Chapter 4

Classification of Partial Differential Equations of Second Order and Canonical Forms

Relevant Information on

1. Second-Order Linear Partial Differential Equations.
2. Classification of Linear Partial Differential Equations of Second Order.
3. Transformation of Coordinates.
4. Canonical Forms.

4.1 Second-Order Linear Partial Differential Equations

The general second-order linear partial differential equation in two independent variables x and y is of the form

$$A(x,y)\frac{\partial^2 z}{\partial x^2} + B(x,y)\frac{\partial^2 z}{\partial x \partial y} + C(x,y)\frac{\partial^2 z}{\partial y^2} + D(x,y)\frac{\partial z}{\partial x} + E(x,y)\frac{\partial z}{\partial y} + F(x,y)z = G(x,y) \qquad (4.1.1)$$

where A, B, C, D, E, F and G are the functions of x and y, defined in some domain $\Omega \subseteq \mathbb{R}^2$.

Here, A, B, C cannot vanish simultaneously in Ω, because (4.1.1) represents a second-order linear partial differential equation. In addition, $z(x, y)$ and all the coefficients of (4.1.1) are twice continuously differentiable in Ω.

Equation (4.1.1) can be expressed in more compact form as

$$Az_{xx} + Bz_{xy} + Cz_{yy} + F(x, y, z, z_x, z_y) = 0 \qquad (4.1.2) \quad (1)$$

i.e., $Ar + Bs + Ct + F(x, y, z, p, q) = 0$.

4.2 Classification of Linear Partial Differential Equations of Second Order

The second-order linear (and quasilinear) partial differential equations can be classified (under some restrictions) into three categories namely, (i) Hyperbolic equations, (ii) Parabolic equations, (iii) Elliptic equations.

In analytical geometry, we have seen that the classification of the conic sections represented by the general equation of second degree

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \qquad (4.2.1)$$

depends on $D = B^2 - 4AC$ and $\Delta = ACF + \frac{1}{4}BDE - \frac{1}{4}AE^2 - \frac{1}{4}CD^2 - \frac{1}{4}FB^2$. The equation (4.2.1) represents a parabola or a hyperbola or an ellipse according as $D = 0$ or $D > 0$ or $D < 0$ when $\Delta \neq 0$. So in order to classify equation (4.1.2), we would like to use the terminology parabolic, hyperbolic and elliptic partial differential equation at a point $(x_0, y_0) \in \Omega$ depending on the sign of $D(x_0, y_0) = [B(x_0, y_0)]^2 - 4A(x_0, y_0)C(x_0, y_0)$, the discriminant of the equation (4.1.2) at the point (x_0, y_0).

(i) *Hyperbolic Equation*: The partial differential equation (4.1.2) is said to be hyperbolic at a point (x_0, y_0) if $D(x_0, y_0) > 0$. It will be hyperbolic everywhere in $\Omega \subseteq \mathbb{R}^2$ if $D(x, y) > 0$ for all $(x, y) \in \Omega$.

► **Example 4.2.1** *Show that the equation* $z_{xx} - 2\sin x z_{xy} - \cos^2 x z_{yy} + \sin x z_y = 0$ *is hyperbolic.*

Solution: Comparing the given equation with (4.1.2) we have, $A = 1$, $B = -2\sin x$, $C = -\cos^2 x$.

$\therefore$ the discriminant of the equation is given by

$D(x, y) = B^2 - 4AC = 4\sin^2 x + 4\cos^2 x = 4 > 0$ for all $(x, y) \in \mathbb{R}^2$.

Hence, the given equation is hyperbolic in $\mathbb{R}^2$.

(ii) *Parabolic Equation*: The partial differential equation (4.1.2) is said to be parabolic at a point (x_0, y_0) if $D(x_0, y_0) = 0$, i.e., $[B(x_0, y_0)]^2 - 4A(x_0, y_0)C(x_0, y_0) = 0$. If $D(x, y) = 0$ for all $(x, y) \in \Omega \subseteq \mathbb{R}^2$, the equation (4.1.2) is called parabolic in Ω.

► **Example 4.2.2** *Show that the partial differential equation*

$$x^2 z_{xx} - 2xyz_{xy} + y^2 z_{yy} + xz_x + yz_y = 0 \text{ is parabolic in } \mathbb{R}^2.$$

Solution: Comparing the given equation with (4.1.2) we have, $A = x^2$, $B = -2xy$ and $C = y^2$.

Thus, the discriminant of the equation is given by

$D(x, y) = B^2 - 4AC = 4x^2y^2 - 4x^2y^2 = 0$ for all $(x, y) \in \mathbb{R}^2$.

It confirms that the given equation is parabolic in $\mathbb{R}^2$.

(iii) *Elliptic Equation*: The partial differential equation (4.1.2) is said to be elliptic at a point (x_0, y_0) if $D(x_0, y_0) < 0$ i.e., $[B(x_0, y_0)]^2 - 4A(x_0, y_0)B(x_0, y_0) < 0$. It will be everywhere elliptic in $\Omega \subseteq \mathbb{R}^2$ if $D(x, y) < 0$ for all $(x, y) \in \Omega$.

► **Example 4.2.3** *Show that the following partial differential equation is elliptic in* $\mathbb{R}^2$:

$$(1 + x^2)z_{xx} + (1 + y^2)z_{yy} + 2yz_x + 2xz_y = 0.$$

Solution: Comparing the given equation with (4.1.2) we have, $A = (1 + x^2)$, $B = 0$, $C = 1 + y^2$.

$\therefore$ the discriminant of the equation is $D(x, y) = B^2 - 4AC$ i.e., $D(x, y) = -4(1 + x^2)(1 + y^2) < 0$ for all $(x, y) \in \mathbb{R}^2$. It follows that the given equation is elliptic in $\mathbb{R}^2$.

Note 4.2.1 *It is remarkable that a particular partial differential equation may be different in nature in a domain* $\Omega \subseteq \mathbb{R}^2$. *For example, let us consider the Euler-Tricomi equation* $z_{xx} + xz_{yy} = 0$.

Comparing it with (4.1.2) we have, $A = 1, B = 0$ and $C = x$.

$\therefore\ D(x, y) = B^2 - 4AC = -4x$.

Thus, the Euler-Tricomi equation will be parabolic if $x = 0$, hyperbolic if $x < 0$ and elliptic if $x > 0$.

Note 4.2.2 *From the physical point of view, parabolic, hyperbolic and elliptic partial differential equations represent the time-dependent diffusion processes, wave propagation and equilibrium processes respectively.*

4.3 Transformation of Coordinates

In order to explore the significance of $D(x, y)$ and to classify the partial differential equation (4.1.2), we would like to reduce it to a normal or canonical form that is the simplest form of the equation (4.1.2). Now, we introduce two new independent variables ξ and η by means of the transformation

$$\xi = \xi(x, y) \text{ and } \eta = \eta(x, y) \tag{4.3.1}$$

where ξ and η are both twice continuously differentiable and the Jacobian of transformation

$$J = \frac{\partial(\xi, \eta)}{\partial(x, y)} \neq 0 \text{ i.e., } \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} \neq 0$$

in the region Ω.

It ensures the existence of a one-to-one transformation between the new and old variables.

