16.1 Second-Order Linear Differential Equations with Constant Coefficients and Constant Term

(Non-homogeneous)

For pedagogic reasons, let us first discuss the method of solution for the *second-order* case ($n = 2$). The relevant differential equation is then the simple one

$$y''(t) + a_1 y'(t) + a_2 y = b \qquad (16.2)$$

where a_1, a_2, and b are all constants. If the term b is identically zero, we have a *homogeneous* equation, but if b is a nonzero constant, the equation is *nonhomogeneous*. Our discussion will proceed on the assumption that (16.2) is nonhomogeneous; in solving the nonhomogeneous version of (16.2), the solution of the homogeneous version will emerge automatically as a by-product.

In this connection, we recall a proposition introduced in Sec. 15.1 which is equally applicable here: If y_c is the *complementary function*, i.e., the general solution (containing arbitrary constants) of the reduced equation of (16.2) and if y_p is the *particular integral*, i.e., any particular solution (containing no arbitrary constants) of the complete equation (16.2), then $y(t) = y_c + y_p$ will be the general solution of the complete equation. As was explained previously, the y_p component provides us with the equilibrium value of the variable y in the intertemporal sense of the term, whereas the y_c component reveals, for each point of time, the deviation of the time path $y(t)$ from the equilibrium.

The Particular Integral

For the case of constant coefficients and constant term, the particular integral is relatively easy to find. Since the particular integral can be *any* solution of (16.2), i.e., any value of y that satisfies this nonhomogeneous equation, we should always try the simplest possible type: namely, y = a constant. If y = a constant, it follows that

$$y'(t) = y''(t) = 0$$

so that (16.2) in effect becomes $a_2 y = b$, with the solution $y = b/a_2$. Thus, the desired particular integral is

$$y_p = \frac{b}{a_2} \qquad (\text{case of } a_2 \neq 0) \qquad (16.3)$$

Since the process of finding the value of y_p involves the condition $y'(t) = 0$, the rationale for considering that value as an intertemporal equilibrium becomes self-evident.

Example 1 Find the particular integral of the equation

$$y''(t) + y'(t) - 2y = -10$$

The relevant coefficients here are $a_2 = -2$ and $b = -10$. Therefore, the particular integral is $y_p = -10/(-2) = 5$.

What if $a_2 = 0$—so that the expression b/a_2 is not defined? In such a situation, since the constant solution for y_p fails to work, we must try some *nonconstant* form of solution. Taking the simplest possibility, we may try $y = kt$. Since $a_2 = 0$, the differential equation is now

$$y''(t) + a_1 y'(t) = b$$

but if $y = kt$, which implies $y'(t) = k$ and $y''(t) = 0$, this equation reduces to $a_1 k = b$. This determines the value of k as b/a_1, thereby giving us the particular integral

$$y_p = \frac{b}{a_1}t \qquad \text{(case of } a_2 = 0;\ a_1 \neq 0) \tag{16.3'}$$

Inasmuch as y_p is in this case a nonconstant function of time, we shall regard it as a moving equilibrium.

xample 2 Find the y_p of the equation $y''(t) + y'(t) = -10$. Here, we have $a_2 = 0$, $a_1 = 1$, and $b = -10$. Thus, by (16.3′), we can write

$$y_p = -10t$$

If it happens that a_1 is also zero, then the solution form of $y = kt$ will also break down, because the expression bt/a_1 will now be undefined. We ought, then, to try a solution of the form $y = kt^2$. With $a_1 = a_2 = 0$, the differential equation now reduces to the extremely simple form

$$y''(t) = b$$

and if $y = kt^2$, which implies $y'(t) = 2kt$ and $y''(t) = 2k$, the differential equation can be written as $2k = b$. Thus, we find $k = b/2$, and the particular integral is

$$y_p = \frac{b}{2}t^2 \qquad \text{(case of } a_1 = a_2 = 0) \tag{16.3''}$$

The equilibrium represented by this particular integral is again a moving equilibrium.

ample 3 Find the y_p of the equation $y''(t) = -10$. Since the coefficients are $a_1 = a_2 = 0$ and $b = -10$, formula (16.3″) is applicable. The desired answer is $y_p = -5t^2$.

