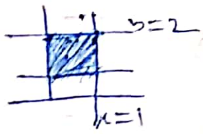
Double and Triple Integration 140901

Ex: 1 Evaluate the integration over R

(a) $\iint_R (x^2+y^2)\,dx\,dy$, $R:\{0\le x\le 1,\ 1\le y\le 2\}$.

$\Rightarrow \iint_R (x^2+y^2)\,dx\,dy$; $R:\{0\le x\le 1,\ 1\le y\le 2\}$



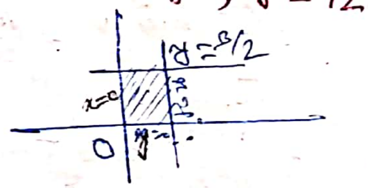
$= \int_{x=0}^{1}\int_{y=1}^{2}(x^2+y^2)\,dy\,dx$

$= \int_0^1 \left[x^2y + \frac{y^3}{3}\right]_1^2 dx = \int_0^1 \left(2x^2 - x^2 + \frac{8}{3} - \frac{1}{3}\right) dx$

$= \left[\frac{x^3}{3} + \frac{7}{3}x\right]_0^1 = \frac{1}{3} + \frac{7}{3} = \frac{8}{3}$.

(b) $\iint_R (4-x^2-y^2)\,dx\,dy$, where R is bounded by $x=0, x=1$, $y=0, y=3/2$

$\Rightarrow \iint_R (4-x^2-y^2)\,dy\,dx$



$= \int_{x=0}^{1}\int_{y=0}^{3/2} (4-x^2-y^2)\,dy\,dx$

$= \int_0^1 \left[4y - x^2y - \frac{1}{3}y^3\right]_0^{3/2} dx = \int_0^1 \left(6 - \frac{3}{2}x^2 + 0 - \frac{9}{8}\right) dx$

$= \int_0^1 \left(\frac{39}{8} - \frac{3}{2}x^2\right) dx = \left[\frac{39}{8}x - \frac{3}{2}\cdot\frac{1}{3}\cdot x^3\right]_0^1 = \frac{35}{8}$. ANS.

(c) $\iint \frac{x}{y}\,dx\,dy$, $R:\{|x|\le 1,\ 1\le y\le 2\}$.

$\Rightarrow \iint_R \frac{x}{y}\,dx\,dy = \int_{x=-1}^{1}\int_{y=1}^{2} \frac{x}{y}\,dy\,dx = \int_{-1}^{1} [x\log y]_1^2\,dx$

$= \int_{-1}^{1} x\log 2\,dx = \log 2\left[\frac{x^2}{2}\right]_{-1}^{1} = \log 2(1-1) = 0.$

(d) $\iiint_R \cos(x+2y+3z)\,dx\,dy\,dz$, $R:\{0\le x\le 2,\ 1\le y\le 3,\ |z|\le 1.\}$

$\Rightarrow \iiint_R \cos(x+2y+3z)\,dx\,dy\,dz$.

$= \int_0^2\int_1^3\int_{-1}^{1} \cos(x+2y+3z)\,dz\,dy\,dx$.

$= \int_0^2\int_1^3 \left[\frac{1}{3}\sin(x+2y+3z)\right]_{-1}^{1} dy\,dz$ (cont)

$$= \frac{1}{3}\int_0^2\int_1^3 \{\sin(x+2y+3) - \sin(x+2y-3)\}\, dy\, dx.$$

$$= \frac{1}{3}\int_0^2 \left[-\frac{1}{2}\cos(x+2y+3) + \frac{1}{2}\cos(x+2y-3)\right]_1^3 dx$$

$$= \frac{1}{6}\int_0^2 \{\cos(x+3) - \cos(x-1) - \cos(x+9) + \cos(x+5)\}\, dx$$

$$= \frac{1}{6}\left[\sin(x+3) - \sin(x-1) - \sin(x+9) + \sin(x+5)\right]_0^2$$

$$= \frac{1}{6}\left[\sin(5) - \sin(1) - \sin(11) + \sin(2) - \sin(3) - \sin(5) + \sin 9 - \sin 5\right]$$

$$= \frac{1}{6}\left[\sin 9 + \sin 7 - \sin 11 - 2\sin 1\right].$$

—o—

(e) $\int_0^1 \int_0^{1-y^2} \{(x-1)^2 + y^2\}\, dx\, dy$

$$\Rightarrow \int_0^1\int_0^{1-y^2} \{(x-1)^2 + y^2\}\, dx\, dy$$

$$= \int_0^1 \left[\frac{(x-1)^3}{3} + y^2 x\right]_0^{1-y^2} dy.$$

$$= \int_0^1 \left[-\frac{1}{3}y^6 + y^2(1-y^2) + \frac{1}{3}\right] dy.$$

$$= \left[-\frac{1}{3}\frac{y^7}{7} + \frac{y^3}{3} - \frac{y^5}{5} + \frac{1}{3}y\right]_0^1$$