Now,

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial z}{\partial \eta}\frac{\partial \eta}{\partial x} = z_\xi \xi_x + z_\eta \eta_x \tag{4.3.2}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial z}{\partial \eta}\frac{\partial \eta}{\partial y} = z_\xi \xi_y + z_\eta \eta_y \tag{4.3.3}$$

$$\therefore\ \frac{\partial}{\partial x} \equiv \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta} \text{ and } \frac{\partial}{\partial y} \equiv \frac{\partial \xi}{\partial y}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial y}\frac{\partial}{\partial \eta}$$

$$\begin{aligned} r = \frac{\partial^2 z}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \left(\frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta}\right)\left(\frac{\partial z}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial z}{\partial \eta}\frac{\partial \eta}{\partial x}\right) \\ &= \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi}\left(\frac{\partial z}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial z}{\partial \eta}\frac{\partial \eta}{\partial x}\right) + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta}\left(\frac{\partial z}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial z}{\partial \eta}\frac{\partial \eta}{\partial x}\right) \\ &= \left(\frac{\partial \xi}{\partial x}\right)^2 \frac{\partial^2 z}{\partial \xi^2} + 2\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x}\frac{\partial^2 z}{\partial \xi \partial \eta} + \left(\frac{\partial \eta}{\partial x}\right)^2 \frac{\partial^2 z}{\partial \eta^2} + \frac{\partial z}{\partial \xi}\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial z}{\partial \eta}\frac{\partial^2 \eta}{\partial x^2} \end{aligned}$$

$$\therefore\ r = z_{xx} = \xi_x^2 z_{\xi\xi} + 2\xi_x \eta_x z_{\xi\eta} + \eta_x^2 z_{\eta\eta} + z_\xi \xi_{xx} + z_\eta \eta_{xx} \tag{4.3.4}$$

$$\begin{aligned} s = \frac{\partial^2 z}{\partial x \partial y} &= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \left(\frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta}\right)\left(\frac{\partial z}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial z}{\partial \eta}\frac{\partial \eta}{\partial y}\right) \\ &= \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi}\left(\frac{\partial z}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial z}{\partial \eta}\frac{\partial \eta}{\partial y}\right) + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta}\left(\frac{\partial z}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial z}{\partial \eta}\frac{\partial \eta}{\partial y}\right) \\ &= \frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y}\frac{\partial^2 z}{\partial \xi^2} + \left(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x}\right)\frac{\partial^2 z}{\partial \xi \partial \eta} + \frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y}\frac{\partial^2 z}{\partial \eta^2} + \frac{\partial z}{\partial \xi}\frac{\partial^2 \xi}{\partial x \partial y} \\ &\quad + \frac{\partial z}{\partial \eta}\frac{\partial^2 \eta}{\partial x \partial y} \end{aligned}$$

$$\therefore\ s = z_{xy} = \xi_x \xi_y z_{\xi\xi} + (\xi_x \eta_y + \xi_y \eta_x) z_{\xi\eta} + \eta_x \eta_y z_{\eta\eta} + z_\xi \xi_{xy} + z_\eta \eta_{xy} \tag{4.3.5}$$

and

$$\begin{aligned} t = \frac{\partial^2 z}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \left(\frac{\partial \xi}{\partial y}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial y}\frac{\partial}{\partial \eta}\right)\left(\frac{\partial z}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial z}{\partial \eta}\frac{\partial \eta}{\partial y}\right) \\ &= \frac{\partial \xi}{\partial y}\frac{\partial}{\partial \xi}\left(\frac{\partial z}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial z}{\partial \eta}\frac{\partial \eta}{\partial y}\right) + \frac{\partial \eta}{\partial y}\frac{\partial}{\partial \eta}\left(\frac{\partial z}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial z}{\partial \eta}\frac{\partial \eta}{\partial y}\right) \\ &= \left(\frac{\partial \xi}{\partial y}\right)^2 \frac{\partial^2 z}{\partial \xi^2} + 2\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y}\frac{\partial^2 z}{\partial \xi \partial \eta} + \left(\frac{\partial \eta}{\partial y}\right)^2 \frac{\partial^2 z}{\partial \eta^2} + \frac{\partial z}{\partial \xi}\frac{\partial^2 \xi}{\partial y^2} + \frac{\partial z}{\partial \eta}\frac{\partial^2 \eta}{\partial y^2} \end{aligned}$$

$$\therefore\ t = z_{yy} = \xi_y^2 z_{\xi\xi} + 2\xi_y \eta_y z_{\xi\eta} + z_{\eta\eta}\eta_y^2 + z_\xi \xi_{yy} + z_\eta \eta_{yy} \tag{4.3.6}$$

Using (4.3.2), (4.3.3), (4.3.4), (4.3.5) and (4.3.6) in (4.1.2) we obtain,

$$a z_{\xi\xi} + b z_{\xi\eta} + c z_{\eta\eta} = f(\xi, \eta, z, z_\xi, z_\eta) \tag{4.3.7}$$

where F becomes f and A, B, C become a, b, c respectively in new coordinates. Here, a, b, c and f are given by

$$\left.\begin{aligned} a &= A\xi_x^2 + B\xi_x \xi_y + C\xi_y^2 \\ b &= 2A\xi_x \eta_x + B(\xi_x \eta_y + \xi_y \eta_x) + 2C\xi_y \eta_y \\ c &= A\eta_x^2 + B\eta_x \eta_y + C\eta_y^2 \\ f &= -\left[z_\xi (A\xi_{xx} + B\xi_{xy} + C\xi_{yy}) \right. \\ &\qquad \left. + z_\eta (A\eta_{xx} + B\eta_{xy} + C\eta_{yy}) + F\right] \end{aligned}\right\} \tag{4.3.8}$$

The system (4.3.8) can be expressed in the following form:

$$\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} = \begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix} \begin{pmatrix} A & B/2 \\ B/2 & C \end{pmatrix} \begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix}^T$$

$$\therefore \begin{vmatrix} a & b/2 \\ b/2 & c \end{vmatrix} = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} \begin{vmatrix} A & B/2 \\ B/2 & C \end{vmatrix} \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix}$$

$$\text{or, } ac - \frac{b^2}{4} = \left(AC - \frac{B^2}{4}\right) J^2$$

$$\text{or, } b^2 - 4ac = (B^2 - 4AC)J^2$$

$$\text{i.e., } d = J^2 D \tag{4.3.9}$$

where $d = b^2 - 4ac$ is called the discriminant of the equation (4.3.7). The relation (4.3.9) ensures that the transformation $\xi = \xi(x, y)$, $\eta = \eta(x, y)$ is unable to change the type of the partial differential equation (4.1.2) if the Jacobian of transformation $J \neq 0$ in Ω.

Now, we would like to find ξ and η so that the equation (4.3.7) reduces to the simplest form. It will be quite easy if $B^2 - 4AC$, the discriminant of the quadratic equation

$$A\lambda^2 + B\lambda + C = 0 \tag{4.3.10}$$

is either everywhere zero or positive or negative in Ω. Here, (4.3.10) is known as characteristic polynomial equation in λ corresponding to the equation (4.1.1).

4.4 Canonical Forms

By a suitable transformation, the partial differential equation (4.1.2) may reduce to a simple form like (4.3.7) in which one or possible two of the coefficients of the leading second-order term of the equation (4.3.7) vanish in the region Ω. Such reduced equation is known as canonical form of the equation (4.1.2). Generally a canonical form specifies a unique representation of a partial differential equation. Now, we would like to introduce the canonical forms of the hyperbolic, parabolic and elliptic partial differential equations of second order.

4.4.1 Canonical Forms of Hyperbolic Equations

Let the equation (4.1.2) be hyperbolic. Then the discriminant $D = B^2 - 4AC > 0$ in the region Ω. Since the sign of the discriminant is invariant under the transformation $\xi = \xi(x, y), \eta = \eta(x, y)$, we must have $d = b^2 - 4ac > 0$.

Case 4.4.1 *Let $a = c = 0$. Then equation (4.3.7) becomes*

$$z_{\xi\eta} = \phi(\xi, \eta, z, z_\xi, z_\eta) \tag{4.4.1}$$

where $\phi = \dfrac{f}{b}$. The equation of the form (4.4.1) is known as the first canonical form of the hyperbolic equation.

Case 4.4.2 *Let $b = 0$ and $c = -a$. Then equation (4.3.7) becomes*

$$z_{\alpha\alpha} - z_{\beta\beta} = \phi(\alpha, \beta, z, z_\alpha, z_\beta) \tag{4.4.2}$$

where $\phi = \dfrac{f}{b}$. Here, the equation of the form (4.4.2) is called the second canonical form of the hyperbolic equation.

First Canonical Forms of the Hyperbolic Equations

In order to reduce the equation (4.1.2) to the first canonical form, we have to choose the new variables ξ, η in such a way that the coefficients a and c vanish simultaneously in the region Ω of the equation (4.3.7).