The Complementary Function

The complementary function of (16.2) is defined to be the general solution of its reduced (homogeneous) equation

$$y''(t) + a_1 y'(t) + a_2 y = 0 \tag{16.4}$$

This is why we stated that the solution of a homogeneous equation will always be a *by-product* in the process of solving a complete equation.

Even though we have never tackled such an equation before, our experience with the complementary function of the first-order differential equations can supply us with a useful hint. From the solutions (15.3), (15.3′), (15.5), and (15.5′), it is clear that exponential expressions of the form Ae^{rt} figure very prominently in the complementary functions of first-order differential equations with constant coefficients. Then why not try a solution of the form $y = Ae^{rt}$ in the second-order equation, too?

If we adopt the trial solution $y = Ae^{rt}$, we must also accept

$$y'(t) = rAe^{rt} \qquad \text{and} \qquad y''(t) = r^2Ae^{rt}$$

as the derivatives of y. On the basis of these expressions for y, $y'(t)$, and $y''(t)$, the reduced differential equation (16.4) can be transformed into

$$Ae^{rt}(r^2 + a_1 r + a_2) = 0 \tag{16.4'}$$

As long as we choose those values of A and r that satisfy (16.4′), the trial solution $y = Ae^{rt}$ should work. Since e^{rt} can never be zero, we must either let $A = 0$ or see to it that r satisfies the equation

$$r^2 + a_1 r + a_2 = 0 \tag{16.4''}$$

Since the value of the (arbitrary) constant A is to be definitized by use of the initial conditions of the problem, however, we cannot simply set $A = 0$ at will. Therefore, it is essential to look for values of r that satisfy (16.4″).

Equation (16.4″) is known as the *characteristic equation* (or *auxiliary equation*) of the homogeneous equation (16.4), or of the complete equation (16.2). Because it is a quadratic equation in r, it yields two roots (solutions), referred to in the present context as *characteristic roots*, as follows:†

$$r_1, r_2 = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2} \tag{16.5}$$

These two roots bear a simple but interesting relationship to each other, which can serve as a convenient means of checking our calculation: The *sum* of the two roots is always equal to $-a_1$, and their *product* is always equal to a_2. The proof of this statement is straightforward:

$$\begin{aligned} r_1 + r_2 &= \frac{-a_1 + \sqrt{a_1^2 - 4a_2}}{2} + \frac{-a_1 - \sqrt{a_1^2 - 4a_2}}{2} = \frac{-2a_1}{2} = -a_1 \\ r_1 r_2 &= \frac{(-a_1)^2 - (a_1^2 - 4a_2)}{4} = \frac{4a_2}{4} = a_2 \end{aligned} \tag{16.6}$$

The values of these two roots are the only values we may assign to r in the solution $y = Ae^{rt}$. But this means that, in effect, there are *two* solutions which will work, namely,

$$y_1 = A_1 e^{r_1 t} \quad \text{and} \quad y_2 = A_2 e^{r_2 t}$$

where A_1 and A_2 are two arbitrary constants, and r_1 and r_2 are the characteristic roots found from (16.5). Since we want only *one* general solution, however, there seems to be one too many. Two alternatives are now open to us: (1) pick either y_1 or y_2 at random, or (2) combine them in some fashion.

The first alternative, though simpler, is unacceptable. There is only one arbitrary constant in y_1 or y_2, but to qualify as a general solution of a *second-order* differential equation, the expression must contain *two* arbitrary constants. This requirement stems from the fact that, in proceeding from a function $y(t)$ to its second derivative $y''(t)$, we "lose" two constants during the two rounds of differentiation; therefore, to revert from a second-order differential equation to the primitive function $y(t)$, two constants should be reinstated. That leaves us only the alternative of combining y_1 and y_2, so as to include both constants

† Note that the quadratic equation (16.4″) is in the normalized form; the coefficient of the r^2 term is 1. In applying formula (16.5) to find the characteristic roots of a differential equation, we must first make sure that the characteristic equation is indeed in the normalized form.

