0)=1$

$$f'(x)=e^x \to f'(0)=1$$

$$f''(x)=e^x \to f''(0)=1$$

$$f'''(x)=e^x \to f'''(0)=1$$

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$$f^{n-1}(x)=e^x \to f^{n-1}(0)=1$$

$$f^n(x)=e^x \to f^n(\theta x)=e^{\theta x}$$

Now,

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\ldots\ldots\ldots.+\frac{x^{n-1}}{(n-1)!}f^{n-1}(0)+\frac{x^n}{n!}f^n(\theta x)$$, where $\frac{x^n}{n!}f^n(\theta x)$ is the remainder in Lagrange's form, $0<\theta<1$.

Hence expansion of the function $f(x)=e^x$ in finite form with remainder in Lagrange's form is

$$\boxed{e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\ldots\ldots+\frac{x^{n-1}}{(n-1)!}+\frac{x^n}{n!}e^{\theta x},\quad 0<\theta<1\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\mathbf{(1)}}$$

From (1) $R_n=\frac{x^n}{n!}e^{\theta x}$, $0<\theta<1$. Now $e^{\theta x}<e^{|x|}$. As $e^{|x|}$ is a finite quantity for a given x, $e^{\theta x}$ is also finite. Also $\frac{x^n}{n!}\to 0$ as $n\to\infty$. Therefore, $R_n\to 0$ as $n\to\infty$.

Thus the conditions of Maclaurin's infinite expansion is valid.

Hence from (1), we have the infinite expansion of the given function $f(x)=e^x$ as

$$\boxed{e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\dots\dots \text{ to } \infty \quad \forall x \in R}$$

EX-(3) Expand the function $f(x)=a^x$ in a finite series in powers of x (i.e., in the neighbourhood of $x=0$) with the remainder in Lagrange's form and also in cauchy's form.

Solution : Given function is $f(x)=a^x \rightarrow f(0)=1$

$$f'(x)=a^x \log_e a \rightarrow f'(0)=\log_e a$$

$$f''(x)=a^x(\log_e a)^2 \rightarrow f''(0)=(\log_e a)^2$$

$$f'''(x)=a^x(\log_e a)^3 \rightarrow f'''(0)=(\log_e a)^3$$

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$$f^{n-1}(x)=a^x(\log_e a)^{n-1} \rightarrow f^{n-1}(0)=(\log_e a)^{n-1}$$

$$f^n(x)=a^x(\log_e a)^n \rightarrow f^n(\theta x)=a^{\theta x}(\log_e a)^n$$

Now,

$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\dots\dots\dots+\frac{x^{n-1}}{(n-1)!}f^{n-1}(0)+\frac{x^n}{n!}f^n(\theta x)$, where $\frac{x^n}{n!}f^n(\theta x)$ is the remainder in Lagrange's form, $0<\theta<1$.

Hence expansion of the function $f(x)=a^x$ in finite form with remainder in Lagrange's form is

$$\boxed{a^x = 1 + x\log_e a + \frac{x^2}{2!}(\log_e a)^2 + \frac{x^3}{3!}(\log_e a)^3 + \dots\dots + \frac{x^{n-1}}{(n-1)!}(\log_e a)^{n-1} + \frac{x^n}{n!}a^{\theta x}(\log_e a)^n}$$

....(1)

Again,

$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\dots\dots\dots+\frac{x^{n-1}}{(n-1)!}f^{n-1}(0)+\frac{x^n(1-\theta)^{n-1}}{(n-1)!}f^n(\theta x)$,

where $\frac{x^n(1-\theta)^{n-1}}{(n-1)!}f^n(\theta x)$ is the remainder in Cauchy's form, $0<\theta<1$.

Hence expansion of the function $f(x)=a^x$ in finite form with remainder in Cauchy's form is

$$e^x = 1 + x\log_e a + \frac{x^2}{2!}(\log_e a)^2 + \dots\dots\dots + \frac{x^{n-1}}{(n-1)!}(\log_e a)^{n-1} + \frac{x^n(1-\theta)^{n-1}}{(n-1)!}a^{\theta x}(\log_e a)^n,$$
$0<\theta<1$ ....**(2)**

EX-(4) Expand the function $f(x) = a^x$ in powers of x (i.e., in the neighbourhood of $x=0$) in infinite series and state the condition under which the expansion is valid.