For $a = c = 0$, the system (4.3.8) gives

$$\left.\begin{aligned} A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 0 \\ A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 = 0 \end{aligned}\right\} \tag{4.4.3}$$

This system can be expressed also in the following form:

$$A\left(\frac{\xi_x}{\xi_y}\right)^2 + B\left(\frac{\xi_x}{\xi_y}\right) + C = 0 \tag{4.4.4}$$

$$A\left(\frac{\eta_x}{\eta_y}\right)^2 + B\left(\frac{\eta_x}{\eta_y}\right) + C = 0 \tag{4.4.5}$$

Clearly, the equation (4.4.4) is a quadratic in $\left(\dfrac{\xi_x}{\xi_y}\right)$. Let the roots of this equation be $\lambda_1(x, y)$ and $\lambda_2(x, y)$ which are given by

$$\lambda_1(x, y) = \frac{-B + \sqrt{B^2 - 4AC}}{2A}$$

$$\text{and } \lambda_2(x, y) = \frac{-B - \sqrt{B^2 - 4AC}}{2A}$$

provided $A(x, y) \neq 0$ in Ω. We see that $\lambda_1(x, y)$ and $\lambda_2(x, y)$ are also the roots of the equation (4.4.5). Since the total number of possible distinct roots of the equations (4.4.4) and (4.4.5) is exactly two, we may consider λ_1 and λ_2 as the roots of the equations (4.4.4) and (4.4.5) respectively, where

$$\left.\begin{aligned}\lambda_1(x,y) &= \frac{\xi_x}{\xi_y} = \frac{-B+\sqrt{B^2-4AC}}{2A}\\ \lambda_2(x,y) &= \frac{\eta_x}{\eta_y} = \frac{-B-\sqrt{B^2-4AC}}{2A}\end{aligned}\right\} \tag{4.4.6}$$

$\therefore$ λ_1 and λ_2 are the roots of the quadratic equation (4.3.10). By (4.4.6), we obtain, the following two equations:

$$\xi_x - \lambda_1(x,y)\xi_y = 0 \tag{4.4.7}$$
$$\eta_x - \lambda_2(x,y)\eta_y = 0 \tag{4.4.8}$$

The Lagrange's auxiliary equations of (4.4.7) are

$$\frac{dx}{1} = \frac{dy}{-\lambda_1} = \frac{d\xi}{0}$$

Considering 3rd ratio we have $d\xi = 0$, i.e., $\xi(x,y) =$ constant.

Again, considering 1st and 2nd ratios we have,

$$\frac{dy}{dx} = -\lambda_1(x,y) \text{ i.e., } \frac{dy}{dx} + \lambda_1(x,y) = 0 \tag{4.4.9}$$

By similar arguments, equation (4.4.8) gives

$$\eta(x,y) = \text{constant and } \frac{dy}{dx} + \lambda_2(x,y) = 0 \tag{4.4.10}$$

Here, equations (4.4.9) and (4.4.10) are called the *characteristic equations* of (4.1.2). Thus, we see that the number of characteristic equations of a hyperbolic equation is two.

Now, solving the ordinary differential equations (4.4.9) and (4.4.10) we obtain, $\xi(x,y) = c_1$ and $\eta(x,y) = c_2$ respectively, where c_1 and c_2 are two arbitrary constants.

The families of curves $\xi(x,y) = c_1$ and $\eta(x,y) = c_2$ are called the characteristic curves or characteristics of the hyperbolic partial differential equation. It follows that every second-order linear hyperbolic partial differential equation has two families of real characteristic curves on the xy-plane.

An Observation

Let A, B, C be constants. Then

$$\lambda_1 = \frac{-B+\sqrt{B^2-4AC}}{2A} \text{ and } \lambda_2 = \frac{-B-\sqrt{B^2-4AC}}{2A}$$

are both constants.

Thus, solving (4.4.9) and (4.4.10) we obtain, respectively

$$\left.\begin{aligned} y+\lambda_1 x = c_1\\ y+\lambda_2 x = c_2\end{aligned}\right\} \tag{4.4.11}$$

where c_1 and c_2 are two arbitrary constants.

Equations in (4.4.11) represent two distinct families of parallel straight lines. Again, we have seen that the families of curves $\xi(x,y) = c_1$ and $\eta(x,y) = c_2$ are the characteristics.

$\therefore$ we can define ξ and η by

$$\xi = y + \frac{-B+\sqrt{B^2-4AC}}{2A}x = y + \lambda_1 x$$
$$\eta = y + \frac{-B-\sqrt{B^2-4AC}}{2A}x = y + \lambda_2 x$$

$$\therefore\ \xi_x = \frac{-B+\sqrt{B^2-4AC}}{2A},\ \xi_y = 1,\ \eta_x = \frac{-B-\sqrt{B^2-4AC}}{2A},\ \eta_y = 1.$$

Thus, the first canonical form of the hyperbolic equation (4.3.7) is

$$z_{\xi\eta} = \phi(\xi, \eta, z, z_\xi, z_\eta)$$

where $\phi = f/b$ and

$$\begin{aligned} b &= 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y, \quad \text{[by (4.3.8)]}\\ &= 2A\frac{B^2-(B^2-4AC)}{4A^2} + B\left(-\frac{B}{A}\right) + 2C\\ &= 4C - \frac{B^2}{A} = -\frac{B^2-4AC}{A} = -\frac{D}{A}.\end{aligned}$$

Second Canonical Forms of the Hyperbolic Equations

In order to reduce the equation (4.1.2) to the second canonical form we need to choose the new variables ξ, η in such a way that $b = 0$ and $c = -a$ hold. In this case, the hyperbolic partial differential equation reduces to the form $z_{\alpha\alpha} - z_{\beta\beta} = \psi(\alpha, \beta, z, z_\alpha, z_\beta)$ by the linear transformation $\alpha = \xi + \eta$, $\beta = \xi - \eta$. The reduced form $z_{\alpha\alpha} - z_{\beta\beta} = \psi(\alpha, \beta, z, z_\alpha, z_\beta)$ is known as the second canonical form of (4.1.2).

Working Rule

(i) Confirm that $D = B^2 - 4AC > 0$ for hyperbolic equation

(ii) Find the roots λ_1 and λ_2 of the quadratic equation
$A\lambda^2 + B\lambda + C = 0$

(iii) Solve the characteristic equation

$$\frac{dy}{dx} + \lambda_1 = 0 \text{ and } \frac{dy}{dx} + \lambda_2 = 0$$

to obtain characteristics $\xi(x, y) = c_1$ and $\eta(x, y) = c_2$

(iv) Choose ξ, η as new variables

(v) Find b and f using (4.3.8), here $a = c = 0$ that may be verified by (4.3.8)

(vi) Find $\phi = \dfrac{f}{b}$

(vii) Write the canonical form $z_{\xi\eta} = \phi$.

► **Example 4.4.1** *Show that the equation* $z_{xx} - x^2 z_{yy} = 0$ $(x \neq 0)$ *is hyperbolic. Obtain its characteristics and hence find its canonical form.*

Solution: Comparing the given equation with (4.1.2) we have $A = 1$, $B = 0$, $C = -x^2$ and $F = 0$.

$\therefore$ discriminant of the equation

$$z_{xx} - x^2 z_{yy} = 0 \tag{4.4.12}$$

is given by $D(x, y) = B^2 - 4AC = 4x^2 > 0$ ($\because$ $x \neq 0$).

Hence the given equation is hyperbolic type.

The λ-quadratic equation corresponding to the equation (4.4.12) is $A\lambda^2 + B\lambda + C = 0$ i.e., $\lambda^2 - x^2 = 0$.

$\therefore$ $\lambda = \pm x$, which are real and distinct.

$\therefore$ the characteristic equations are

$$\frac{dy}{dx} + x = 0 \text{ and } \frac{dy}{dx} - x = 0.$$

Solving we have $y + \frac{1}{2}x^2 = c_1$ and $y - \frac{1}{2}x^2 = c_2$, where c_1 and c_2 are two arbitrary constants.

Hence the characteristics of the given equation are $y + \frac{1}{2}x^2 = c_1$ and $y - \frac{1}{2}x^2 = c_2$ which represent two different families of curves in xy-plane.