A_1 and A_2. As it turns out, we can simply take their *sum*, $y_1 + y_2$, as the general solution of (16.4). Let us demonstrate that, if y_1 and y_2, respectively, satisfy (16.4), then the sum $(y_1 + y_2)$ will also do so. If y_1 and y_2 are indeed solutions of (16.4), then by substituting each of these into (16.4), we must find that the following two equations hold:

$$y_1''(t) + a_1 y_1'(t) + a_2 y_1 = 0$$
$$y_2''(t) + a_1 y_2'(t) + a_2 y_2 = 0$$

By adding these equations, however, we find that

$$\underbrace{[y_1''(t) + y_2''(t)]}_{=\frac{d^2}{dt^2}(y_1+y_2)} + a_1 \underbrace{[y_1'(t) + y_2'(t)]}_{=\frac{d}{dt}(y_1+y_2)} + a_2(y_1 + y_2) = 0$$

Thus, like y_1 or y_2, the sum $(y_1 + y_2)$ satisfies the equation (16.4) as well. Accordingly, the general solution of the homogeneous equation (16.4) or the complementary function of the complete equation (16.2) can, in general, be written as $y_c = y_1 + y_2$.

A more careful examination of the characteristic-root formula (16.5) indicates, however, that as far as the values of r_1 and r_2 are concerned, three possible cases can arise, some of which may necessitate a modification of our result $y_c = y_1 + y_2$.

Case 1 (distinct real roots) When $a_1^2 > 4a_2$, the square root in (16.5) is a real number, and the two roots r_1 and r_2 will take *distinct* real values, because the square root is added to $-a_1$ for r_1, but subtracted from $-a_1$ for r_2. In this case, we can indeed write

$$y_c = y_1 + y_2 = A_1 e^{r_1 t} + A_2 e^{r_2 t} \qquad (r_1 \neq r_2) \tag{16.7}$$

Because the two roots are distinct, the two exponential expressions must be linearly independent (neither is a multiple of the other); consequently, A_1 and A_2 will always remain as separate entities and provide us with two constants, as required.

nple 4 Solve the differential equation

$$y''(t) + y'(t) - 2y = -10$$

The particular integral of this equation has already been found to be $y_p = 5$, in Example 1. Let us find the complementary function. Since the coefficients of the equation are $a_1 = 1$ and $a_2 = -2$, the characteristic roots are, by (16.5),

$$r_1, r_2 = \frac{-1 \pm \sqrt{1+8}}{2} = \frac{-1 \pm 3}{2} = 1, -2$$

(*Check:* $r_1 + r_2 = -1 = -a_1$; $r_1 r_2 = -2 = a_2$.) Since the roots are distinct real numbers, the complementary function is $y_c = A_1 e^t + A_2 e^{-2t}$. Therefore, the general solution can be written as

$$y(t) = y_c + y_p = A_1 e^t + A_2 e^{-2t} + 5 \tag{16.8}$$

In order to definitize the constants A_1 and A_2, there is need now for *two* initial conditions. Let these conditions be $y(0) = 12$ and $y'(0) = -2$. That is, when $t = 0$, $y(t)$ and $y'(t)$ are, respectively, 12 and -2. Setting $t = 0$ in (16.8), we find that

$$y(0) = A_1 + A_2 + 5$$

Differentiating (16.8) with respect to t and then setting $t = 0$ in the derivative, we find that

$$y'(t) = A_1 e^t - 2A_2 e^{-2t} \qquad \text{and} \qquad y'(0) = A_1 - 2A_2$$

To satisfy the two initial conditions, therefore, we must set $y(0) = 12$ and $y'(0) = -2$, which results in the following pair of simultaneous equations:

$$A_1 + A_2 = 7$$
$$A_1 - 2A_2 = -2$$

with solutions $A_1 = 4$ and $A_2 = 3$. Thus the definite solution of the differential equation is

$$y(t) = 4e^t + 3e^{-2t} + 5 \qquad (16.8')$$

As before, we can check the validity of this solution by differentiation. The first and second derivatives of (16.8′) are

$$y'(t) = 4e^t - 6e^{-2t} \qquad \text{and} \qquad y''(t) = 4e^t + 12e^{-2t}$$

When these are substituted into the given differential equation along with (16.8′), the result is an identity $-10 = -10$. Thus the solution is correct. As you can easily verify, (16.8′) also satisfies both of the initial conditions.

