4.2. Production Function

'Function' is a mathematical device to show the relationship between at least two variables. Let x and y are two variables and the relationship between these two is such that, y depends on x. It can be stated as, $y = f(x)$.

Now y is the dependent variable and x, the independent variable.

Similarly, a production function is a purely technical relation which connects factor inputs and outputs, i.e., *technological relationship between physical inputs and physical outputs of a firm at any particular period of time.*

Production function shows the efficiency relationship, because it states,

(i) the maximum level of output that can be produced from any given amount of resources, or

(ii) the minimum amount of the factors that are required to yield a given level of output.

Let Q represents the quantity of output which can be produced by the employment of labour (L), capital (K), raw materials (R), land (N) etc. Then the production function can be written as, $Q = f(L, K, R, N)$ where Q is the dependent variable and independent variables are L, K, R, N. It is to be noted that, some inputs, like technology (T) may be assumed to be fixed and will not enter into the equation.

Different types of production functions are used in the economic analysis :

(i) According to the *Classical economists*, production mainly depends on the employment of labour. Therefore, the production function is, $Q = f(L)$.

(ii) According to the *Neo-classical economists*, production depends on labour as well as capital. Hence, $Q = f(L, K)$.

(iii) One specific production function commonly used in the economic analysis is the *Cobb-Dauglas production* function. The standard form is, $Q = A.L^a.K^b$ where A, a, b are three positive constants and $a + b = 1$ Following the function, if labour and capital both are doubled then Q will also be doubled. So it follows the constant returns to Scale.

$$A.(2L)^a.(2K)^b = A.2^a.L^a.2^b.K^b$$
$$= 2^{a+b}.AL^aK^b = 2.Q \quad [\because a + b = 1]$$

(iv) One another production function is used which is Leontief production function or *Fixed Co-efficient production function* where both the inputs are employed in a fixed proportion. Let 3 units of labour and 4 units of capital are used to produce 50 units of a product. Then the proportion is 3 : 4 which is maintained, i.e. fixed proportion. The general form of such function is $Q = \text{minimum}\left(\frac{K}{\alpha}, \frac{K}{\beta}\right)$ which means the proportion should be maintained for the expansion of the output. If output is expanded or contracted, alll the inputs must be expanded or contracted so as to maintain the fixed input ratio, eg. two parts hydrogen and one part oxygen can produce water (H_2O) and this proportion must be maintained.

different input combinations can be obtained for constant output level.

◈ 4.2.1. Short Run Production Function

Short run represents such a period of time during the production process where all the existing factors can be decomposed in two categories, i.e. fixed and variable. A fixed input is one which is required in the production process but whose quantity employed in the process is constant over given period of time regardless of the quantity of output produced.

The ***Law of variable proportion*** can be developed in the short run where the production function can be written as, Q = f(L) where other factors are constant.

So, labour is the single variable factor. Since in the short run, we have fixed factors, therefore in order to increase output, the firm must employ more of the variable factors, e.g. labour with the given quantity of fixed factor.

◈ 4.2.2. Long Run Production Function

Long run represents such a period of time during the production process where all the existing factors are variable, i.e. as the time is lengthened, more of the fixed factors become variable.

The behaviour of production now is subject to the ***Law of Returns to Scale***. Now increase in production is possible though the use of more labour or through plant expansion depending on one optimum or efficient combination of these two to produce the desired output. Now, the production function can be written as, Q = f(L, K) where all the factors are variable.

From the above analysis, the **importance of the** concept of production function in managerial decision can be stated as follows :

(i) Least Cost Combination for output can be determined.

(ii) Marginal Revenue Productivity of a variable factor can be determined, where, Marginal Revenue Productivity = Marginal Revenue × Marginal Productivity.

(iii) For increasing returns to scale, the longrun decision will be increase in production.

4.3. Law of Variable Proportion

As stated before, the law of variable proportion is a short run theory of production; where one of the factor is variable with other factors are kept constant. Let in the short run, labour (L) is the variable factor where the constant factor is capital (K). Hence, the short run production function can be written as, $Q = f(L)$ where $K = \bar{K}$.