$\therefore$ by the transformation $\xi = y + \frac{1}{2}x^2$, $\eta = y - \frac{1}{2}x^2$ the given equation reduces to its canonical form.

Clearly, $\xi_x = x$, $\xi_y = 1$, $\xi_{xx} = 1$, $\xi_{xy} = 0$, $\xi_{yy} = 0$, $\eta_x = -x$, $\eta_y = 1$, $\eta_{xx} = -1$, $\eta_{xy} = 0$, $\eta_{yy} = 0$.

$\therefore$ $a = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 0$, $c = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 = 0$ and $b = 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y = -4x^2$.

Again,

$$f = -[z_\xi(A\xi_{xx} + B\xi_{xy} + C\xi_{yy}) + z_\eta(A\eta_{xx} + B\eta_{xy} + C\eta_{yy}) + F]$$
$$= -(z_\xi - z_\eta)$$

$$\therefore \ \phi = \frac{f}{b} = \frac{z_\xi - z_\eta}{4(\xi - \eta)}$$

Hence desired canonical form is

$$z_{\xi\eta} = \frac{z_\xi - z_\eta}{4(\xi - \eta)}.$$

► **Example 4.4.2** *Reduce the following equation to its canonical form and hence solve:*

$$r - 2\sin x s - \cos^2 x t - \cos x q = 0$$

Solution: Comparing the given equation with (4.1.2) we have, $A = 1$, $B = -2\sin x$, $C = -\cos^2 x$ and $F = -\cos xq$.

$\therefore$ the discriminant of the equation is given by $D = B^2 - 4AC$ i.e., $D = 4\sin^2 x + 4\cos^2 x = 4 > 0$ for all $(x, y) \in \mathbb{R}^2$.

It follows that the equation is hyperbolic.

The λ-quadratic equation is $A\lambda^2 + B\lambda + C = 0$ i.e., $\lambda^2 - 2\sin x \cdot \lambda - \cos^2 x = 0$ or, $\lambda^2 - 2\sin x \cdot \lambda + \sin^2 x = 1$.

$\therefore$ $\lambda = \sin x \pm 1$

$\therefore$ the characteristic equations are

$$\frac{dy}{dx} + \sin x + 1 = 0 \text{ and } \frac{dy}{dx} + \sin x - 1 = 0.$$

Solving these two equations we have, $y - \cos x + x = c_1$ and $y - \cos x - x = c_2$, where c_1, c_2 are two arbitrary constants. These two families of curves are known as the characteristics of the given partial differential equation.

Now, let us choose the transformation $\xi = y - \cos x + x$, $\eta = y - \cos x - x$. Then $\xi_x = \sin x + 1$, $\xi_y = 1$, $\xi_{xx} = \cos x$, $\xi_{xy} = 0$, $\xi_{yy} = 0$, $\eta_x = \sin x - 1$, $\eta_y = 1$, $\eta_{xx} = \cos x$, $\eta_{xy} = 0$, $\eta_{yy} = 0$.

It is quite easy to verify that $a = 0$ and $c = 0$.

Again, $b = -4$ and $F = -\cos xq = -\cos x(z_\xi \xi_y + z_\eta \eta_y) = -\cos x(z_\xi + z_\eta)$.

$\therefore\ f = -[z_\xi(A\xi_{xx} + B\xi_{xy} + C\xi_{yy}) + z_\eta(A\eta_{xx} + B\eta_{xy} + C\eta_{yy}) + F] = 0.$

Consequently, $\phi = f/b = 0$.

Hence desired canonical form of the given equation is $z_{\xi\eta} = 0$. Now, integrating both sides of the equation $z_{\xi\eta} = 0$ with respect to ξ we obtain, $z_\eta = f_1(\eta)$, f_1 being an arbitrary function.

Integrating again with respect to η we have,

$$z = \int f_1(\eta)d\eta + \psi_1(\xi)$$

ψ_1 being another arbitrary function.

$\therefore\ z = \psi_2(\eta) + \psi_1(\xi)$, where

$$\psi_2(\eta) = \int f_1(\eta)\,d\eta \text{ i.e., } z = \psi_1(y - \cos x + x) + \psi_2(y - \cos x - x),$$

which is the required general solution.

4.4.2 Canonical Forms of the Parabolic Equations

Let the equation (4.1.2) be parabolic. Then the discriminant is given by $D = B^2 - 4AC = 0$ in the region Ω.

$\therefore$ by (4.3.9) we have, $d = b^2 - 4ac = 0$. In this case, a (or c) $= 0$ and $b = 0$. Thus, we need to choose the new variables ξ and η in such a way that a (or c) and b vanish. Then the equation (4.3.7) reduces to the form

$$z_{\eta\eta} = \phi(\xi, \eta, z, z_\xi, z_\eta) \qquad (4.4.13)$$

where $\phi = f/c$. Here, the equation (4.4.13) is called the canonical form of the parabolic equation (4.1.2).

Now, for $a = 0$, (4.3.8) gives

$$A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 0$$

$$\text{or, } A\left(\frac{\xi_x}{\xi_y}\right)^2 + B\left(\frac{\xi_x}{\xi_y}\right) + C = 0 \qquad (4.4.14)$$

The roots of this quadratic equation in $\left(\frac{\xi_x}{\xi_y}\right)$ are real and equal.

$$\therefore\ \lambda = \frac{\xi_x}{\xi_y} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \text{ if } A(x,y) \neq 0 \text{ in } \Omega$$

$$= -\frac{B}{2A} \quad (\because\ B^2 - 4AC = 0)$$

which represents the root of the equation (4.4.14).

$$\therefore\ \xi_x - \lambda\xi_y = 0 \qquad (4.4.15)$$

The Lagrange's auxiliary equations of (4.4.15) are

$$\frac{dx}{1} = \frac{dy}{-\lambda} = \frac{d\xi}{0}$$

Considering 3rd ratio we have, $d\xi = 0$.

$\therefore\ \xi(x,y) = \text{constant}$.

Again, considering 1st and 2nd ratios,

$$\frac{dy}{dx} + \lambda(x,y) = 0 \qquad (4.4.16)$$

Here, the new variable ξ is determined by the equation (4.4.16) that is called the characteristic equation of (4.1.2). Equation (4.4.16) represents an ordinary differential equation corresponding to the family of curves (known as characteristics) in the xy-plane along which $\xi(x,y) = \text{constant}$.

In order to determine another variable η, we put $b = 0$ in (4.3.8) that gives

$$2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y = 0$$

$$\text{or, } 2A\left(\frac{\xi_x}{\xi_y}\right)\eta_x + B\left[\left(\frac{\xi_x}{\xi_y}\right)\eta_y + \eta_x\right] + 2C\eta_y = 0$$

$$\text{or, } 2A\left(-\frac{B}{2A}\right)\eta_x + B\left(-\frac{B}{2A}\cdot\eta_y + \eta_x\right) + 2C\eta_y = 0$$

$$\text{or, } (B^2 - 4AC)\eta_y = 0$$

Since $B^2 - 4AC = 0$, η_y will be an arbitrary function of x and y. It confirms that the transformation variable η can be chosen arbitrarily satisfying

$$J = \frac{\partial(\xi, \eta)}{\partial(x, y)} \neq 0 \text{ in } \Omega.$$

An Observation

Let A, B, C be constants. Then $\lambda = -\dfrac{B}{2A}$ is constant.

$\therefore$ solving (4.4.16) we obtain, $y + \lambda x = c_1$, c_1 being arbitrary constant i.e., $y - \dfrac{B}{2A}x = c_1$.

Since $\xi(x, y)$ = constant represents the characteristic curves, we set, $\xi = y - \dfrac{B}{2A}x$. Let us choose $\eta = x$.

Then

$$J = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = \begin{vmatrix} -\dfrac{B}{2A} & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0.$$

By (4.3.8) we have,

$$b = 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y$$
$$= 2A\left(-\frac{B}{2A}\right) + B = 0.$$

Thus, by $\xi = y - \dfrac{B}{2A}x$, $\eta = x$, equation (4.3.7) reduces to the canonical form $z_{\eta\eta} = \phi(\xi, \eta, z, z_\xi, z_\eta)$, where $\phi = f/c$.