Case 2 (repeated real roots) When the coefficients in the differential equation are such that $a_1^2 = 4a_2$, the square root in (16.5) will vanish, and the two characteristic roots take an identical value:

$$r(= r_1 = r_2) = -\frac{a_1}{2}$$

Such roots are known as *repeated roots*, or *multiple* (here, *double*) *roots*.

If we attempt to write the complementary function as $y_c = y_1 + y_2$, the sum will in this case collapse into a single expression

$$y_c = A_1 e^{rt} + A_2 e^{rt} = (A_1 + A_2)e^{rt} = A_3 e^{rt}$$

leaving us with only one constant. This is not sufficient to lead us from a second-order differential equation back to its primitive function. The only way out is to find another eligible component term for the sum—a term which satisfies (16.4) and yet which is linearly independent of the term $A_3 e^{rt}$, so as to preclude such "collapsing."

An expression that will satisfy these requirements is $A_4 t e^{rt}$. Since the variable t has entered into it multiplicatively, this component term is obviously linearly independent of the $A_3 e^{rt}$ term; thus it will enable us to introduce another constant, A_4. But does $A_4 t e^{rt}$ qualify as a solution of (16.4)? If we try $y = A_4 t e^{rt}$, then, by the product rule, we can find its first and second derivatives to be

$$y'(t) = (rt + 1)A_4 e^{rt} \qquad \text{and} \qquad y''(t) = (r^2 t + 2r)A_4 e^{rt}$$

Substituting these expressions of y, y', and y'' into the left side of (16.4), we get the expression

$$[(r^2 t + 2r) + a_1(rt + 1) + a_2 t]A_4 e^{rt}$$

Inasmuch as, in the present context, we have $a_1^2 = 4a_2$ and $r = -a_1/2$, this last expression vanishes identically and thus is always equal to the right side of (16.4); this shows that A_4te^{rt} does indeed qualify as a solution.

Hence, the complementary function of the double-root case can be written as

$$y_c = A_3e^{rt} + A_4te^{rt} \tag{16.9}$$

Example 5 Solve the differential equation

$$y''(t) + 6y'(t) + 9y = 27$$

Here, the coefficients are $a_1 = 6$ and $a_2 = 9$; since $a_1^2 = 4a_2$, the roots will be repeated. According to formula (16.5), we have $r = -a_1/2 = -3$. Thus, in line with the result in (16.9), the complementary function may be written as

$$y_c = A_3e^{-3t} + A_4te^{-3t}$$

The general solution of the given differential equation is now also readily obtainable. Trying a constant solution for the particular integral, we get $y_p = 3$. It follows that the general solution of the complete equation is

$$y(t) = y_c + y_p = A_3e^{-3t} + A_4te^{-3t} + 3$$

The two arbitrary constants can again be definitized with two initial conditions. Suppose that the initial conditions are $y(0) = 5$ and $y'(0) = -5$. By setting $t = 0$ in the preceding general solution, we should find $y(0) = 5$; that is,

$$y(0) = A_3 + 3 = 5$$

This yields $A_3 = 2$. Next, by differentiating the general solution and then setting $t = 0$ and also $A_3 = 2$, we must have $y'(0) = -5$. That is,

$$y'(t) = -3A_3e^{-3t} - 3A_4te^{-3t} + A_4e^{-3t}$$

and

$$y'(0) = -6 + A_4 = -5$$

This yields $A_4 = 1$. Thus we can finally write the definite solution of the given equation as

$$y(t) = 2e^{-3t} + te^{-3t} + 3$$

Case 3 (complex roots) There remains a third possibility regarding the relative magnitude of the coefficients a_1 and a_2, namely, $a_1^2 < 4a_2$. When this eventuality occurs, formula (16.5) will involve the square root of a *negative* number, which cannot be handled before we are properly introduced to the concepts of *imaginary* and *complex* numbers. For the time being, therefore, we shall be content with the mere cataloging of this case and shall leave the full discussion of it to Secs. 16.2 and 16.3.

