2. $f(x) = \text{sgn}\, x$ is ~~co~~ not continuous at $x=0$. 10:55

$$f(x) = \text{sgn}\, x = \begin{cases} 1 & , x>0 \\ 0 & , x=0 \\ -1 & , x<0 \end{cases}$$

$\lim_{x\to 0} \text{sgn}\, x \to 1$ when $x \to 0^+$

$\to -1$, $x \to 0^-$

$\Rightarrow$ limit does not exist at $x=0$.

So, $\text{sgn}\, x$ is not continuous at $x=0$.

3. Let $A = \mathbb{R}$, f is defined as

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

We claim f is not continuous at any point of $\mathbb{R}$.

$\therefore$ Test at c, which is rational, $\exists \{x_n\}$ of irrational in $\mathbb{R}$ whose ~~Every rational number can be approximated by the sequence of~~ limit is. That is $\lim_{n\to\infty} x_n = c$.

But $f(x_n) = 0 \;\forall n$

$\Rightarrow \lim_{n\to\infty} f(x_n) = 0 \neq f(c) = 1$.

Therefore f is not continuous at any rational points in $\mathbb{R}$.

Similarly we can show that it is also discontinuous at every irrational points in $\mathbb{R}$.

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4. $A = \{x \in \mathbb{R} \,/\, x > 0\}$. Define $h(x)$ on A as follows.

$$h(x) = \begin{cases} 0 & \text{if } x > 0\text{, irrational number} \\ 1/n & \text{if } x = \frac{m}{n} \text{ (rational number)}, \ \gcd(m,n) = 1. \end{cases}$$

We claim that $h(x)$ is ~~to~~ dis continuous at every rational number in A but it is continuous at every irrationals in A.

Sol^n:- Take $a > 0$, rational pt. in A, $\exists \{x_n\}$ of irrational points which converges to a.

But $h(x_n) = 0 \ \forall n \Rightarrow \lim_{n\to\infty} h(x_n) = 0$

$h(a) = \frac{1}{n} > 0$ if $a = \frac{m}{n}$.

$\lim_{n\to\infty} h(x_n) = 0 \neq h(a) = 1/n$.

$\therefore$ h is not continuous at any rational number in A.

Suppose $b \in A$ is an irrational number.

choose $\varepsilon > 0$. So $\exists\, n_0 \in \mathbb{N}$ s.t. $\frac{1}{n_0} < \varepsilon$.

[by Archimedian property of $\mathbb{R}$]

Choose an interval $(b-1, b+1)$.

$b-1 \quad b-\delta \quad b+\delta \quad b+1$

Then there are only finite number of rationals with denominator less than n_0.

Hence we can choose a nbd

$(b-\delta, b+\delta)$ which contains no rational numbers with denominator less than ~~no~~ n_0.

Then for any x, $|x-b|<\delta$, $x \in A$, we have $|h(x)-h(b)| = |h(x)| \leq 1/n \leq \frac{1}{n_0} < \epsilon$

$\Rightarrow$ h is continuous at all irrational points in A.

Remark:- 1. Suppose a function $f: A \to \mathbb{R}$ is not continuous at a pt. c because the value of $f(c)$ is not defined. But $\lim_{x \to c} f(x)$ exists. Then we can redefine the fun^n F on $A \cup \{c\}$ as follows—

$$F(x) = \begin{cases} \lim_{x \to c} f(x) = L & \text{for } x = c \\ f(x) & \text{for } x \in A. \end{cases}$$

Then F is continuous at c.

$$\lim_{x \to c} F(x) = F(c)$$

2. If a fun^n $f: A \to \mathbb{R}$ does not have a limit at c, then, there is no way that we can obtain a function $G: A \cup \{c\}$ that is continuous at $\{c\}$.

By defining

$$G(x) = \begin{cases} C & \text{for } x = c \\ g(x) & \text{`` } x \in A \end{cases}$$

$\therefore \lim_{x \to c} G(x) = C \Rightarrow$ ~~if~~ it exists.

~~It it is~~ the $\lim_{x \to c} g(x)$ exists and equal to C.

Which contradicts.

Ex:- $f(x) = \sin 1/x$ for $x \neq 0$

Then ~~fun~~ this function is not continuous at $x = 0$.

Since f is not defined at $x = 0$ because $\lim_{x \to 0} [\sin 1/x]$ does not exist.

$\left[x_n = \frac{1}{n\pi}, f(x_n) \to 0 \right.$
$\left. x_n = \frac{1}{(2n+1)\pi/2}, f(x_n) \to 1 \right]$

Ex:- ~~$f(x) = \sin x$~~

$f(x) = x\sin(1/x)$, $x \neq 0$

$f(x)$ is not not defined at $x=0$.

Then $f(x)$ cannot be continuous at $x = 0$.

If we redefine ~~the~~ the funⁿ $F : \mathbb{R} \to \mathbb{R}$

$$F(x) = \begin{cases} 0 & \text{for } x = 0 \\ x\sin(1/x) & , x \neq 0 \end{cases}$$

Clearly F is continuous at $x = 0$, because $\lim_{x \to 0} x\sin 1/x = 0 = F(0)$.

'D' is removable continuity.