The canonical form depends on the choice of the new variable η. If we choose $c = 0$ instead of $a = 0$, the form will be

$$z_{\xi\xi} = \phi(\xi, \eta, z, z_\xi, z_\eta), \text{ where } \phi = \frac{f}{a}.$$

Working Rule

(i) Confirm that $D = B^2 - 4AC = 0$ for parabolic equation.

(ii) Find the real and equal roots (λ_1) of the quadratic equation $A\lambda^2 + B\lambda + C = 0$.

(iii) Solve the characteristic equation $\dfrac{dy}{dx} + \lambda_1 = 0$ to obtain the characteristic $\xi(x, y) = c_1$.

(iv) Select ξ as a new variable.

(v) Choose another new variable η so that $J = \dfrac{\partial(\xi, \eta)}{\partial(x, y)} \neq 0$.

(vi) Find a, b, c and f using (4.3.8).

(vii) Find $\phi = f/c$ if $c \neq 0$, otherwise $\phi = f/a$.

(viii) Write the canonical equation $z_{\eta\eta} = \phi$ (or $z_{\xi\xi} = \phi$).

► **Example 4.4.3** *Reduce the equation $z_{xx} - 4z_{xy} + 4z_{yy} + z = 0$ to its canonical form.*

Solution: Comparing the given equation with (4.1.2) we have,

$$A = 1, B = -4, C = 4, F = z$$

$\therefore$ the discriminant of the equation

$$z_{xx} - 4z_{xy} + 4z_{yy} + z = 0 \tag{4.4.17}$$

is given by $D = B^2 - 4AC = 0$. It follows that the equation (4.4.17) is parabolic.

The λ-quadratic equation corresponding to the equation (4.4.17) is $A\lambda^2 + B\lambda + C = 0$ i.e., $\lambda^2 - 4\lambda + 4 = 0$.

$\therefore$ $\lambda = 2, 2$ which are real and equal.

$\therefore$ the characteristic equation is

$$\frac{dy}{dx} + 2 = 0 \text{ i.e., } dy + 2dx = 0$$

Integrating we have, $y + 2x = c_1$, c_1 being arbitrary constant.

Let us choose $\xi(x, y) = y + 2x$ and $\eta = y$.

Then $\xi_x = 2$, $\xi_y = 1$, $\xi_{xx} = 0$, $\xi_{xy} = 0$, $\xi_{yy} = 0$, $\eta_x = 0$, $\eta_y = 1$, $\eta_{xx} = 0$, $\eta_{xy} = 0$, $\eta_{yy} = 0$.

$$\therefore\ J = \frac{\partial(\xi, \eta)}{\partial(x, y)} = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = 2 \neq 0$$

Thus, the transformation $\xi = y + 2x, \eta = y$ is able to reduce the equation (4.4.17) to its canonical form.

We see that

$$a = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 0$$
$$b = 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y = 0$$
$$c = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 = 4$$
$$\text{and } f = -[z_\xi(A\xi_{xx} + B\xi_{xy} + C\xi_{yy}) + z_\eta(A\eta_{xx} + B\eta_{xy} + C\eta_{yy}) + F]$$
$$= z$$
$$\therefore \phi = \frac{f}{c} = \tfrac{1}{4}z.$$

Hence desired canonical form is $z_{\eta\eta} = \frac{1}{4}z$.

4.4.3 Canonical Forms of the Elliptic Equations

Let the partial differential equation (4.1.2) be elliptic. Then the discriminant of this equation is $D = B^2 - 4AC < 0$ in the region Ω.

$\therefore$ by (4.3.9), $d = b^2 - 4ac = J^2D < 0$. It is possible only when $b = 0$ and $a = c$. Thus, in order to reduce the equation (4.1.2) to its canonical form, we have to choose the new variables ξ, η in such a way that $b = 0$ and $a = c$ i.e., $a - c = 0$.

Now, from the system (4.3.8) we obtain the following equations:

$$\left.\begin{aligned} A(\xi_x^2 - \eta_x^2) + B(\xi_x\xi_y - \eta_x\eta_y) + C(\xi_y^2 - \eta_y^2) &= 0 \\ \text{and } 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y &= 0 \end{aligned}\right\}$$

Clearly the above system of equations is represented by the equation

$$A(\xi_x + i\eta_x)^2 + B(\xi_x + i\eta_x)(\xi_y + i\eta_y) + C(\xi_y + i\eta_y)^2 = 0,$$
where $i = \sqrt{-1}$

$$\text{or, } A\left(\frac{\xi_x + i\eta_x}{\xi_y + i\eta_y}\right)^2 + B\left(\frac{\xi_x + i\eta_x}{\xi_y + i\eta_y}\right) + C = 0 \qquad (4.4.18)$$

$$\text{or, } A\lambda^2 + B\lambda + C = 0, \text{ where } \lambda = \frac{\xi_x + i\eta_x}{\xi_y + i\eta_y}.$$

$$\therefore \lambda = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}, \text{ if } A(x,y) \neq 0 \text{ in } \Omega$$
$$= \frac{-B \pm i\sqrt{4AC - B^2}}{2A} \quad (\because B^2 - 4AC < 0).$$

Let $\beta(x,y)$ be the complex conjugate of $\alpha(x,y)$ so that

$$\alpha(x,y) = \xi(x,y) + i\eta(x,y)$$
$$\beta(x,y) = \xi(x,y) - i\eta(x,y).$$

Then the roots of the equation (4.4.18) are given by

$$\lambda_1 = \frac{\alpha_x}{\alpha_y} = \frac{-B + i\sqrt{4AC - B^2}}{2A}$$
$$\text{and } \lambda_2 = \frac{\beta_x}{\beta_y} = \frac{-B - i\sqrt{4AC - B^2}}{2A}.$$

$$\therefore \alpha_x - \lambda_1\alpha_y = 0 \qquad (4.4.19)$$
$$\text{and } \beta_x - \lambda_2\beta_y = 0 \qquad (4.4.20)$$

Now, the Lagrange's auxiliary equations of (4.4.19) are

$$\frac{dx}{1} = \frac{dy}{-\lambda_1} = \frac{d\alpha}{0}$$

Considering 3rd ratio we have, $d\alpha = 0$.

$\therefore$ $\alpha =$ constant.

Again, considering 1st and 2nd ratios we have,

$$\frac{dy}{dx} + \lambda_1 = 0 \qquad (4.4.21)$$

By similar arguments, $\beta=$ constant and

$$\frac{dy}{dx} + \lambda_2 = 0 \qquad (4.4.22)$$

Equations (4.4.21) and (4.4.22) are known as the characteristic equation of (4.1.2). Clearly the solutions of these two equations are necessarily complex-valued. Thus, in this case, there does not exist real characteristic of the equation (4.1.2).

Solving the ordinary differential equations (4.4.21) and (4.4.22) we obtain, $\alpha(x,y) = c_1$ and $\beta(x,y) = c_2$, where c_1 and c_2 are arbitrary complex constants. Here, $\mathrm{Re}\,\alpha$ and $\mathrm{Im}\,\beta$ will give the new variables ξ and η.

$$\therefore \xi = \frac{1}{2}(\alpha + \beta) \text{ and } \eta = \frac{1}{2i}(\alpha - \beta)$$

by which the equation (4.1.2) reduces to the canonical form $z_{\xi\xi} + z_{\eta\eta} = \phi(\xi, \eta, z, z_\xi, z_\eta)$, where $\phi = f/a$.

Working Rule

(i) Confirm that $D = B^2 - 4AC < 0$ for elliptic equation

(ii) Find the complex roots λ_1, λ_2 of the quadratic equation $A\lambda^2 + B\lambda + C = 0$

(iii) Solve the equations $\frac{dy}{dx} + \lambda_1 = 0$ and $\frac{dy}{dx} + \lambda_2 = 0$ to find the solutions $\alpha(x, y) = c_1$ and $\beta(x, y) = c_2$ respectively

(iv) Choose the new variables ξ and η such that $\xi = \frac{1}{2}(\alpha + \beta)$ and $\eta = \frac{1}{2i}(\alpha - \beta)$

(v) Find $a(= c)$ and f by (4.3.8), where $b = 0$ may be verified by (4.3.8)

(vi) Find $\phi = f/a$ and write the canonical form $z_{\xi\xi} + z_{\eta\eta} = \phi$.