Solution : Given function is $f(x) = a^x \to f(0) = 1$

$$f'(x) = a^x \log_e a \to f'(0) = \log_e a$$

$$f''(x) = a^x (\log_e a)^2 \to f''(0) = (\log_e a)^2$$

$$f'''(x) = a^x (\log_e a)^3 \to f'''(0) = (\log_e a)^3$$

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$$f^{n-1}(x) = a^x (\log_e a)^{n-1} \to f^{n-1}(0) = (\log_e a)^{n-1}$$

$$f^n(x) = a^x (\log_e a)^n \to f^n(\theta x) = a^{\theta x}(\log_e a)^n$$

Now,

$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \dots\dots\dots. + \frac{x^{n-1}}{(n-1)!}f^{n-1}(0) + \frac{x^n}{n!}f^n(\theta x)$, where $\frac{x^n}{n!}f^n(\theta x)$ is the remainder in Lagrange's form, $0<\theta<1$.

Hence expansion of the function $f(x) = a^x$ in finite form with remainder in Lagrange's form is

$$a^x = 1 + x\log_e a + \frac{x^2}{2!}(\log_e a)^2 + \frac{x^3}{3!}(\log_e a)^3 + \dots\dots\dots + \frac{x^{n-1}}{(n-1)!}(\log_e a)^{n-1} + \frac{x^n}{n!}a^{\theta x}(\log_e a)^n$$

....(1)

From (1), the remaider (R_n) term after nth term is $R_n = \frac{x^n}{n!}f^n(\theta x) = \frac{x^n}{n!}a^{\theta x}(\log_e a)^n$, $0<\theta<1$, Let $P = \log_e a$. Then

$$R_n = \frac{x^n}{n!}a^{\theta x}(P)^n = \frac{(Px)^n}{n!}a^{\theta x} = \frac{K^n}{n!}a^{\theta x} = \frac{K^n}{n!}e^{\log_e a^{\theta x}} = \frac{K_n}{n!}e^{\theta x \log_e a} = \frac{\log_e aK^n}{n!}e^{\theta x} = \frac{PK^n}{n!}e^{\theta x}$$

Now $e^{\theta x} < e^{|x|}$, $0<\theta<1$. As $e^{|x|}$ is a finite quantity for a give**n** x, $e^{\theta x}$ is also finite. Also $\frac{PK^n}{n!} \to 0$ as $n \to \infty$. Therefore, $R_n \to 0$ as $n \to \infty$. Thus the condition of Maclaurin's infinite expansion is valid.

Hence from (1), we have the infinite expansion of the given function $f(x)=a^x$ as

$$a^x = 1 + x\log_e a + \frac{x^2}{2!}(\log_e a)^2 + \frac{x^3}{3!}(\log_e a)^3 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{ to } \infty \quad \forall x \in R$$

EX-(5) **Expand the function $f(x)=\sin x$ in a finite series in powers of x (i.e., in the neighbourhood of $x=0$) with the remainder in Lagrange's form and also in cauchy's form.**

Solution : Given function is $f(x)=\sin x \qquad \rightarrow f(0)=0$

$$f'(x)=\cos x=\sin\left(\frac{\pi}{2}+x\right) \qquad \rightarrow f'(0)=1$$

$$f''(x)=\cos\left(\frac{\pi}{2}+x\right)=\sin\left(\frac{2\pi}{2}+x\right) \qquad \rightarrow f''(0)=0$$

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$$f^{n-1}(x)=\sin\left(\frac{(n-1)\pi}{2}+x\right) \qquad \rightarrow f^{n-1}(0)=$$

$$f^{n}(x)=\sin\left(\frac{n\pi}{2}+x\right) \qquad \rightarrow f^{n}(\theta x)=\sin\left(\frac{n\pi}{2}+\theta x\right)$$

Now,

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\ldots\ldots\ldots.+\frac{x^{n-1}}{(n-1)!}f^{n-1}(0)+\frac{x^n}{n!}f^n(\theta x)\text{ , where }\frac{x^n}{n!}f^n(\theta x)$$

is the remainder in Lagrange's form, $0<\theta<1$.