The three cases cited can be illustrated by the three curves in Fig. 16.1, each of which represents a different version of the quadratic function $f(r) = r^2 + a_1r + a_2$. As we learned earlier, when such a function is set equal to zero, the result is a quadratic *equation* $f(r) = 0$, and to solve the latter equation is merely to "find the zeros of the quadratic *function*." Graphically, this means that the roots of the equation are to be found on the horizontal axis, where $f(r) = 0$.

The position of the lowest curve in Fig. 16.1, is such that the curve intersects the horizontal axis twice; thus we can find two distinct roots r_1 and r_2, both of which satisfy the

16.1

1. Find the particular integral of each equation:
   (*a*) $y''(t) - 2y'(t) + 5y = 2$
   (*b*) $y''(t) + y'(t) = 7$
   (*c*) $y''(t) + 3y = 9$
   (*d*) $y''(t) + 2y'(t) - y = -4$
   (*e*) $y''(t) = 12$
2. Find the complementary function of each equation:
   (*a*) $y''(t) + 3y'(t) - 4y = 12$
   (*b*) $y''(t) + 6y'(t) + 5y = 10$
   (*c*) $y''(t) - 2y'(t) + y = 3$
   (*d*) $y''(t) + 8y'(t) + 16y = 0$
3. Find the general solution of each differential equation in Prob. 2, and then definitiz the solution with the initial conditions $y(0) = 4$ and $y'(0) = 2$.
4. Are the intertemporal equilibriums found in Prob. 3 dynamically stable?
5. Verify that the definite solution in Example 5 indeed (*a*) satisfies the two initial cond tions and (*b*) has first and second derivatives that conform to the given differenti equation.
6. Show that, as $t \to \infty$, the limit of te^{rt} is zero if $r < 0$, but is infinite if $r \geq 0$.

nplex Numbers and Circular Functions

When the coefficients of a second-order linear differential equation, $y''(t) + a_1 y'(t$ $a_2 y = b$, are such that $a_1^2 < 4a_2$, the characteristic-root formula (16.5) would call for ing the square root of a *negative* number. Since the square of any positive or negative number is invariably positive, whereas the square of zero is zero, only a *nonnegative* number can ever yield a real-valued square root. Thus, if we confine our attention to eal number system, as we have so far, no characteristic roots are available for this ca Case 3). This fact motivates us to consider numbers outside of the real-number syste

maginary and Complex Numbers

Conceptually, it is possible to define a number $i \equiv \sqrt{-1}$, which when squared will equ -1. Because i is the square root of a negative number, it is obviously not real-valued; it i erefore referred to as an *imaginary number*. With it at our disposal, we may write a hos f other imaginary numbers, such as $\sqrt{-9} = \sqrt{9}\sqrt{-1} = 3i$ and $\sqrt{-2} = \sqrt{2}i$.

Extending its application a step further, we may construct yet another type of number— that contains a *real* part as well as an *imaginary* part, such as $(8 + i)$ and $(3 + 5i)$. these can be represented generally in the form $(h + vi)$. in case $v = 0$, the complex number will imaginary number. Thus the ... *complex numbers* (call ... of C

... whether the variables are separable and (2) ... linear or else can be linearized:

(a) $2t\,dy + 2y\,dt = 0$ (c) $\dfrac{dy}{dt} = -\dfrac{t}{y}$

(b) $\dfrac{y}{y+t}\,dy + \dfrac{2t}{y+t}\,dt = 0$ (d) $\dfrac{dy}{dt} = 3y^2t$

2. Solve (*a*) and (*b*) in Prob. 1 by separation of variables, taking y and t to be positive. Check your answers by differentiation.
3. Solve (*c*) in Prob. 1 as a separable-variable equation and, also, as a Bernoulli equation.
4. Solve (*d*) in Prob. 1 as a separable-variable equation and, also, as a Bernoulli equation.
5. Verify the correctness of the intermediate solution $z(t) = At^2 + 2t$ in Example 4 by showing that its derivative dz/dt is consistent with the linearized differential equation.