► **Example 4.4.4** *Reduce the equation $z_{xx} + x^2 z_{yy} = 0$ to its canonical equation for $x \neq 0$.*

Solution: Comparing the given equation with (4.1.2) we have, $A = 1$, $B = 0$, $C = x^2$.

$\therefore$ the discriminant, $D = B^2 - 4AC = -4x^2 < 0$ for $x \neq 0$.

It follows that the given equation is elliptic.

Here, the λ-quadratic equation is $A\lambda^2 + B\lambda + C = 0$ i.e., $\lambda^2 + x^2 = 0$.

$\therefore \ \lambda = \pm ix$.

Thus, the characteristic equations are $\frac{dy}{dx} \pm ix = 0$.

Integrating we obtain, $y + \frac{1}{2}ix^2 = c_1$ and $y - \frac{1}{2}ix^2 = c_2$, where c_1 and c_2 are arbitrary constants.

Let $\alpha = y + \frac{1}{2}ix^2$ and $\beta = y - \frac{1}{2}ix^2$.

Then we choose $\xi = \frac{1}{2}(\alpha + \beta) = y$ and $\eta = \frac{1}{2i}(\alpha - \beta) = \frac{x^2}{2}$.

$\therefore \ \xi_x = 0$, $\xi_y = 1$, $\xi_{xx} = \xi_{xy} = \xi_{yy} = 0$, $\eta_x = x$, $\eta_y = 0$, $\eta_{xx} = 1$, $\eta_{xy} = \eta_{yy} = 0$.

Now,

$$
\begin{aligned}
a &= A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = x^2\\
b &= 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y = 0\\
c &= A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 = x^2
\end{aligned}
$$

and $f = -[z_\xi(A\xi_{xx} + B\xi_{xy} + C\xi_{yy}) + z_\eta(A\eta_{xx} + B\eta_{xy} + C\eta_{yy}) + F]$

$= -z_\eta$

$$\therefore \ \phi = \frac{f}{a} = -\frac{z_\eta}{x^2} = -\frac{1}{2\eta}z_\eta.$$

Hence required canonical form is

$$z_{\xi\xi} + z_{\eta\eta} = -\frac{1}{2\eta}z_\eta.$$

4.5 Further Solved Problems

► **Example 4.5.1** *Classify the following partial differential equations:*

(i) $z_{xx} - 2\cot x\, z_{xy} - z_{yy} = 0, x \neq n\pi, n \in \mathbb{Z}$

(ii) $(a - b)(z_{xx} - z_{yy}) + 4hz_{xy} = 0$, a, b, h are constants

(iii) $xz_{xx} - 4z_{xy} + z_{yy} + z_y = 0$.

Solution:

(i) Comparing the given equation with (4.1.2) we have, $A = 1$, $B = -2\cot x$, $C = -1$.

$\therefore \ B^2 - 4AC = 4\cot^2 x + 4 = 4(\cot^2 x + 1) = 4\operatorname{cosec}^2 x > 0$.

Hence the given equation is hyperbolic.

(ii) Here, $A = a - b$, $B = 4h$, $C = -(a - b)$.

$\therefore \ B^2 - 4AC = 16h^2 + 4(a - b)^2 > 0$, which shows that the given equation is hyperbolic.

(iii) Comparing the given equation with (4.1.2) we have, $A = x$, $B = -4$, $C = 1$.

$\therefore \ B^2 - 4AC = 16 - 4x = 4(4 - x)$.

$$\therefore \ B^2 - 4AC \begin{cases} > 0, & \text{if } x < 4\\ < 0, & \text{if } x > 4\\ = 0, & \text{if } x = 4 \end{cases}$$

Hence the given equation is hyperbolic in $\Omega_1 = \{(x,y) \in \mathbb{R}^2 : x < 4\}$, parabolic in $\Omega_2 = \{(x,y) \in \mathbb{R}^2 : x = 4\}$ and elliptic in $\Omega_3 = \{(x,y) \in \mathbb{R}^2 : x > 4\}$.

► **Example 4.5.2** *Find the nature of the partial differential equation $z_{xx} + (1-k^2)z_{yy} = 0$. Reduce this equation to its canonical form and hence find the general solution of the equation for $k > 1$.*

Solution: Comparing the given equation with (4.1.2) we have, $A = 1$, $B = 0$, $C = 1 - k^2$ and $F = 0$.

$\therefore$ the discriminant of the equation is given by

$$D = B^2 - 4AC = -4(1-k^2).$$

Thus, the given equation will be hyperbolic if $D > 0$ i.e., $|k| > 1$, parabolic if $k = \pm 1$ and elliptic if $|k| < 1$.

The λ-quadratic equation is $A\lambda^2 + B\lambda + C = 0$ i.e., $\lambda^2 + (1-k^2) = 0$ or, $\lambda^2 = k^2 - 1$.

$\therefore$ $\lambda = \pm\sqrt{k^2-1}$, which is real and distinct for $|k| > 1$.

The characteristic equations are

$$\frac{dy}{dx} + \sqrt{k^2-1} = 0 \text{ and } \frac{dy}{dx} - \sqrt{k^2-1} = 0.$$

Solving we obtain, $y + \sqrt{k^2-1}x = c_1$ and $y - \sqrt{k^2-1}x = c_2$, where c_1 and c_2 are two arbitrary constants.

Let us choose new variables ξ and η such that $\xi = y + \sqrt{k^2-1} \cdot x$ and $\eta = y - \sqrt{k^2-1} \cdot x$.

Then $\xi_x = \sqrt{k^2-1}$, $\xi_y = 1$, $\xi_{xx} = \xi_{xy} = \xi_{yy} = 0$, $\eta_x = -\sqrt{k^2-1}$, $\eta_y = 1$, $\eta_{xx} = \eta_{xy} = \eta_{yy} = 0$.

$$\therefore\ a = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 0$$

$$c = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 = 0$$

$$\text{and } b = -\frac{D}{A} = -4(k^2-1).$$

$\therefore$ $f = -[z_\xi(A\xi_{xx} + B\xi_{xy} + C\xi_{yy}) + z_\eta(A\eta_{xx} + B\eta_{xy} + C\eta_{yy}) + F] = 0.$

$\therefore$ $\phi = f/b = 0.$

Hence the canonical form of the given equation is $z_{\xi\eta} = 0$.

Integrating with respect to ξ we obtain, $z_\eta = g(\eta)$, g being arbitrary function.

Integrating again with respect to η we obtain,

$$z = \int g(\eta)d\eta + \psi_2(\xi), \psi_2, \text{ being arbitrary function}$$

$$= \psi_1(\eta) + \psi_2(\xi), \text{ where } \psi_1(\eta) = \int g(\eta)d\eta$$

$$= \psi_1\left(y - \sqrt{k^2-1} \cdot x\right) + \psi_2\left(y + \sqrt{k^2-1} \cdot x\right).$$

Hence required general solution is

$$z = \psi_1\left(y - \sqrt{k^2-1} \cdot x\right) + \psi_2\left(y + \sqrt{k^2-1} \cdot x\right)$$

where ψ_1 and ψ_2 are two arbitrary functions.

► **Example 4.5.3** *Reduce the following equation to its canonical equation:*

$$2z_{xx} - 4z_{xy} + 2z_{yy} + 3z = 0.$$

Solution: Comparing the given equation with (4.1.2) we have, $A = 2, B = -4, C = 2, F = 3z$.

$\therefore$ the discriminant, $D = B^2 - 4AC = 16 - 4 \cdot 2 \cdot 2 = 0$.

It confirms that the equation is parabolic in $\mathbb{R}^2$.

The λ-quadratic equation is $A\lambda^2 + B\lambda + C = 0$ i.e., $2\lambda^2 - 4\lambda + 2 = 0$.

$\therefore$ $\lambda = 1, 1$, which are real and equal.

Thus, the characteristic equation is

$$\frac{dy}{dx} + 1 = 0 \text{ or, } dy + dx = 0$$

Integrating we obtain, $y + x = c_1$, c_1 being arbitrary constant.