$$= -\frac{1}{21} + \frac{1}{3} - \frac{1}{5} + \frac{1}{3}$$

$$= \frac{-5-21+70}{105} = \frac{44}{105} \quad \text{ANS}$$

—o—

(f) $\int_0^2 \int_{-y}^{\sqrt{y}} (1+x+y)\, dx\, dy$

$$\Rightarrow \int_0^2\int_{-y}^{\sqrt{y}} (1+x+y)\, dx\, dy$$

$$= \int_0^2 \left[x + \frac{1}{2}x^2 + yx\right]_{-y}^{\sqrt{y}} dy$$

$$= \int_0^2 \left(y^{1/2} + \frac{y}{2} + y^{3/2} + y - \frac{y^2}{2} + y^2\right) dy$$

$$= \left[\frac{2}{3}y^{3/2} + \frac{1}{4}y^2 + \frac{2}{5}y^5 + \frac{1}{2}y^2 - \frac{1}{6}y^6 + \frac{y^3}{2}\right]_0^2$$

$$= \left[\frac{2}{3}y^{3/2} + \frac{3}{4}y^2 + \frac{2}{5}y^{3/2} + \frac{1}{6}y^3\right]_0^2$$

$$= \frac{44\sqrt{2}+65}{15} \quad \text{ANS}$$

—o—

(g) $\int_0^{2\pi}\int_{a\sin\theta}^{a} r\, dr\, d\theta$

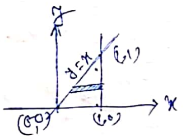
$$\Rightarrow \int_0^{2\pi}\int_{a\sin\theta}^{a} r\, dr\, d\theta = \int_0^{2\pi}\left[\frac{r^2}{2}\right]_{a\sin\theta}^{a}$$

$$= \int_0^{2\pi}\left[\frac{a^2}{2} - \frac{a^2}{2}\sin^2\theta\right] d\theta = \frac{a^2}{2}\int_0^{2\pi}\cos^2\theta\, d\theta.$$

$$= \frac{a^2}{4}\int_0^{2\pi}(1+\cos 2\theta)\, d\theta = \frac{a^2}{4}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{2\pi}$$

$$= \frac{a^2}{4}\cdot 2\pi = \frac{a^2\pi}{2} \quad \text{ANS}.$$

EX: 2 Show that $\iint (x^2+y^2)\, dx\, dy$ over the region enclosed by the triangle having its vertices at $(0,0)$, $(1,0)$, $(1,1)$ is $1/3$.



$\Rightarrow$ we evaluate $\iint (x^2+y^2)\, dx\, dy$.

$$\int_{y=0}^{1}\int_{x=y}^{1} (x^2+y^2)\, dx\, dy$$

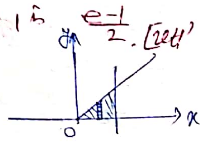
$$= \int_{y=0}^{1}\int_{x=y}^{1} (x^2+y^2)\, dx\, dy = \int_0^1 \left[\frac{x^3}{3} + y^2 x\right]_y^1 dy$$

$$= \int_0^1 \left(\frac{1}{3} + y^2 - \frac{y^3}{3} - y^3\right) dy$$

$$= \left[\frac{1}{3}y + \frac{y^3}{3} - \frac{y^4}{12} - \frac{y^4}{4}\right]_0^1 = 1/3.$$

—o—

EX: 3 Show that $\iint_R e^{y/x}\, dx\, dy$, where R is a triangle bounded by $y = x$, $y = 0$ and $x = 1$ is $\frac{e-1}{2}$. [2011]



$$\Rightarrow \int_0^1\int_{y=0}^{x} e^{y/x}\, dy\, dx$$

$$= \int_0^1 \left[x\, e^{y/x}\right]_0^x dx$$

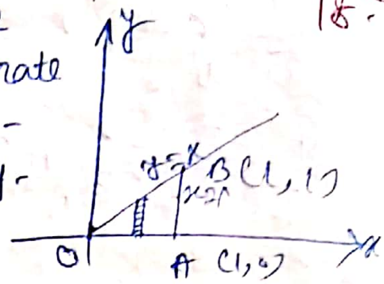
$$= \int_0^1 (ex - x)\, dx = e\int_0^1 x\, dx - \int_0^1 x\, dx = e\left[\frac{x^2}{2}\right]_0^1 - \left[\frac{x^2}{2}\right]_0^1$$

$$= e\left[\frac{x^2}{2}\right]_0^1 - \left[\frac{x^2}{2}\right]_0^1$$

$$= \frac{e}{2} - \frac{1}{2} = \left(\frac{e-1}{2}\right) \quad \text{ANS}$$

E14 Show that $\iint \sqrt{4x^2-y^2}\,dx\,dy$ extended over the triangle formed by $y=0$, $x=1$, $y=x$ is $\frac{3\sqrt{3}+2\pi}{18}$.