Hence expansion of the function $f(x)=\sin x$ in finite form with remainder in Lagrange's form is

$$\sin x = 0 + x\cdot 1 + \frac{x^2}{2!}\times 0 + \frac{x^3}{3!}\times(-1) + \ldots\ldots\ldots\ldots\ldots\ldots + \frac{x^n}{n!}\sin\left(\frac{n\pi}{2}+\theta x\right), \quad 0<\theta<1.$$

OR

$$\sin x = x - \frac{x^3}{3!} + .\frac{x^5}{5!}. - \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \frac{x^n}{n!}\sin\left(\frac{n\pi}{2}+\theta x\right), \quad 0<\theta<1 \ldots\ldots\ldots\ldots\ldots\ldots\text{(1)}$$

Again,

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\ldots\ldots\ldots.+\frac{x^{n-1}}{(n-1)!}f^{n-1}(0)+\frac{x^n(1-\theta)^{n-1}}{(n-1)!}f^n(\theta x),$$

where $\dfrac{x^n(1-\theta)^{n-1}}{(n-1)!}f^n(\theta x)$ is the remainder in Cauchy's form, $0<\theta<1$.

Hence expansion of the function $f(x)=\sin x$ in finite form with remainder in Cauchy's form is

$$\boxed{\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\ldots\ldots + \frac{x^n}{(n-1)!}(1-\theta)^{n-1}\sin\left(\frac{n\pi}{2} + \theta x\right),\ 0<\theta<1 \ldots\ldots\ldots(1)}$$

EX-(6) **Expand the function $f(x) = \sin x$ in powers of x (i.e., in the neighbourhood of $x = 0$) in infinite series and state the condition under which the expansion is valid.**

Solution : Given function is $f(x) = \sin x \qquad \rightarrow f(0) = 0$

$$f'(x) = \cos x = \sin\left(\frac{\pi}{2} + x\right) \qquad \rightarrow f'(0) = 1$$

$$f''(x) = \cos\left(\frac{\pi}{2} + x\right) = \sin\left(\frac{2\pi}{2} + x\right) \qquad \rightarrow f''(0) = 0$$

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$$f^{n-1}(x) = \sin\left(\frac{(n-1)\pi}{2} + x\right) \qquad \rightarrow f^{n-1}(0) =$$

$$f^{n}(x) = \sin\left(\frac{n\pi}{2} + x\right) \qquad \rightarrow f^{n}(\theta x) = \sin\left(\frac{n\pi}{2} + \theta x\right)$$

Now,

$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots\ldots\ldots + \frac{x^{n-1}}{(n-1)!}f^{n-1}(0) + \frac{x^n}{n!}f^{n}(\theta x)$, where $\frac{x^n}{n!}f^{n}(\theta x)$ is the remainder in Lagrange's form, $0<\theta<1$.

Hence expansion of the function $f(x) = \sin x$ in finite form with remainder in Lagrange's form is

$$\sin x = 0 + x \cdot 1 + \frac{x^2}{2!}\times 0 + \frac{x^3}{3!}\times(-1) + \ldots\ldots\ldots\ldots\ldots + \frac{x^n}{n!}\sin\left(\frac{n\pi}{2} + \theta x\right),\quad 0<\theta<1.$$

OR

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\ldots\ldots\ldots\ldots\ldots + \frac{x^n}{n!}\sin\left(\frac{n\pi}{2} + \theta x\right),\ 0<\theta<1 \ldots\ldots\ldots(1)$$

From (1), the remaider (R_n) term after nth term is $R_n = \frac{x^n}{n!}\sin\left(\frac{n\pi}{2} + \theta x\right)$.