The Qualitative-Graphic Approach

The several cases of nonlinear differential equations previously discussed (exact differential equations, separable-variable equations, and Bernoulli equations) have all been solved *quantitatively*. That is, we have in every case sought and found a time path $y(t)$ which, for each value of t, tells the specific corresponding value of the variable y.

At times, we may not be able to find a quantitative solution from a given differential equation. Yet, in such cases, it may nonetheless be possible to ascertain the *qualitative* properties of the time path—primarily, whether $y(t)$ converges—by directly observing the differential equation itself or by analyzing its graph. Even when quantitative solutions are available, moreover, we may still employ the techniques of qualitative analysis if the qualitative aspect of the time path is our principal or exclusive concern.

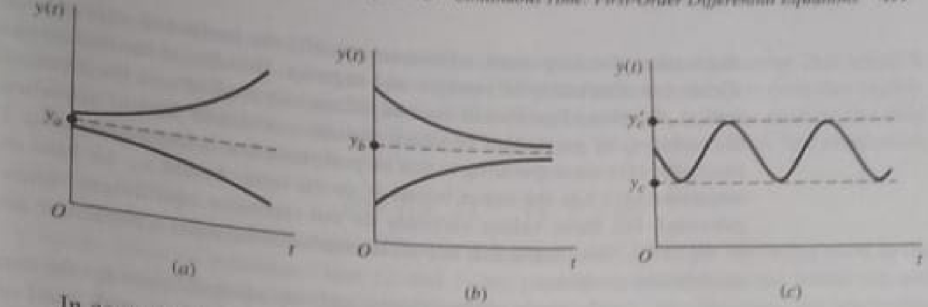
The Phase Diagram

Given a first-order differential equation in the general form

$$\frac{dy}{dt} = f(y)$$

either linear or nonlinear in the variable y, we can plot dy/dt against y as in Fig. 15.3. Such a geometric representation, feasible whenever dy/dt is a function of y alone, is called a *phase diagram*, and the graph representing the function f, a *phase line*. (A differential equation of this form—in which the time variable t does not appear as a separate argument of

FIGURE 15.4



In contrast, phase line B implies a stable equilibrium at y_b. If $y(0) = y_b$, equilibrium prevails at once. But the important feature of phase line B is that, even if $y(0) \neq y_b$, the movement along the phase line will guide y toward the level of y_b. The time path $y(t)$ corresponding to this type of phase line should therefore be of the form shown in Fig. 15.4*b*, which is reminiscent of the dynamic market model.

The preceding discussion suggests that, in general, it is the slope of the phase line at its intersection point which holds the key to the dynamic stability of equilibrium or the convergence of the time path. A (finite) *positive* slope, such as at point y_a, makes for dynamic *instability*; whereas a (finite) *negative* slope, such as at y_b, implies dynamic *stability*.

This generalization can help us to draw qualitative inferences about given differential equations without even plotting their phase lines. Take the linear differential equation in (15.4), for instance:

$$\frac{dy}{dt} + ay = b \qquad \text{or} \qquad \frac{dy}{dt} = -ay + b$$

Since the phase line will obviously have the (constant) slope $-a$, here assumed nonzero, we may immediately infer (without drawing the line) that

$$a \gtrless 0 \quad \Leftrightarrow \quad y(t) \begin{Bmatrix} \text{converges to} \\ \text{diverges from} \end{Bmatrix} \text{equilibrium}$$

As we may expect, this result coincides perfectly with what the quantitative solution of this equation tells us:

$$y(t) = \left[y(0) - \frac{b}{a}\right]e^{-at} + \frac{b}{a} \qquad \text{[from (15.5')]}$$

We have learned that, starting from a nonequilibrium position, the convergence of $y(t)$ hinges on the prospect that $e^{-at} \to 0$ as $t \to \infty$. This can happen if and only if $a > 0$; if $a < 0$, then $e^{-at} \to \infty$ as $t \to \infty$, and $y(t)$ cannot converge. Thus, our conclusion is one and the same, whether it is arrived at quantitatively or qualitatively.