Now factor proportion or input ratio means the fraction $\frac{L}{K}$ or $\frac{K}{L}$. Now for the first fraction or proportion, increase or decrease in L, K remains fixed means increase or decrease in the proporton. Thus the proportion is variable. So, the law of variable proportion exhibits the direction and the rate of change in the firms output when the amount of only one factor of production is varied.

❑ Assumptions of the Law

(i) It operates purely in the short run where one of the existing factors is variable and others are fixed.

(ii) The technology must be constant.

(iii) It is not applied for the 'fixed-coefficient' type of production function, stated before.

(iv) Each unit of the variable factor must be homogeneous in nature, i.e., identical in amount and quality.

❑ Explanation of the Law

The law can be explained by the statement of LEFTWITCH. According to him, **"The law of variable proportion states that if the input of one resource is increased by equal increments per unit of time while the inputs of other resources are held constant then total production will increase but beyond some point the resulting output increase will become smaller and smaller."**

According to Prof. Paul. A. Samuelson, the Law of variable proportion can be developed with the help of three phases : increasing return to factor, constant return to factor and diminishing return to factor. Following him, **"An increase in some inputs relative to the other comparatively fixed inputs will cause output to increase, but after a point the extra output resulting from the same additions of inputs will become less and less, this falling off of extra returns is a consequence of the fact that the new 'doses' of the varying resources have less and less of the constant resources to work with."**

❑ Graphical Representation

The law of variable proportion can be explained by the following graph, which will be the total product curve. Since one of the factor is variable, say labour (L) with the fixed one capital (K), therefore, the short run production function can be written as : $Q = f(L)$ where $K = \bar{K}$.

Hence, different combinations between Q and L can be obtained from the above function. Joining these combinations, the shape of total product for labour (TP_L) can be derived.

(i) At $L = 0$ with $K = \bar{K}$, Q must be zero. Therefore, TP_L curve starts from origin.

(ii) If labour increases, there will be a corresponding increase in quantitiy (Q). So Q and L both are directly related. Due to this direct relation, the TP_L curve must be positively sloped.
But positively sloped means the curve may be a positively sloped straight line or curveture, i.e, either convex to the vertical axis (concave to the horizontal axis) or convex to the horizontal axis (concave to the vertical axis). The actual shape depends on the rate of change. So there are three possibilities :

(a) Q and L both are increasing at a same rate, i.e, if L is doubled, then Q will also be doubled. *A straight line represents such relationship.*

(b) If Q is increasing at a higher rate, i.e. L is doubled, but Q is more than doubled. Then a positively sloped curveture, *convex to the horizontal axis represents this relationship.*

(c) If Q is increasing at a lower rate, i.e. L is doubled, but Q is less than doubled. Then a positively sloped curveture, *convex to the vertical axis represents this relationship.*

Hence the shape of total production (TP_L) curve depends on Law of Variable proportion which is divided in three returns with in the scale or to factor.

(iii) Initially due to the **Increasing Return**, as the employment of labour increases, total production will increase at a higher rate subject to the fixed factor capital. It is to be noted that initially, at the first phase (not stage, which is defined latter) the amount of capital is fixed but exists with a greater volume. Thus the total production curve is convex to the horizontal axis. The range OA in the Fig. 4.1 represents this phase.

(iv) At the second phase, there will be **Constant Return**, i.e. as the employment of labour increases, total production will increase at a same rate. It means the fixed factor, capital is now more exhausted or utilised than before, which will lead to a slower growth. The point of inflection, A, in Fig. 4.1 represents this phase.

(v) At the third phase, there will be **Diminishing Return**, i.e. as the employment of labour increases, total production will increase at a lower rate because the fixed factor capital is now more exhausted or utilised. This is the reason by which the growth rate becomes slower than before because the fixed factor in this phase is in smaller volume. The total production curve is concave to the horizontal axis. The range AB in fig-1 represents this phase.

(vi) Point B represents the maximum amount of production in Fig. 4.1.

(viii) Beyond point B, employment of more labour, keeping the fixed factor as constant, the total production will fall. This phase is called the **Negative Return**, which is not accepted. The range BC in Fig. 4.1 represents this phase.