As $|R_n| = \left|\frac{x^n}{n!}\sin\left(\frac{n\pi}{2} + \theta x\right)\right| = \left|\frac{x^n}{n!}\right|\left|\sin\left(\frac{n\pi}{2} + \theta x\right)\right| \le \left|\frac{x^n}{n!}\right| \quad \left(\because \left|\sin\left(\frac{\pi}{2} + \theta x\right)\right| \le 1\right)$.

$\frac{x^n}{n!} \to 0$ as $n \to \infty$. Therefore, $R_n \to 0$ as $n \to \infty$. Thus the conditions for Maclaurin's infinite expansion are satisfied.

Hence from (1), we have the infinite expansion of the given function $f(x) = \sin x$.

$$\boxed{\sin x = x - \frac{x^3}{3!} + .\frac{x^5}{5!}. - \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{ to } \infty \quad \forall x \in R}$$

EX-(7) Expand the function $f(x) = \cos x$ in a finite series in powers of x (i.e., in the neighbourhood of $x = 0$) with the remainder in Lagrange's form and also in cauchy's form.

Solution : Given function is $f(x) = \sin x \qquad \to f(0) = 1$

$$f'(x) = \cos\left(\frac{\pi}{2} + x\right) = -\sin x \qquad \to f'(0) = 0$$

$$f''(x) = \cos\left(\frac{2\pi}{2} + x\right) = -\cos x \qquad \to f''(0) = -1$$

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$$f^n(x) = \cos\left(\frac{n\pi}{2} + x\right) \qquad \to f^n(\theta x) = \cos\left(\frac{n\pi}{2} + \theta x\right)$$

Now,

$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots\ldots\ldots. + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0) + \frac{x^n}{n!} f^n(\theta x)$, where $\frac{x^n}{n!} f^n(\theta x)$ is the remainder in Lagrange's form, $0 < \theta < 1$.

Hence expansion of the function $f(x) = \cos x$ in finite form with remainder in Lagrange's form is

$$\boxed{\cos x = 1 - \frac{x^2}{2!} + .\frac{x^4}{4!}. - \ldots\ldots\ldots\ldots\ldots\ldots\ldots + \frac{x^n}{n!}\cos\left(\frac{n\pi}{2} + \theta x\right), \quad 0 < \theta < 1 \ldots\ldots\ldots\ldots\ldots\ldots\text{(1)}}$$

Again,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots\ldots\ldots. + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0) + \frac{x^n(1-\theta)^{n-1}}{(n-1)!} f^n(\theta x),$$

where $\frac{x^n(1-\theta)^{n-1}}{(n-1)!} f^n(\theta x)$ is the remainder in Cauchy's form, $0 < \theta < 1$.

Hence expansion of the function $f(x) = \cos x$ in finite form with remainder in Cauchy's form is

$$\cos x = 1 - \frac{x^2}{2!} + .\frac{x^4}{4!} . - .... + \frac{x^n}{(n-1)!}(1-\theta)^{n-1}\cos\left(\frac{n\pi}{2}+\theta x\right), \quad 0<\theta<1 \text{.................}(1)$$

EX-(8) Expand the function $f(x)=\cos x$ in powers of x (i.e., in the neighbourhood of $x=0$) in infinite series and state the condition under which the expansion is valid.

Solution : Given function is $f(x)=\sin x \quad \to f(0)=1$

$$f'(x)=\cos\left(\frac{\pi}{2}+x\right)=-\sin x \quad \to f'(0)=0$$

$$f''(x)=\cos\left(\frac{2\pi}{2}+x\right)=-\cos x \quad \to f''(0)=-1$$

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$$f^n(x)=\cos\left(\frac{n\pi}{2}+x\right) \quad \to f^n(\theta x)=\cos\left(\frac{n\pi}{2}+\theta x\right)$$

Now,

$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+..........+\frac{x^{n-1}}{(n-1)!}f^{n-1}(0)+\frac{x^n}{n!}f^n(\theta x)$, where $\frac{x^n}{n!}f^n(\theta x)$ is the remainder in Lagrange's form, $0<\theta<1$.