It remains to discuss phase line C, which, being a closed loop sitting across the horizontal axis, does not qualify as a *function* but shows instead a *relation* between dy/dt and y.† The interesting new element that emerges in this case is the possibility of a periodically fluctuating time path. The way that phase line C is drawn, we shall find y fluctuating between the two values y_c and y_c' in a perpetual motion. In order to generate the periodic

† This can arise from a second-degree differential equation $(dy/dt)^2 = f(y)$.

the system. All these variables, we recall, are assigned the same base b in the trial solutions, so they must end up with the same b_i^t expressions in the complementary functions and share the same convergence properties. Thus a single application of the Schur theorem will enable us to determine the convergence or divergence of the time path of every variable in the system.

Simultaneous Differential Equations

The method of solution just described can also be applied to a first-order linear *differential-equation* system. About the only major modification needed is to change the trial solutions to

$$x(t) = me^{rt} \qquad \text{and} \qquad y(t) = ne^{rt} \tag{19.11}$$

which imply that

$$x'(t) = rme^{rt} \qquad \text{and} \qquad y'(t) = rne^{rt} \tag{19.12}$$

In line with our notational convention, the characteristic roots are now denoted by r instead of b.

Suppose that we are given the following equation system:

$$\begin{aligned} x'(t) + 2y'(t) + 2x(t) + 5y(t) &= 77 \\ y'(t) + \phantom{2y'(t)+{}} x(t) + 4y(t) &= 61 \end{aligned} \tag{19.13}$$

First, let us rewrite it in matrix notation as

$$Ju + Mv = g \tag{19.13'}$$

where the matrices are

$$J = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \qquad u = \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} \qquad M = \begin{bmatrix} 2 & 5 \\ 1 & 4 \end{bmatrix} \qquad v = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \qquad g = \begin{bmatrix} 77 \\ 61 \end{bmatrix}$$

Note, that, in view of the appearance of the $2y'(t)$ term in the first equation of (19.13), we have to use the matrix J in place of the identity matrix I, as in (19.4''). Of course, if J is nonsingular (so that J^{-1} exists), then we can in a sense *normalize* (19.13') by premultiplying every term therein by J^{-1}, to get

$$J^{-1}Ju + J^{-1}Mv = J^{-1}g \qquad \text{or} \qquad Iu + Kv = d$$
$$(K \equiv J^{-1}M;\ d \equiv J^{-1}g) \tag{19.13''}$$

This new format is an exact duplicate of (19.4''), although it must be remembered that the vectors u and v have altogether different meanings in the two different contexts. In the ensuing development, we shall adhere to the $Ju + Mv = g$ formulation given in (19.13').

To find the particular integrals, let us try constant solutions $x(t) = \bar{x}$ and $y(t) = \bar{y}$—which imply that $x'(t) = y'(t) = 0$. If these solutions hold, the vectors v and u will become $v = \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix}$ and $u = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, and (19.13') will reduce to $Mv = g$. Thus the solution for $\bar{x}$ and $\bar{y}$ can be written as

$$\begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} = \bar{v} = M^{-1}g \tag{19.14}$$

which you should compare with (19.5′). In numerical terms, our present problem yields th following particular integrals:

$$\begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 1 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 77 \\ 61 \end{bmatrix} = \begin{bmatrix} \frac{4}{3} & -\frac{5}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 77 \\ 61 \end{bmatrix} = \begin{bmatrix} 1 \\ 15 \end{bmatrix}$$

Next, let us look for the complementary functions. Using the trial solutions suggested i (19.11) and (19.12), the vectors u and v become

$$u = \begin{bmatrix} m \\ n \end{bmatrix} re^{rt} \qquad \text{and} \qquad v = \begin{bmatrix} m \\ n \end{bmatrix} e^{rt}$$

Substitution of these into the reduced equation

$$Ju + Mv = 0$$

yields the result

$$J \begin{bmatrix} m \\ n \end{bmatrix} re^{rt} + M \begin{bmatrix} m \\ n \end{bmatrix} e^{rt} = 0$$

or, after multiplying through by the scalar e^{-rt} and factoring,

$$(rJ + M) \begin{bmatrix} m \\ n \end{bmatrix} = 0 \tag{19.15}$$

You should compare this with (19.8′). Since our objective is to find *nontrivial* solutions o m and n (so that our trial solutions will also be nontrivial), it is necessary that

$$|rJ + M| = 0 \tag{19.16}$$

The analog of (19.9′), this last equation—the characteristic equation of the given equatio system—will yield the roots r_i that we need. Then, we can find the corresponding (non trivial) values of m_i and n_i.

