Theorem:- Let $A \subseteq \mathbb{R}$ and f and g be functions on ~~D~~ ~~D~~ A to $\mathbb{R}$. Let c be the cluster pt. of A. If f is bounded on $N'(c) \cap D$ for some deleted neighbourhood $N'(c)$ of c and $\lim_{x \to c} f(x) = 0$ then $\lim_{x \to c} (f \cdot g)(x) = 0$.

Proof:-

Let $\exists B > 0$ & $\delta_1 > 0$ s.t.

$$|f(x)| < B \quad \forall x \in N_c(\delta_1) \cap A,\ x \neq c$$

Let $\varepsilon > 0$.

Since $\lim_{x \to c} g(x) = 0$, there exists a $\delta_2 > 0$

s.t. $|g(x) - 0| < \varepsilon/B \quad \forall x \in N_c(\delta_2) \cap A,\ x \neq c$

$(0 < |x - c| < \delta_2,\ x \in A,\ x \neq c)$

let $\delta = \min\{\delta_1, \delta_2\}$

Then $|f(x)| < B$ & $|g(x)| < \varepsilon/B$

$\forall 0 < |x - c| < \delta,\ x \in A$.

Now,

$$|f \cdot g(x) - 0| = |f(x)|\,|g(x)| < \varepsilon \quad \forall x \in A \ \& \ 0 < |x - c| < \delta$$

$\Rightarrow \lim_{x \to c} (f g)(x)$.

(i) $\lim_{x \to 0} x \sin 1/x = 0$ (ii) $\lim_{x \to 0} x^2 \cos 1/x = 0$

Note:- If $\lim_{x \to c} f(x) = l\ (\neq 0)$ then we can't say $\lim_{x \to c} f(x)\, g(x) = l$.

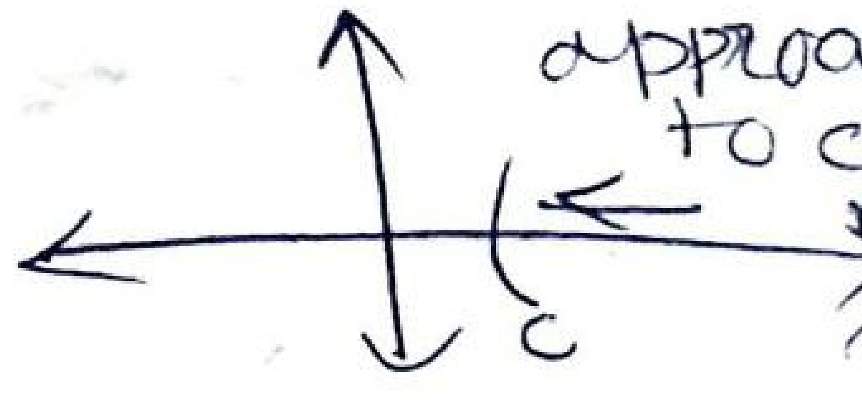
when $\lim_{x \to c} g(x)$ does not exists but $g(x)$ is bdd in some deleted nbd of c.

Def^n:- Let $A \subseteq \mathbb{R}$ and $f: A \to \mathbb{R}$ -

(i) If $c \in \mathbb{R}$ is a cluster point of the set $A \cap (c, \infty) = \{x \in A \mid c < x < \infty\}$

then we say that $l \in \mathbb{R}$ is a right hand limit of f at c and is defined by $\lim_{x \to c^+} f(x) = l$. If ~~for~~

$$\lim_{x \to c^+} f(x) = l$$



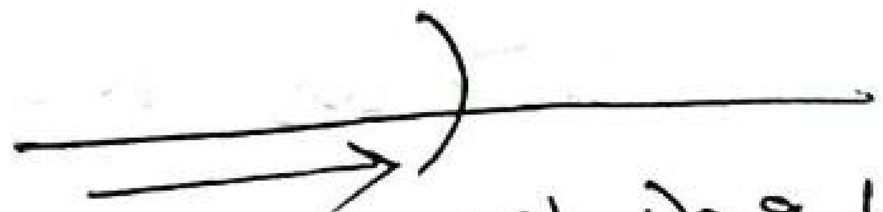
approaching to c from right side

If for any $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that ~~fo~~ $\forall x \in A$ and with $0 < x - c < \delta$ then $|f(x) - l| < \varepsilon$.