Hence expansion of the function $f(x)=\cos x$ in finite form with remainder in Lagrange's form is

$$\cos x = 1 - \frac{x^2}{2!} + .\frac{x^4}{4!} . - .... + \frac{x^n}{n!}\cos\left(\frac{n\pi}{2}+\theta x\right), \quad 0<\theta<1 \text{.................}(1)$$

From (1), the remaider (R_n) term after nth term is $R_n=\frac{x^n}{n!}\cos\left(\frac{n\pi}{2}+\theta x\right)$.

As $|R_n|=\left|\frac{x^n}{n!}\cos\left(\frac{n\pi}{2}+\theta x\right)\right|=\left|\frac{x^n}{n!}\right|\left|\cos\left(\frac{n\pi}{2}+\theta x\right)\right|\le\left|\frac{x^n}{n!}\right| \quad \left(\because \left|\cos\left(\frac{\pi}{2}+\theta x\right)\right|\le 1\right)$.

$\frac{x^n}{n!}\to 0$ as $n\to\infty$. Therefore, $R_n\to 0$ as $n\to\infty$. Thus the conditions for Maclaurin's infinite expansion are satisfied.

Hence from (1), we have the infinite expansion of the given function $f(x)=\cos x$ as

$$\cos x = 1 - \frac{x^2}{2!} + .\frac{x^4}{4!} . - .... \textbf{ to } \infty \quad \forall x\in R$$

EX-(9) Expand the function $f(x)=\log(1+x)$, $(-1\le x\le 1)$ in a finite series in powers of x (i.e., in the neighbourhood of $x=0$) with the remainder in Lagrange's form and also in cauchy's form.

Solution : Given function is $f(x)=\log(1+x)$, $(-1\le x\le 1)$ $\quad\rightarrow f(0)=0$

$$f'(x)=\frac{1}{1+x}=(1+x)^{-1} \quad \rightarrow f'(0)=1$$

$$f''(x)=(-1)(1+x)^{-2} \quad \rightarrow f''(0)=-1$$

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$$f^{n}(x)=(-1)(-2)......\{-(n-1)\}(1+x)^{-n}$$

$$=\frac{(-1)^{n-1}(n-1)!}{(1+x)^{n}} \quad \rightarrow =\frac{(-1)^{n-1}(n-1)!}{(1+\theta x)^{n}}$$

Now,

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+..........+\frac{x^{n-1}}{(n-1)!}f^{n-1}(0)+\frac{x^n}{n!}f^{n}(\theta x)$$, where $\frac{x^n}{n!}f^{n}(\theta x)$

is the remainder in Lagrange's form, $0<\theta<1$.

Hence expansion of the function $f(x)=\log(1+x)$, $(-1\le x\le 1)$in finite form with remainder in Lagrange's form is

$$\log(1+x)=x-\frac{x^2}{2}+.\frac{x^3}{3}.-.\frac{x^4}{4}.+.............+\frac{x^n}{n!}(-1)^{n-1}\frac{(n-1)!}{(1+\theta x)^n},\ 0<\theta<1$$

Or

$$\log(1+x)=x-\frac{x^2}{2}+.\frac{x^3}{3}.-.\frac{x^4}{4}.+.............+\frac{x^n}{n}\frac{(-1)^{n-1}}{(1+\theta x)^n},\ 0<\theta<1 \text{.............(1)}$$

Again,

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+..........+\frac{x^{n-1}}{(n-1)!}f^{n-1}(0)+\frac{x^n(1-\theta)^{n-1}}{(n-1)!}f^{n}(\theta x),$$

where $\frac{x^n(1-\theta)^{n-1}}{(n-1)!}f^{n}(\theta x)$ is the remainder in Cauchy's form, $0<\theta<1$.