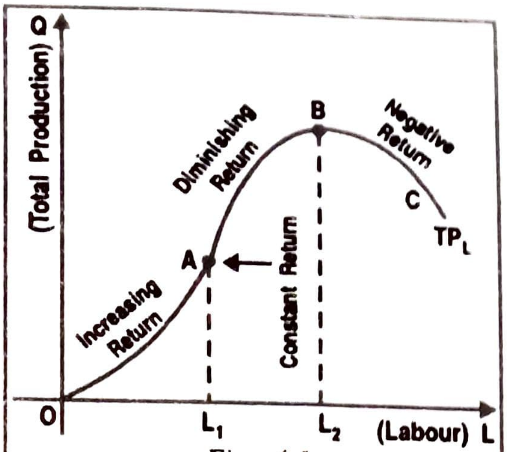

Fig. 4.1

Hence the total production curve is 'S' shaped — starts from origin, then convex to L axis due to increasing return, point of inflection due to constant return and concave to the L-axis due to diminishing return. This is the graphical representation of the law of variable proportion.

◈ 4.3.1. Derivation of Average Productivity of Labour

The Average productivity (AP) represents 'per unit' productivity of a factor. So if 10 labour can produce 100 units of a product, then AP_L must be $(100 \div 10) = 10$. Therefore it represents one fraction of total output and no. of labour employed in production. For the production function like, $Q = f(L)$ where $K = \overline{K}$;

$$AP_L = \frac{\text{Total Production}}{\text{No. of Labour Employed}} = \frac{Q}{L}.$$

Now from this fraction, it is quite easy to derive the relationship between AP_L and L. By the Law of variable proportion, we can draw the following conclusions :

(a) Initially for increasing return, Q is increasing at a higher rate than L. Since the numerator is increasing faster than the ultimate value of that fraction must be increasing. Therefore, AP_L is increasing.

(b) Then, due to the constant return, Q and L both are increasing at a constant rate, by which the fraction or AP_L must be constant.

(c) Due to the diminishing return, Q is increasing at a diminishing rate. Now the numerator is increasing at a slower rate which means the value of the fraction must be diminishing. Therefore AP_L is diminishing.

So AP_L is rising, then constant or reaches at its' maxima and then falling. Hence the AP_L curve must be 'Inverse–U' shaped. But one should remember or note the important difference between the simple derivation, mentioned above and the actual graphical derivation, mentioned below.

The shape of AP_L curve can be derived from the total production curve.

(i) At point A on the Total Production Curve :

$$Q = AB,\ L = OB. \therefore AP_L = \frac{Q}{L} = \frac{AB}{OB} = \text{slope of OA line.}$$

(ii) At point C on the Total production curve :

$$Q = CD,\ L = OD. \therefore AP_L = \frac{Q}{L} = \frac{CD}{OD} = \text{slope of OC line.}$$

(iii) At point E on the total production curve :

$$Q = EF,\ L = OF; \therefore AP_L = \frac{Q}{L} = \frac{EF}{OF} = \text{slope of OE line.}$$

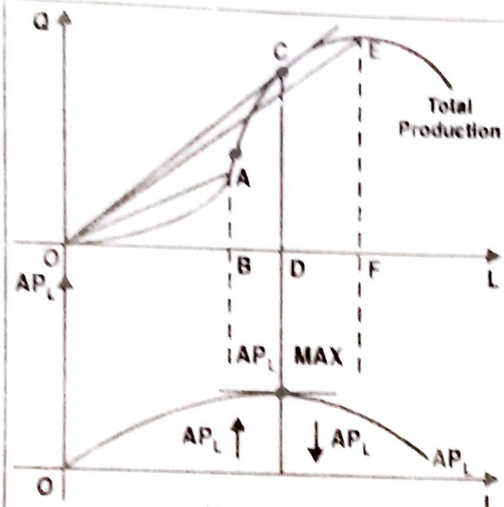

Fig. 4.2

Movement along the TP curve, from A to C means $\frac{AB}{OB} < \frac{CD}{OD}$ because the OC line is steeper than the OA line.

But movement along the TP curve, from C to E means $\frac{CD}{OD} > \frac{EF}{OF}$ because the OE line is flatter than the OC line.

Hence, we can conclude that movement from A to E means $\frac{AB}{OB} < \frac{CD}{OD} > \frac{EF}{OF} \Rightarrow$ slope of OA line < slope of OC line > slope of OE line $\Rightarrow AP_L$ at point A < AP_L at point C > AP_L at point E.