(ii) If $c \in \mathbb{R}$ is a cluster pt. of the set $A \cap (-\infty, c) = \{x \in A \mid x < c\}$, then we say $l \in \mathbb{R}$ is a left hand limit of f at c, denoted by $\lim_{x \to c^-} f = l$,

If given any $\varepsilon > 0$ there exists a $\delta > 0$ s.t. then for all $x \in A$ with $0 < c - x < \delta$

~~$|f(x) - l| < \varepsilon$~~. $|f(x) - l| < \varepsilon$



approaching to c from left hand side.

Ex:- $f(x) = \operatorname{sign} x$, $c = 0$

$$\operatorname{sign}(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$

$$\lim_{x \to 0^-} \operatorname{sgn}(x) = -1, \quad \lim_{x \to 0^+} \operatorname{sgn} x = 1$$

① $f(x) = \begin{cases} 1+x^2, & 0 \le x \le 1 \\ 2-x, & x>1 \end{cases}$

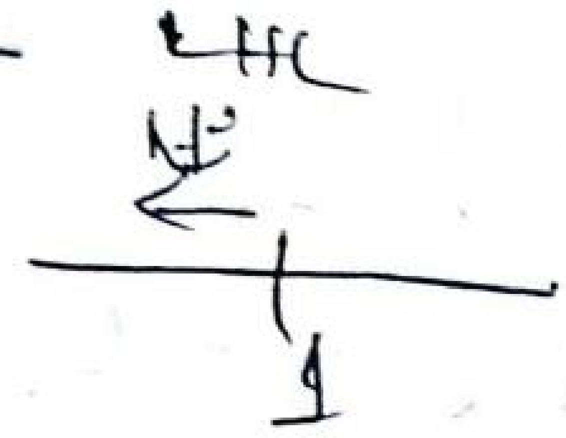
LHL

$\lim_{x\to 1^-} f(x)$

$= \lim_{h\to 0} f(1-h) = \lim_{h\to 0} 1+(1-h)^2$

$= \lim_{h\to 0} 1+1-2h+h^2$

$= 2-0 = 2$



RHL

$\lim_{x\to 1^+} f(x)$

$= \lim_{h\to 0^+} f(1+h)$

$= \lim_{h\to 0^+} 2-(1+h)$

$= 1$.

② $f(x) = \begin{cases} \frac{x-|x|}{x}; & x\neq 0 \\ 2, & x=0 \end{cases}$ at $x=0$

LHL $\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(-h)$

$= \lim_{h\to 0} \frac{-h-|-h|}{-h} = \lim_{h\to 0} \frac{-h-h}{-h}$

$= 2$

RHL

$\lim_{h\to 0^+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} \frac{h-h}{h}$

$= 0$

(3) $f(x) = \begin{cases} 5x-4, & 0 < x \leq 1 \\ 4x^3-3x, & 1 < x < 2 \end{cases}$ at $x=1$

LHL

$\lim_{x \to 1^-} f(x)$

$= \lim_{h \to 0} f(1-h) = \lim_{h \to 0} 5(1-h)-4)$

$= 1$

RHL

$\lim_{x \to 1^+} f(x)$

$= \lim_{h \to 0} f(1+h) = \lim_{h \to 0} \left[4(1+h)^3 - 3(1+h)\right]$

$= 4-3 = 1$

LHL = RHL

So, $\lim_{x \to 1} f(x) = 1$.

(4) $f(x) = \begin{cases} \dfrac{e^{1/x}-1}{e^{1/x}+1} \end{cases}$ at $x=0$

LHL

$\lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h)$

$= \cancel{e} \lim_{h \to 0} \dfrac{e^{-1/h}-1}{e^{-1/h}+1}$ $\quad h \to 0$, $e^{-1/h} \to 0$

$= \dfrac{-1}{1} = 0 - 1$

RHL

$\lim_{h \to 0} f(0+h) = \lim_{h \to 0} \dfrac{e^{1/h}-1}{e^{1/h}+1}$

$= \lim_{h \to 0} \dfrac{1-e^{-1/h}}{1+e^{-1/h}} = 1.$

② $f(x) = \frac{1}{x}, c = 0$

$\lim_{x \to 0^-} \frac{1}{x} \to -\infty \quad \lim_{x \to 0^+} \frac{1}{x} \to \infty$

Both limit does not exist.