Hence expansion of the function $f(x)=\log(1+x)$, $(-1\le x\le 1)$in finite form with remainder in Cauchy's form is

$$\log(1+x)=x-\frac{x^2}{2}+.\frac{x^3}{3}.-.\frac{x^4}{4}.+.............+\frac{x^n}{(n-1)!}(1-\theta)^{n-1}(-1)^{n-1}\frac{(n-1)!}{(1+\theta x)^n},\ 0<\theta<1$$

Or

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots\dots\dots + (-1)^{n-1}(1-\theta)^{n-1}\frac{x^n}{(1+\theta x)^n},\quad 0<\theta<1 \text{..........(2).}$$

EX-(10) **Expand the function $f(x)=\log(1+x)$, $(-1\le x\le 1)$ in powers of x (i.e., in the neighbourhood of $x=0$) in infinite series and state the condition under which the expansion is valid.**

Solution : Given function is $f(x)=\log(1+x)$, $(-1\le x\le 1)$ $\quad\rightarrow f(0)=0$

$$f'(x)=\frac{1}{1+x}=(1+x)^{-1} \quad\rightarrow f'(0)=1$$

$$f''(x)=(-1)(1+x)^{-2} \quad\rightarrow f''(0)=-1$$

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$$f^n(x)=(-1)(-2)\dots\dots\{-(n-1)\}(1+x)^{-n}$$

$$=\frac{(-1)^{n-1}(n-1)!}{(1+x)^n} \quad\rightarrow =\frac{(-1)^{n-1}(n-1)!}{(1+\theta x)^n}$$

Now,

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\dots\dots\dots+\frac{x^{n-1}}{(n-1)!}f^{n-1}(0)+\frac{x^n}{n!}f^n(\theta x)$$, where $\frac{x^n}{n!}f^n(\theta x)$

is the remainder in Lagrange's form, $0<\theta<1$.

Hence expansion of the function $f(x)=\log(1+x)$, $(-1\le x\le 1)$in finite form with remainder in Lagrange's form is

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots\dots\dots + \frac{x^n}{n!}(-1)^{n-1}\frac{(n-1)!}{(1+\theta x)^n},\quad 0<\theta<1$$

Or

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots\dots\dots + \frac{x^n}{n}\frac{(-1)^{n-1}}{(1+\theta x)^n},\quad 0<\theta<1 \text{.............(1)}$$

Again,

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\dots\dots\dots+\frac{x^{n-1}}{(n-1)!}f^{n-1}(0)+\frac{x^n(1-\theta)^{n-1}}{(n-1)!}f^n(\theta x),$$

where $\frac{x^n(1-\theta)^{n-1}}{(n-1)!}f^n(\theta x)$ is the remainder in Cauchy's form, $0<\theta<1$.

Hence expansion of the function $f(x)=\log(1+x)$, $(-1\le x\le 1)$in finite form with remainder in Cauchy's form is

$$\log(1+x) = x - \frac{x^2}{2} + \cdot\frac{x^3}{3} \cdot - \cdot\frac{x^4}{4} \cdot + \dots\dots\dots + \frac{x^n}{(n-1)!}(1-\theta)^{n-1}(-1)^{n-1}\frac{(n-1)!}{(1+\theta x)^n}, \quad 0<\theta<1$$

Or

$$\log(1+x) = x - \frac{x^2}{2} + \cdot\frac{x^3}{3} \cdot - \cdot\frac{x^4}{4} \cdot + \dots\dots\dots + (-1)^{n-1}(1-\theta)^{n-1}\frac{x^n}{(1+\theta x)^n}, \quad 0<\theta<1 \text{..........(2).}$$

From (1), the remaider (R_n) term after nth term is $R_n = \frac{x^n}{n}\frac{(-1)^{n-1}}{(1+\theta x)^n}$. Evidently, $f(x)$ and all its derivatives exist and are continuous in $-1 \le x \le 1$.

CASE-1 : **When** $0 \le x \le 1$. Then $0 < \frac{x}{1+\theta x} < 1 \Rightarrow \left(\frac{x}{1+\theta x}\right)^n < 1$

$$\Rightarrow \left(\frac{x}{1+\theta x}\right)^n \to 0 \text{ as } n \to \infty.$$

Also, $\frac{1}{n} \to 0$ as $n \to \infty$. Hence $R_n \to 0$ as $n \to \infty$ for $0 \le x \le 1$. Thus the conditions of Maclaurin's Series are satisfied when $0 \le x \le 1$.