$\Rightarrow$ Here the needed not change the order, since we are to integrate w.r.t y treating x constant, we consider the strips parallel to y-axis whose extremities lies on $y=0$, $y=x$, so that the limits of y are given to be 0 to x and those of y are clearly $x=0$ to O as $x=1$ for A.



$$\therefore I = \int_{x=0}^{1}\int_{y=0}^{x} \sqrt{(4x^2-y^2)}\,dx\,dy$$

$$= \int_0^1 \left[\frac{y}{2}\sqrt{(4x^2-y^2)} + \frac{x^2}{2}\sin^{-1}\frac{y}{2x}\right]_0^x dx.$$

$$= \int_0^1 \left(\frac{x}{2}\sqrt{3}\cdot x + \frac{(2x)^2}{2}\sin^{-1}\left(\frac{1}{2}\right)\right] dx$$

$$= \int_0^1 \left(\frac{\sqrt{3}}{2}x^2 + \frac{4x^2}{2}\cdot\frac{\pi}{6}\right) dx$$

$$= \left[\frac{\sqrt{3}}{2}\cdot\frac{x^3}{3} + \frac{x^3}{3}\cdot\frac{\pi}{3}\right)_0^1$$

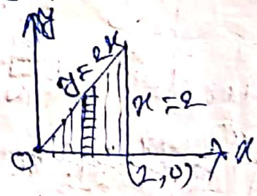
$$= \frac{\sqrt{3}}{2}\cdot\frac{1}{3} + \frac{1}{6}\cdot\frac{\pi}{3}$$

$$= \frac{1}{3}\left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right) = \frac{3\sqrt{3}+2\pi}{18}.$$

—o—

Ex 5 verify that $\iint (x^3+y^2)\,dx\,dy$ taken over the triangle bounded by $x=2$, $y=0$, $y=2x$ is $\frac{352}{15}$

$\Rightarrow$ The domain of the integral being the shaddow reasion in the figher.



$$\int_{x=0}^{2}\int_{y=0}^{2x} (x^3+y^2)\,dx\,dy$$

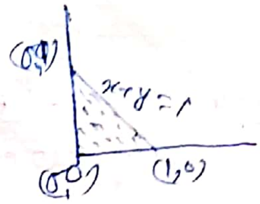
$$= \int_0^2 \left[x^3 y + \frac{y^3}{3}\right]_0^{2x} dx$$

$$= \int_0^2 \left(2x^4 + \frac{8}{3}x^3\right) dx$$

$$= \left[\frac{2}{5}x^5 + \frac{8\cdot x^4}{3\cdot 4}\right]_0^2 = \left[\frac{2}{5}\cdot 32 + \frac{2}{3}\cdot 16\right]$$

$$= \frac{64}{5} + \frac{32}{3} = \frac{352}{15}$$ **ANS**

Ex) Show that the integral $\iint e^{\frac{y-x}{y+x}}\,dxdy$, taken over the triangle with vertices at $(0,0)$, $(1,0)$, $(0,1)$ is $\frac{1}{4}\left(e-\frac{1}{e}\right)$.



$\Rightarrow$ The region R is bounded by the triangle whose sides are $x=0$, $y=0$, $x+y=1$.

putting, $x+y=u$, $x=uv$

$\therefore y = u(1-v)$

then the region becomes $u: 0$ to 1, $v: 0$ to 1

we know $dxdy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv$

Here $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ 1-v & -u \end{vmatrix} = -u$

$\therefore dxdy = |-u|\, dudv$

$= u\, dudv.$

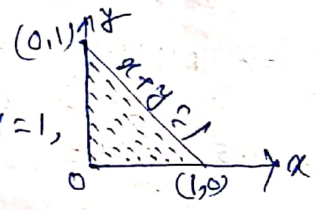
Now, $\iint_R e^{\frac{y-x}{y+x}}\,dxdy$

$= \int_{u=0}^{1}\int_{v=0}^{1} e^{\frac{u(1-v)-uv}{u}} \cdot u\, dudv = \int_0^1\int_0^1 e^{(1-2v)} \cdot u\, dudv.$

$= \int_0^1 u\,du \cdot \int_0^1 e^{(1-2v)}\,dv = \left[\frac{u^2}{2}\right]_0^1 \left[-\frac{1}{2}e^{1-2v}\right]_0^1$

$= \frac{1}{2}\cdot\frac{1}{2}(-e^{-1}+e) = \frac{1}{4}\left(e-\frac{1}{e}\right).$ **ANS**

Ex 8 Show that $\iint x^{1/2}(1-x-y)^{2/3}\,dxdy$, over the triangle bounded by $x=0$, $y=0$, $x+y=1$ is $\beta(17/6, 5/3)\,\beta(4/3, 3/2)$.