From this relationship three conclusions can be drawn :

(i) From A to C, the AP_L is increasing.

(ii) At C, AP_L is maximum.

(iii) From C to E, the AP_L is diminishing.

Hence in the lower panel of Fig. 4.2, the AP_L curve is inverse–U shaped.

◈ 4.3.2. Derivation of Marginal Productivity Of Labour

The Marginal Productivity of labour (MP_L) represents the 'last unit' labour productivity, eg. let 10 men can produce 100 units of a product and 11 men can produce 109 units, then additional 9 units can be produced by the additional or 11th labour. So the marginal productivity of labour will be 9.

By the definition, marginal productivity of labour (MP_L) represents addition to the total production due to a change in the employment of labour by extra one unit, other factors remain constant. Hence MP_L is the ratio of change in output and change in employment of labour.

$$\therefore MP_L = \frac{\text{Change in output}}{\text{Change in labour employed}} = \frac{\Delta Q}{\Delta L}$$

Let Q_1 unit of output can be produced by the employment of L_1 no. of labour. Then if the employment of labour goes up to L_2, the volume of production changes to the amount of Q_2.

$$\therefore MP_L = \frac{\Delta Q}{\Delta L} = \frac{Q_2 - Q_1}{L_2 - L_1}$$

Relation between Total and Marginal Production : The sign of MP_L depends on Q_2.

(i) If $Q_2 > Q_1 \Rightarrow \Delta Q > 0 \Rightarrow MP_L > 0$.

(ii) If $Q_2 = Q_1 \Rightarrow \Delta Q = 0 \Rightarrow MP_L = 0$.

(iii) If $Q_2 < Q_1 \Rightarrow \Delta Q < 0 \Rightarrow MP_L < 0$.

Therefore, mathematically MP_L may be positive, zero or even negative. A rational producer will choose only that portion where $MP_L > 0$. Hence in a diagram, the relevant portion of MP_L is the positivity segment.

❐ Logical Derivation

From the above fraction or from the Law of Variable proportion the shape of MP_L can be derived logically as AP_L. The process is as follows :

(i) Due to the increasing return to factor, the first phase of law of variable proportion, production or Q is increasing at a higher rate compare to the employment of labour. Now, by the fraction, MP_L represents this rate of change. Therefore MP_L must be increasing due to the increasing return. As labour increases MP_L also increases. Due to this direct relationship the MP_L curve must be positively sloped.

(ii) Due to the constant return to factor, the second phase of law of variable proportion, production or Q is increasing at a constant rate with the employment of labour. Now MP_L represent this rate of change. So MP_L must be constant with the increase in L. This is the situation where MP_L reaches at its maxima.

(iii) Due to the diminishing return to factor, the third phase of law of variable proportion, production or Q is increasing at a diminishing rate with the employment of labour. So MP_L must be decreasing with the increase in L. Due to this inverse relationship the MP_L curve must be negatively sloped.

Therefore, MP_L is increasing, reaches at it's maxima and then decreasing. So, like AP_L, the MP_L curve is also inverse–U shaped. The relationship and the difference between these two inverse–U shaped curves is discussed in latter part of this section.

❐ Graphical Derivation

The following diagram can be developed to derive the shape of MP_L curve from the total production Curve.

Since, $MP_L = \frac{\Delta Q}{\Delta L}$, slope of the total production curve, therefore, we have one alternative explanation to derive the shape of MP_L from the total production curve with the help of slope. Now different targets can be drawn at different points of the total production curve to calculate the slope at that point.