CASE-2: When $-1 < x < 0$. In this case, $\left(\frac{x}{1+\theta x}\right)$ may not be numerically less than 1 and hence $\left(\frac{x}{1+\theta x}\right)^n$ may not tend to 0 as $n \to \infty$. Thus we fail to draw any definite conclusion from Lagrange's form of remainder. In this case we have to try with Cauchy's form of remainder. The we have from (2),

$$R_n = (-1)^{n-1}(1-\theta)^{n-1}\frac{x^n}{(1+\theta x)^n}$$

$= (-1)^{n-1}\frac{x^n}{(1+\theta x)}\left(\frac{1-\theta}{1+\theta x}\right)^{n-1}$. As $(1-\theta) < (1+\theta x)$, we have $\left(\frac{1-\theta}{1+\theta x}\right)^{n-1} \to 0$ as $n \to \infty$. Also $x^n \to 0$ as $n \to \infty$ ($\because x < 1$) and $\frac{1}{1+\theta x} < \frac{1}{1-|x|}$. That is, $\frac{1}{1+\theta x}$ is bounded and moreover it is independent of x. Thus $R_n \to 0$ as $n \to \infty$ for $-1 < x < 0$. Thus the conditions of Maclaurin's Series are satisfied.

$$\boxed{\log(1+x) = x - \frac{x^2}{2} + \cdot\frac{x^3}{3} \cdot - \cdot\frac{x^4}{4} \cdot + \dots\dots\dots \text{ to } \infty \text{ is valid for } -1 < x \le 1}$$

EX-(11) Expand the function $f(x) = (1+x)^m$, in a finite series in powers of x (i.e., in the neighbourhood of $x = 0$) with the remainder in Lagrange's form and also in cauchy's form.

Solution : Given function is $f(x) = (1+x)^m$ $\qquad \to f(0) = 1$

$$f'(x) = m(1+x)^{m-1} \qquad \rightarrow f'(0) = m$$
$$f''(x) = m(m-1)(1+x)^{m-2} \qquad \rightarrow f''(0) = m(m-1)$$
$$\vdots$$
$$f^{n}(x) = m(m-1)(m-2)......\{m-(n-1)\}(1+x)^{m-n}$$
$$\rightarrow f^{n}(\theta x) = m(m-1)(m-2)......\{m-n+1)\}(1+\theta x)^{m-n}$$

Now,

$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + ... + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0) + \frac{x^n}{n!} f^{n}(\theta x)$, where $\frac{x^n}{n!} f^{n}(\theta x)$ is the remainder in Lagrange's form, $0 < \theta < 1$.

Hence expansion of the function $f(x) = (1+x)^m$, ($-1 \le x \le 1$)in finite form with remainder in Lagrange's form is

$$\boxed{(1+x)^m = 1 + mx + .m(m-1)\frac{x^2}{2!} + ... + m(m-1)(m-2)....(m-n+1)\frac{x^n}{n!}(1+\theta x)^{m-n}, \quad 0 < \theta < 1 \text{.............}(1)}$$

Again,

$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + ... + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0) + \frac{x^n (1-\theta)^{n-1}}{(n-1)!} f^{n}(\theta x)$,

where $\frac{x^n (1-\theta)^{n-1}}{(n-1)!} f^{n}(\theta x)$ is the remainder in Cauchy's form, $0 < \theta < 1$.

Hence expansion of the function $f(x) = (1+x)^m$, ($-1 \le x \le 1$)in finite form with remainder in Cauchy's form is

$$\boxed{(1+x)^m = 1 + mx + .m(m-1)\frac{x^2}{2!} + ... + m(m-1)(m-2)....(m-n+1)(1-\theta)^{n-1}\frac{x^n}{(n-1)!}(1+\theta x)^{m-n}, \quad 0 < \theta < 1 \text{........................}(2)}$$

EX-(12) Expand the function $f(x) = (1+x)^m$, in powers of x (i.e., in the neighbourhood of $x = 0$) in infinite series and state the condition under which the expansion is valid.