In our present example, the characteristic equation is

$$|rJ + M| = \begin{vmatrix} r+2 & 2r+5 \\ 1 & r+4 \end{vmatrix} = r^2 + 4r + 3 = 0 \tag{19.16′}$$

with roots $r_1 = -1, r_2 = -3$. Substituting these into (19.15), we get

$$\begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} m_1 \\ n_1 \end{bmatrix} = 0 \qquad (\text{for } r_1 = -1)$$

$$\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} m_2 \\ n_2 \end{bmatrix} = 0 \qquad (\text{for } r_2 = -3)$$

It follows that $m_1 = -3n_1$ and $m_2 = -n_2$, which we may also express as

$$\begin{aligned} m_1 &= 3A_1 \\ n_1 &= -A_1 \end{aligned} \qquad \text{and} \qquad \begin{aligned} m_2 &= A_2 \\ n_2 &= -A_2 \end{aligned}$$

Now that r_i, m_i, and n_i have all been found, the complementary functions can be wri ten as the following linear combinations of exponential expressions:

$$\begin{bmatrix} x_c \\ y_c \end{bmatrix} = \begin{bmatrix} \Sigma m_i e^{r_i t} \\ \Sigma n_i e^{r_i t} \end{bmatrix} \qquad [\text{distinct real roots}]$$

And the general solution will emerge in the form

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} x_c \\ y_c \end{bmatrix} + \begin{bmatrix} \overline{x} \\ \overline{y} \end{bmatrix}$$

In our present example, the solution is

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 3A_1e^{-t} + A_2e^{-3t} + 1 \\ -A_1e^{-t} - A_2e^{-3t} + 15 \end{bmatrix}$$

Moreover, if we are given the initial conditions $x(0) = 6$ and $y(0) = 12$, the arbitrary constants can be found to be $A_1 = 1$ and $A_2 = 2$. These will serve to definitize the preceding solution.

Once more we may observe that, since the $e^{r_i t}$ expressions are shared by both time paths $x(t)$ and $y(t)$, the latter must either both converge or both diverge. The roots being -1 and -3 in the present case, both time paths converge to their respective equilibria, namely, $\overline{x} = 1$ and $\overline{y} = 15$.

Even though our example consists of a two-equation system only, the method certainly extends to the general n-equation system. When n is large, quantitative solutions may again be difficult, but once the characteristic equation is found, a qualitative analysis will always be possible by resorting to the Routh theorem.

Further Comments on the Characteristic Equation

The term "characteristic equation" has now been encountered in *three* separate contexts: In Sec. 11.3, we spoke of the characteristic equation of a matrix; in Secs. 16.1 and 18.1, the term was applied to a single linear differential equation and difference equation; now, in this section, we have just introduced the characteristic equation of a system of linear difference or differential equations. Is there a connection between the three?

There indeed is, and the connection is a close one. In the first place, given a single equation and an equivalent equation system—as exemplified by the equation (19.1) and the system (19.1′), or the equation (19.3) and the system (19.3′)—their characteristic equations must be identical. For illustration, consider the difference equation (19.1), $y_{t+2} + a_1y_{t+1} + a_2y_t = c$. We have earlier learned to write its characteristic equation by directly transplanting its constant coefficients into a quadratic equation:

$$b^2 + a_1b + a_2 = 0$$

What about the equivalent system (19.1′)? Taking that system to be in the form of I[illegible]$= d$, as in (19.4″), we have the matrix $K = \begin{bmatrix} a_1 & a_2 \\ -1 & 0 \end{bmatrix}$. So the characteristic