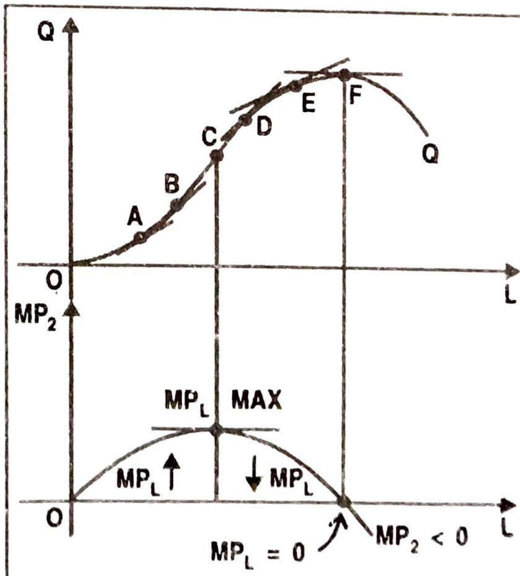

Fig. 4.3

Following conclusions can be drawn from the above diagram :

(i) At the phase of increasing return to a single factor the law of variable proportion, the total production curve is convex to the horizontal or labour axis. Now tangent at point B is steeper than the tanget at point A. It means the value of the slope is increasing. Since MP_L represents the slope, therefore, movement along the total production curve from A to C means MP_L must be increasing with the increase in L. Due to this direct relationship, the MP_L curve must be positively sloped in the lower panel of the diagram.

(ii) At the point of inflection C, we have the phase of constant return to a single factor. It represents the maximum value of the slope of the total production curve. Hence, MP_L is maximize or reaches at its maxima in the lower panel.

(iii) At the phase of diminishing return to a single factor, the total porduction curve is concave to the horizontal or labour axis. Now tangent at point E is flatter than the tangent at point D. It means the value of the slope or MP_L is diminishing. Therefore, movement along the total production curve from D to E means MP_L must be diminishing with the increase in L. Due to this inverse relationship, the MP_L curve must be negatively sloped, drawn in the lower panel.

(iv) At point F, the tangent drawn is horizontal means the value of the slope or MP_L becomes zero. Therefore, MP_L cuts the horizontal or L axis in the lower panel.

(v) Beyond point F, the total production curve is falling, represents negative return. Hence corresponding MP_L lies in the negative quadrant.

It is to be noted that the first three cases represent the meaningful shape of MP_L. Hence following the law of variable proportion, the MP_L curve is rising, reaches at its maxima and then falling — so 'inverse–U' shaped.

Hence we can conclude that the portion of IQ which lies between the two ridge lines is called the economic region where marginal productivities of both the factors like labour as well as capital are positive. The region outside the ridge lines is called the non-economic region, So economic region or range means negatively sloped IQ. [discussed latter]

◈ 4.3.3. Relation between average And Marginal Production

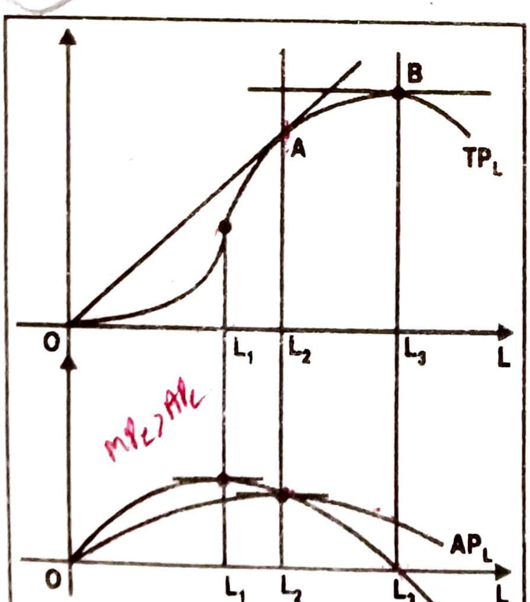

Fig. 4.4

The relationship between total, average and marginal production can be developed by a single diagram. The diagram is as under :

(i) At the range of OL_1 employment, both AP_L and MP_L are rising. But the rising rate of MP_L is greater than the rising rate of AP_L. Therefore MP_L is steeper than AP_L. Since for this range, MP_L curve lies above the AP_L curve.

Therefore, $MP_L > AP_L$.

(ii) At the OL_2 level of employment, the falling portion of MP_L passes through the maximum point of AP_L. At that intersection, $MP_L = AP_L$.

(iii) Beyond OL_2 level of employment, both AP_L and MP_L are diminishing. But the diminishing rate of MP_L is greater than the diminishing rate of AP_L. Therefore, MP_L is steeper than AP_L. Since for this range, MP_L curve lies below the AP_L curve, Therefore, $MP_L < AP_L$