Solution : Given function is $f(x) = (1+x)^m \qquad \rightarrow f(0) = 1$

$$f'(x) = m(1+x)^{m-1} \qquad \rightarrow f'(0) = m$$
$$f''(x) = m(m-1)(1+x)^{m-2} \qquad \rightarrow f''(0) = m(m-1)$$
$$\vdots$$

$$f^n(x) = m(m-1)(m-2)......\{m-(n-1)\}(1+x)^{m-n}$$

$$\rightarrow f^n(\theta x) = m(m-1)(m-2)......\{m-n+1)\}(1+\theta x)^{m-n}$$

Now,

$f(x) = f(0) + xf'(0) + \dfrac{x^2}{2!}f''(0) + \dfrac{x^3}{3!}f'''(0) + ... + \dfrac{x^{n-1}}{(n-1)!}f^{n-1}(0) + \dfrac{x^n}{n!}f^n(\theta x)$, where $\dfrac{x^n}{n!}f^n(\theta x)$ is the remainder in Lagrange's form, $0 < \theta < 1$.

Hence expansion of the function $f(x) = (1+x)^m$, in finite form with remainder in Lagrange's form is

$$(1+x)^m = 1 + mx + .m(m-1)\frac{x^2}{2!} + ... + m(m-1)(m-2)....(m-n+1)\frac{x^n}{n!}(1+\theta x)^{m-n},$$
$0 < \theta < 1$**.............(1)**

Again**,**

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + ... + \frac{x^{n-1}}{(n-1)!}f^{n-1}(0) + \frac{x^n(1-\theta)^{n-1}}{(n-1)!}f^n(\theta x),$$

where $\dfrac{x^n(1-\theta)^{n-1}}{(n-1)!}f^n(\theta x)$ is the remainder in Cauchy's form, $0 < \theta < 1$.

Hence expansion of the function $f(x) = (1+x)^m$, in finite form with remainder in Cauchy's form is

$$(1+x)^m = 1 + mx + .m(m-1)\frac{x^2}{2!} + ... + m(m-1)(m-2)....(m-n+1)(1-\theta)^{n-1}\frac{x^n}{(n-1)!}(1+\theta x)^{m-n},$$
$0 < \theta < 1$**.........................(2)**

<u>CASE-1</u> : When m is positive(+) integer.

Then $f^n(x) = 0$ for $n > m$ and for every value of x. Hence the expansion stops after $(m+1)th$ term and the expansion is finite.

$(1+x)^m = 1 + mx + .m(m-1)\dfrac{x^2}{2!} + ... + x^m$ for all x which is a finite expansion.

<u>CASE-2</u> : When m is negative(-) integer or a fraction.

From (2),

the Cauchy's form of remainder is $R_n = m(m-1)(m-2)....(m-n+1)\dfrac{x^n}{(n-1)!}(1+\theta x)^{m-1}\left(\dfrac{1-\theta}{1+\theta x}\right)^{n-1}$

Let $-1 < x < 1$ That is, $|x| < 1$.

Since $0 < \theta < 1$, $0 < 1-\theta < 1+\theta x \Rightarrow 0 < \dfrac{1-\theta}{1+\theta x} < 1 \Rightarrow 0 < \left(\dfrac{1-\theta}{1+\theta x}\right)^{n-1} < 1$.

Therefore, $\left(\frac{1-\theta}{1+\theta x}\right)^{n-1} \to 0$ as $n \to \infty$. Also $(1+\theta x)^{m-1}$ is finite. Again, if $|x| < 1$, $x^n \to 0$ as $n \to \infty$. Therefore, when $|x| < 1$, $R_n \to 0$ as $n \to \infty$. Hence for $|x| < 1$, Maclaurin's infinite expansion for $f(x) = (1+x)^m$ is valid, m being a negative integer or fraction.

REMARK : If $|x| > 1$ then x^n does not tend to 0 as $n \to \infty$. Also $\frac{m(m-1)(m-2)....(m-n+1)}{(n-1)!}$ does not tend to 0 as $n \to \infty$ and hence R_n does not tend to 0.

INSTRUCTIONS FOR STUDENTS

All the definitions, theorems, results, examples highlighted by yellow colour are very very important.