③ $f(x) = e^{1/x}, c = 0$

$\lim_{x \to 0^+} e^{1/x} = \infty \quad \lim_{x \to 0^-} e^{1/x} = 0$

$\to$ right hand limit does not exist. $\to$ left hand limit exist

<u>For any $0<t$</u>, $0 < t < e^t \quad \forall t > 0$

Replace t by $1/x$ $\quad e^t = 1 + t + t^2/2! + \cdots$

Then $0 < 1/x < e^{1/x}$ for $x > 0$

Take a sequence $x_n = 1/n \to 0$

$0 < 1/x_n < e^{1/x_n}$

$\Rightarrow \cancel{0 < e^{x}} < e'$

$\Rightarrow 0 < n < e^n < \infty$

$\Rightarrow$ as $n \to \infty \quad e^n \to \infty$

$\Rightarrow \lim_{x \to 0} e^{1/x} = \infty$

<u>For $x<0$</u>, $t = -1/x$ if $x < 0$

$0 < -1/x < e^{-1/x}$ for $x < 0$

when $x \to 0^-$ $\lim_{x \to 0^-} e^{1/x} = \frac{1}{\infty} = 0.$

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Equivalent Def^n of left/right hand limit of func^n f at c

Theorem:- Let $A \subseteq \mathbb{R}$, let $f: A \to \mathbb{R}$ and $c \in \mathbb{R}$ be cluster point of $A \cap (c, \infty)$. Then the following statements are equivalent:

(i) $\lim_{x \to c^+} f = l$

(ii) For every sequence $\{x_n\}$ that converges to c such that $x_n \in A$ and $x_n > c$ $\forall n \in \mathbb{N}$, then the sequence $\{f(x_n)\}$ converges to l.

Theorem:- Let $A \subseteq \mathbb{R}$, let $f: A \to \mathbb{R}$ and let c be a cluster pt. of both the sets $A \cap (-\infty, c)$ & $A \cap (c, \infty)$. Then $\lim_{x \to c} f = l$ iff $\lim_{x \to c^+} f = l = \lim_{x \to c^-} f$.

Determine the give limit, if it exists.

① $f(x) = \begin{cases} 2x - 1, & x \le -2 \\ -x + 2, & x > -2 \end{cases}$ find $\lim_{x \to 2^-} f(x)$

② $f(x) = \begin{cases} -x^2 + 4x - 3, & x < 1 \\ x - 7, & x \geq 1 \end{cases}$, $\lim_{x \to 1^-} f(x)$

③ $f(x) = \begin{cases} x^2 - 2x + 1, & x < -1 \\ -\frac{x}{2} + \frac{7}{2}, & x \geq -1 \end{cases}$, find $\lim_{x \to 1} f(x)$

(4) $f(x) = \begin{cases} x+3, & x \in (-\infty, 0) \\ -x+2, & x \in [0, 2) \\ (x-2)^2, & x \in [2, \infty) \end{cases}$

(5) $\lim_{x \to 0} f(x)$ & $\lim_{x \to 2} f(x)$

(5) $f(x) = \begin{cases} (x+1)^2 - 1, & -2 \leq x < 0 \\ \frac{5}{4} \sin\left(\frac{\pi x}{2}\right), & 0 \leq x < 2 \\ (x-3)^2 - 1, & 2 \leq x \leq 4 \end{cases}$

$\lim_{x \to 2} f(x)$

(6) $f(x) = \begin{cases} 2x-1, & x \leq -1 \\ x^2+1, & -1 < x \leq 1 \\ -x+3, & x > 1 \end{cases}$

$\lim_{x \to 1^-} f(x)$ & $\lim_{x \to 1} f(x)$

(7) $f(x) = \begin{cases} -x^2 - 4x - 2, & x \leq 0 \\ (x-1)^2 - 1, & x > 0 \end{cases}$

(8) $f(x) = \begin{cases} x^2 + 6x + 8 & x \leq -1 \\ -x + 4 & x > -1 \end{cases}$