Let us choose new variables ξ, η such that $\xi = y + x$, $\eta = y - x$.

Then $\xi_x = 1$, $\xi_y = 1$, $\xi_{xx} = \xi_{xy} = \xi_{yy} = 0$, $\eta_x = -1$, $\eta_y = 1$, $\eta_{xx} = \eta_{xy} = \eta_{yy} = 0$.

$$\therefore\ J = \frac{\partial(\xi,\eta)}{\partial(x,y)} = \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 2 \neq 0$$

Consequently,

$$b = 0$$
$$a = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 0$$
$$c = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 = 8$$
$$\text{and } f = -[z_\xi \cdot 0 + z_\eta \cdot 0 + 3z] = -3z$$

$$\therefore \ \phi = \frac{f}{c} = \frac{-3z}{8}.$$

Hence required canonical form is $z_{\eta\eta} = -\frac{3}{8}z$.

Alternative

Here, $A = 2, B = -4, C = 2$.

$\therefore$ the discriminant, $D = B^2 - 4AC = 0$, which shows that the given equation is parabolic in $\mathbb{R}^2$.

The λ-quadratic equation is $A\lambda^2+B\lambda+C = 0$ i.e., $2\lambda^2-4\lambda+2 = 0$.

$\therefore$ $\lambda = 1, 1$, which are real and distinct.

The characteristic equation is $\frac{dy}{dx} + 1 = 0$.

The solution of this ordinary equation is $y + x = c_1, c_1$ being an arbitrary constant.

Let us choose the transformation $\xi = y + x, \eta = y - x$.

Then $\xi_x = 1, \xi_y = 1, \xi_{xx} = \xi_{xy} = \xi_{yy} = 0, \eta_x = -1, \eta_y = 1, \eta_{xx} = \eta_{xy} = \eta_{yy} = 0$.

$$\therefore \ J = \frac{\partial(\xi,\eta)}{\partial(x,y)} = 2 \neq 0$$

$$\therefore \ z_x = z_\xi\xi_x + z_\eta\eta_x = z_\xi - z_\eta, \text{ by (4.3.2)}$$
$$z_y = z_\xi\xi_y + z_\eta\eta_y = z_\xi + z_\eta, \text{ by (4.3.3)}$$

$$z_{xx} = \xi_x^2 z_{\xi\xi} + 2\xi_x\eta_x z_{\xi\eta} + \eta_x^2 z_{\eta\eta} + z_\xi\xi_{xx} + z_\eta\eta_{xx}, \text{ by (4.3.4)}$$
$$= z_{\xi\xi} - 2z_{\xi\eta} + z_{\eta\eta}$$
$$z_{xy} = \xi_x\xi_y z_{\xi\xi} + (\xi_x\eta_y + \xi_y\eta_x)z_{\xi\eta} + \eta_x\eta_y z_{\eta\eta} + z_\xi\xi_{xy} + z_\eta\eta_{xy}, \text{ by (4.3.5)}$$
$$= z_{\xi\xi} + (1-1)z_{\xi\eta} - z_{\eta\eta}$$
$$= z_{\xi\xi} - z_{\eta\eta}$$

$$\text{and } z_{yy} = \xi_y^2 z_{\xi\xi} + 2\xi_y\eta_y z_{\xi\eta} + z_{\eta\eta}\eta_y^2 + z_\xi\xi_{yy} + z_\eta\eta_{yy}, \text{ by (4.3.6)}$$
$$= z_{\xi\xi} + 2z_{\xi\eta} + z_{\eta\eta}.$$

Using above results in the given equation we obtain,

$$2(z_{\xi\xi} - 2z_{\xi\eta} + z_{\eta\eta}) - 4(z_{\xi\xi} - z_{\eta\eta}) + 2(z_{\xi\xi} + 2z_{\xi\eta} + z_{\eta\eta}) + 3z = 0$$

or, $8z_{\eta\eta} + 3z = 0$.

$\therefore \ z_{\eta\eta} = -\frac{3z}{8}$, which is the required canonical form.

▶ **Example 4.5.4** *Reduce the following equation to its canonical form and find its general solution:*

$$x^2 z_{xx} + 2xyz_{xy} + y^2 z_{yy} + xyz_x + y^2 z_y = 0, \ x \neq 0, y \neq 0.$$

Solution: Comparing the given equation with (4.1.2) we get, $A = x^2$, $B = 2xy$, $C = y^2$, $F = xyz_x + y^2 z_y$.

$\therefore$ the discriminant of the equation is given by

$$D = B^2 - 4AC = 4x^2y^2 - 4x^2y^2 = 0.$$

It follows that the given equation is parabolic.

The λ-quadratic equation is $A\lambda^2 + B\lambda + C = 0$ i.e., $x^2\lambda^2 + 2xy\lambda + y^2 = 0$ or, $(x\lambda + y)^2 = 0$.

$$\therefore \ \lambda = -\frac{y}{x}, -\frac{y}{x}$$

which are real and equal.

Thus, the characteristic equation is

$$\frac{dy}{dx} - \frac{y}{x} = 0$$

Solving this ordinary differential equation we obtain, $y/x = c_1, c_1$ being an arbitrary constant.

Let us choose the new variables ξ and η such that $\xi = y/x, \eta = y$.

$\therefore \ \xi_x = -\frac{y}{x^2}, \ \xi_y = \frac{1}{x}, \ \xi_{xx} = \frac{2y}{x^3}, \ \xi_{xy} = -\frac{1}{x^2}, \ \xi_{yy} = 0, \ \eta_x = 0, \ \eta_y = 1, \ \eta_{xx} = \eta_{xy} = \eta_{yy} = 0.$

$$\therefore \ J = \frac{\partial(\xi,\eta)}{\partial(x,y)} = -\frac{y}{x^2} \neq 0, \text{ for } x \neq 0, y \neq 0.$$

Consequently,

$$b = 0$$
$$a = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 0$$
$$c = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 = y^2$$
$$z_x = z_\xi \xi_x + z_\eta \eta_x = -\frac{y}{x^2} z_\xi$$
$$z_y = z_\xi \xi_y + z_\eta \eta_y = \frac{1}{x} z_\xi + z_\eta$$
$$\therefore\ F = xy\left(-\frac{y}{x^2} z_\xi\right) + y^2\left(\frac{1}{x} z_\xi + z_\eta\right)$$
$$= y^2 z_\eta = \eta^2 z_\eta.$$

$$\therefore\ f = -[z_\xi(A\xi_{xx} + B\xi_{xy} + C\xi_{yy}) + z_\eta(A\eta_{xx} + B\eta_{xy} + C\eta_{yy}) + F]$$
$$= -\eta^2 z_\eta$$

$\therefore\ \phi = f/c = -z_\eta$.

Hence desired canonical form is $z_{\eta\eta} + z_\eta = 0$.

$$\therefore\ \frac{\partial}{\partial \eta}\left(\frac{\partial z}{\partial \eta}\right) + \frac{\partial z}{\partial \eta} = 0$$

Integrating with respect to η we have,

$$\ln\left(\frac{\partial z}{\partial \eta}\right) + \eta = \ln \psi_1(\xi),\ \psi_1 \text{ being arbitrary function}$$

or, $\dfrac{\partial z}{\partial \eta} = \psi_1(\xi)e^{-\eta}$.

Integrating with respect to η we have,

$$z = -\psi_1(\xi)e^{-\eta} + \psi_2(\eta),\ \psi_2 \text{ being arbitrary function}$$
$$= \psi_2\left(\frac{y}{x}\right) - \psi_1\left(\frac{y}{x}\right)e^{-y}.$$

Hence required general solution is

$$z = \psi_2\left(\frac{y}{x}\right) - \psi_1\left(\frac{y}{x}\right)e^{-y}.$$

► **Example 4.5.5** *Reduce the equation* $z_{xx} - (1+x)^2 z_{yy} = 0,\ x \neq -1$ *to its canonical form.*

Solution: Comparing the given equation with the equation (4.1.2) we have, $A = 1,\ B = 0,\ C = -(1+x)^2,\ F = 0$.