Results on combinations of continuous functions :-

Theorem:- Let ~~$A \subseteq \mathbb{R}$, let f and g be continuous on $A \to \mathbb{R}$~~

Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ & $g : A \to \mathbb{R}$. and let $b \in \mathbb{R}$. Suppose $c \in A$ and that f and g are continuous at c.

Then —

(a) $f+g$, $f-g$, fg, & bf are all continuous at c.

(b) If $h : A \to \mathbb{R}$ is continuous at $c \in A$ and if $h(x) \neq 0$ $\forall x \in A$, then $\frac{f}{h}$ is continuous at c.

Theorem:- Let $A \subseteq \mathbb{R}$, let f and g be continuous on A to $\mathbb{R}$ and $b \in \mathbb{R}$. Then

(c) $f+g$, $f-g$, fg & bf are continuous on A

(d) ~~Als~~ $h : A \to \mathbb{R}$ is continuous on A if $h(x) \neq 0$ $\forall x \in A$, then $\frac{f}{h}$ is continuous on A.

Thm:- Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$ and $|f|$ be defined by $|f|(x) = |f(x)|$ for $x \in A$.

(a) If f is continuous at $c \in A$ then $|f|$ is continuous at c.

(b) If f is continuous on A, then $|f|$ is continuous on A.

If $f(x) \geq 0$ $\forall x \in A$. Then

(c) If f is continuous at a pt. $c \in A$, then $\sqrt{f}$ is also continuous at c.

(d) If f is continuous on A then $\sqrt{f}$ is also continuous on A.

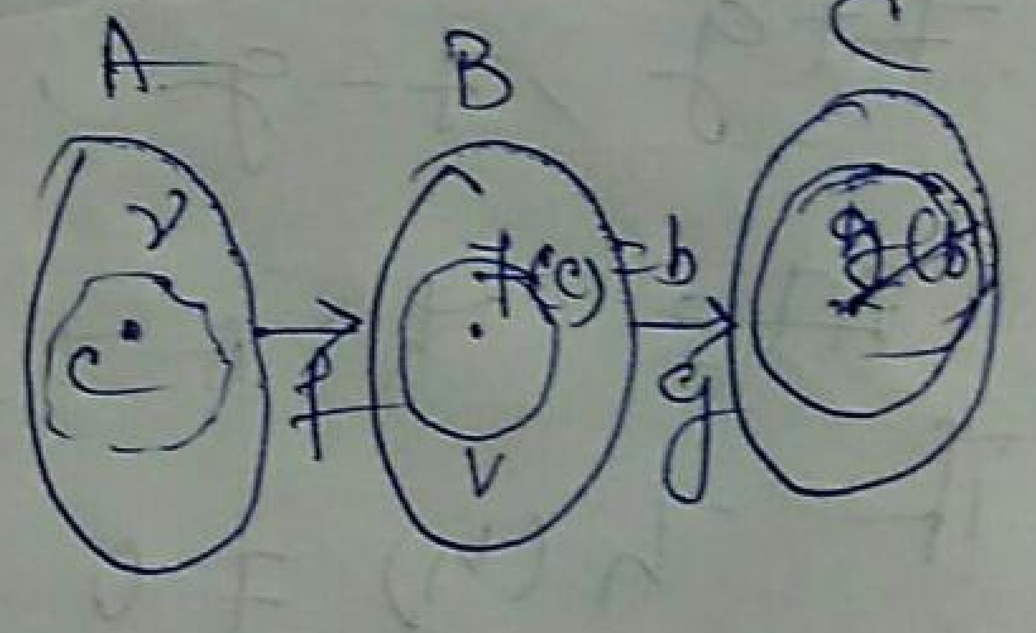
Where $(\sqrt{f})(x) = \sqrt{f(x)}$, $x \in A$, $f(x) \geq 0$.

Composition of continuous functions:-

Thm:- ~~Let~~ If $f : A \to \mathbb{R}$ is continuous at a pt. c and if $g : B \to \mathbb{R}$ is continuous at a pt. $b = f(c)$, then the composition $g \circ f$ is continuous at c.

Here we assume $f(A) \subseteq B$. ($g \circ f$)

$(g \circ f)(x) = g(f(x))$



Proof:- g is continuous at a pt. b.

Let W be an ϵ-nbd of $g(b)$.

Since g is continuous at b, $\exists$ a δ-nbd V of $b = f(c)$ s.t. if $y \in B \cap V$ then $g(y) \in W$.

Further, f is continuous at a pt. c (given).
So, $\exists$ a γ-nbd of & U of c such that
if $x \in A \cap U$, then $f(x) \in B \cap V$, so that

$(g \circ f)(x) = g[f(x)] \in W$. —(1)

Since W is an arbitrary ε-nbd of $g(b)$
$\Rightarrow$ $g \circ f$ is continuous at c s.t.
(1) holds $\Rightarrow$ $g \circ f$ is continuous at c.

Maximum and Minimum:-

Defn:- Let $A \subseteq \mathbb{R}$ and let $f: A \to \mathbb{R}$,
f has an absolute maximum on A if
there is a point x^* in A such that
$f(x^*) \geq f(x) \ \forall \ x \in A$, we
We say, f has an absolute minimum
on A if there is a point $x_* \in A$ s.t.
$f(x_*) \leq f(x) \ \forall \ x \in A$.
$\Rightarrow$ x_* is a absolute minimum pt.
& x^* is absolute maximum pt.