$\therefore$ the discriminant, $D = B^2 - 4AC = 4(1+x)^2 > 0$.

It follows that the equation is hyperbolic.

The λ-quadratic equation is $A\lambda^2 + B\lambda + C = 0$ i.e., $\lambda^2 - (1+x)^2 = 0$.

$\therefore\ \lambda = \pm(1+x)$, which are real and distinct.

Now, the characteristic equations are

$$\frac{dy}{dx} + 1 + x = 0 \text{ and } \frac{dy}{dx} - 1 - x = 0.$$

The solutions of these two ordinary differential equations are $y + x + \frac{1}{2}x^2 = c_1$ and $y - x - \frac{1}{2}x^2 = c_2$, where c_1 and c_2 are two arbitrary constants.

Now, let us choose the transformation $\xi = y + x + \frac{1}{2}x^2$ and $\eta = y - x - \frac{1}{2}x^2$.

Then $\xi_x = 1 + x,\ \xi_y = 1,\ \xi_{xx} = 1,\ \xi_{xy} = 0,\ \xi_{yy} = 0,\ \eta_x = -1 - x,\ \eta_y = 1,\ \eta_{xx} = -1,\ \eta_{xy} = 0,\ \eta_{yy} = 0$.

It is quite easy to verify that $a = 0,\ c = 0,\ b = -4(1+x^2)$

$$\text{and } f = -[z_\xi(A\xi_{xx} + B\xi_{xy} + C\xi_{yy}) + z_\eta(A\eta_{xx} + B\eta_{xy} + C\eta_{yy}) + F]$$
$$= -[z_\xi - z_\eta]$$

$$\therefore\ \phi = \frac{f}{b} = \frac{z_\xi - z_\eta}{4(1+x)^2} = \frac{z_\xi - z_\eta}{4(1+\xi-\eta)}.$$

Hence desired canonical form is

$$z_{\xi\eta} = \frac{1}{4}\cdot\frac{z_\xi - z_\eta}{1+\xi-\eta}.$$

Examples IV

1. Find the nature of the following partial differential equations:

(a) $z_{xx} - xz_{xy} + 4z_{yy} + 4z_y = 0$

(b) $z_{xx} + yz_{xy} - 2z_{yy} = 0$

(c) $az_{xx} + 2hz_{xy} + bz_{yy} = h(z_{xx} + z_{yy})$; a, b, h are real constants
(d) $h(z_{xx} - z_{yy}) - (a - b)z_{xy} = 0$; a, b, h are real constants
(e) $z_{xx} - 2\sec x z_{xy} + z_{yy} = 0$
(f) $z_{xx} + 2z_{xy} + \operatorname{cosec}^2 y z_{yy} = 0$
(g) $2(x^2 + y^2)z_{xx} + 2(x + y)z_{xy} + z_{yy} = 0$.

2. Find the characteristics of the following equations:

(a) $z_{xx} - 4z_{xy} + z_{yy} + 6z_x - 14z_y + 12z = 0$
(b) $2z_{xx} - 3z_{xy} - 2z_{yy} + 7z_x - 9z_y = 0$
(c) $4z_{xx} - 4z_{xy} + z_{yy} + 2z_x - 26z_y + 9z = 0$
(d) $3z_{xx} + 10z_{xy} + 3z_{yy} - 14z_y = 0$
(e) $9z_{xx} + 6\sin x z_{xy} - \cos^2 x z_{yy} - 2z_x = 0$
(f) $\sin^2 x z_{xx} + 2\cos x z_{xy} - z_{yy} = 0$.

3. Show that the one-dimensional diffusion equation

$$k\frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}, \quad -\infty < x < \infty, \; t > 0$$

is parabolic. Write the equation in equivalent canonical form.

4. Show that the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0$$

is hyperbolic. Find the characteristics.

Hence solve it.

5. Show that the equation $z_{xx} + x^2 z_{yy} = 0$ is elliptic everywhere except on the straight line $x = 0$. Write the equation in equivalent canonical form.

6. Find the canonical form of the equation $z_{xx} - xz_{yy} = 0$ in the region where it is hyperbolic.

7. Reduce the following equation to its canonical form:

$$x^2 z_{xx} - 2xyz_{xy} + y^2 z_{yy} + xz_x + yz_y = 0, \; x > 0$$

8. Find the canonical form of the equation $yz_{xx} + z_{yy} = 0, y > 0$.

9. Write the equation $x^2 z_{xx} - y^2 z_{yy} = 0$ $(x \neq 0, y \neq 0)$ in equivalent canonical form.

10. Write the equation $yz_{xx} + (x+y)z_{xy} + xz_{yy} = 0$ in equivalent canonical form.

11. Write the equation $z_{xx} + 2z_{xy} + z_{yy} = 0$ in equivalent canonical form and hence solve it.

12. Find the canonical form of the equation $x^2 z_{xx} + 2xyz_{xy} + y^2 z_{yy} = 0$ and hence solve it.

13. Find the canonical form of the equation $y^2 z_{xx} + x^2 z_{yy} = 0$.

14. Write the following equation in equivalent canonical form:

$$(1 - c^2)z_{xx} + z_{yy} = 0, \; c > 1$$

15. Find the characteristics of the equation $u_{xx} - 2\sin x u_{xy} - \cos^2 x u_{yy} - \cos x u_y = 0$. Write the equation in equivalent canonical form and hence solve it.

Answers

1. (a) Hyperbolic in $\Omega = \{(x, y) \in \mathbb{R}^2 : |x| > 4\}$
Parabolic in $\Omega = \{(x, y) \in \mathbb{R}^2 : x = \pm 4\}$
Elliptic in $\Omega = \{(x, y) \in \mathbb{R}^2 : |x| < 4\}$
(b) Hyperbolic
(c) Hyperbolic, if $(a + b)h > ab$
Parabolic, if $(a + b)h = ab$
Elliptic, if $(a + b)h < ab$
(d) Hyperbolic
(e) Hyperbolic
(f) Elliptic
(g) Elliptic in $\Omega = \{(x, y) \in \mathbb{R}^2 : x \neq y\}$
Parabolic in $\Omega = \{(x, y) \in \mathbb{R}^2 : x = y\}$.

2. (a) $y + (2 + \sqrt{3})x = c_1, y + (2 - \sqrt{3})x = c_2$
(b) $y + 2x = c_1, y - \frac{1}{2}x = c_2$
(c) $y + \frac{1}{2}x = c$
(d) $y - 3x + c_1, y - \frac{1}{3}x = c_2$
(e) $3y + \cos x + x = c_1, 3y + \cos x - x = c_2$
(f) $y + \operatorname{cosec} x - \cot x = c_1, y + \operatorname{cosec} x + \cot x = c_2$.

3. x = constant.

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4. $u_{\xi\eta} = 0; u = f(x - ct) + g(x + ct)$.

5. $z_{\xi\xi} + z_{\eta\eta} + \frac{1}{2\eta} z_\eta = 0$.

6. $z_{\xi\eta} = \frac{1}{6} \cdot \frac{z_\xi - z_\eta}{\xi - \eta}$ in the region $\Omega = \{(x, y) \in \mathbf{R}^2 : x > 0\}$.

7. $z_{\eta\eta} + \frac{1}{\eta} z_\eta = 0$.

8. $z_{\xi\xi} + z_{\eta\eta} + \frac{3}{\xi} z_\xi = 0$.

9. $2\xi z_{\xi\eta} - z_\eta = 0$.

10. $\xi z_{\xi\eta} + z_\eta = 0$.

11. $z_{\eta\eta} = 0; z = (x + y) f(x - y) + g(x - y)$ (choosing $\eta = x + y$).

12. $z_{\eta\eta} = 0; z = y f\left(\frac{y}{x}\right) + g\left(\frac{y}{x}\right)$ (choosing $\eta = y$).

13. $z_{\xi\xi} + z_{\eta\eta} + \frac{1}{2}\left(\frac{1}{\xi} z_\xi + \frac{1}{\eta} z_\eta\right) = 0$.

14. $z_{\xi\eta} = 0$.

15. $u_{\xi\eta} = 0; u = f(\cos x + x - y) - g(\cos x - x - y)$.