$\Rightarrow$ Let the region be $R: x=0, y=0, x+y=1$,

let, $x+y=u$, $x=uv$, $\therefore y = u(1-v)$.

$\therefore dxdy = u\,dudv$.

The new region becomes $u: 0$ to 1, $v: 0$ to 1.

Hence $\iint_R (uv)^{1/2}\{u(1-v)\}^{1/3}\cdot(1-u)^{2/3}\,u\,dudv$

$= \int_0^1\int_0^1 u^{1/2+1/3+1}\cdot(1-u)^{2/3}\cdot v^{1/2}(1-v)^{1/3}\,dv$

$= \int_0^1 u^{11/6}(1-u)^{2/3}\,du \cdot \int_0^1 v^{1/2}(1-v)^{1/3}\,dv.$

$= \int_0^1 u^{(17/6-1)}(1-u)^{(5/3-1)}\,du \cdot \int_0^1 v^{3/2-1}(1-v)^{1/3-1}\,dv.$

$= \beta(\frac{17}{6}, \frac{5}{3}) \cdot \beta(3/2, 4/3)$

$= \beta(\frac{17}{6}, 5/3) \cdot \beta(4/3, 3/2)$ as $\beta(m,n) = \beta(n,m)$

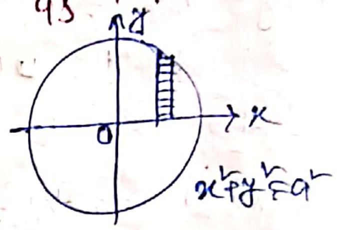
— 0 —

Ex: 9 Verify that $\iint y^2\sqrt{a^2-x^2}\, dx\,dy$ extended over the disc $x^2+y^2 \le a^2$ is $\frac{32}{45}a^5$.

⇒ We integrating the first quadrant, integrating first by w.r.t y and the region is

$y:\ 0$ to $\sqrt{a^2-x^2}$

$x:\ 0$ to a.



∴ The given integral becomes -

$$\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} y^2\sqrt{a^2-x^2}\, dy\, dx$$

$$= \int_0^a \left[\sqrt{a^2-x^2}\cdot\frac{y^3}{3}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{3}\int_0^a (a^2-x^2)^2\, dx$$

$$= \frac{1}{3}\int_0^a (a^4 - 2a^2x^2 + x^4)\, dx$$

$$= \frac{1}{3}\left[a^4x - \frac{2}{3}a^2x^3 + \frac{x^5}{5}\right]_0^a$$

$$= \frac{1}{3}\left[a^5 - \frac{2a^5}{3} + \frac{1}{5}a^5\right] = \frac{8a^5}{45}$$ **ANS**

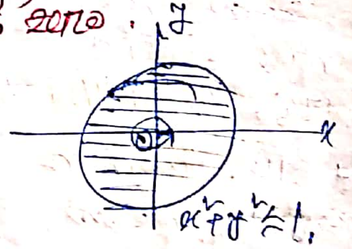
$\therefore \iint y^2(\sqrt{a^2-x^2})\, dxdy = 4\times\frac{8}{4}$
$= \frac{32}{4}$

— 0 —

Ex: 10 Show that $\iint \frac{x^3+y^3-3xy(x^2+y^2)}{(x^2+y^2)^{3/2}}\, dxdy$ over the circle is $x^2+y^2 \le 1$ is zero.

⇒ putting $x = r\cos\theta$
$y = r\sin\theta$.

and $dxdy = r\, drd\theta$.



the region in r, θ plane is

$r:\ 0$ to 1, $\theta:\ 0$ to 2π.

∴ The given integral becomes

$$\int_0^1\int_0^{2\pi} \frac{r^3\cos^3\theta + r^3\sin^3\theta - 3r^4\sin\theta\cos\theta}{r^3} \cdot r\, drd\theta.$$

$$= \int_0^1\int_0^{2\pi} \{r(\cos^3\theta + \sin^3\theta) - 3r^2\sin\theta\cos\theta\}\, drd\theta.$$

$= \int_0^1 r\,dr \int_0^{2\pi} (\cos^3\theta + \sin^3\theta) - \frac{3}{2}\int_0^1 r^2\,dr \int_0^{2\pi} \sin 2\theta\, d\theta.$

$= \left[\frac{r^2}{2}\right]_0^1 \cdot \int_0^{2\pi} \frac{1}{4}\left(\cos 3\theta + 3\cos\theta + \sin 3\theta - 3\sin\theta\right) d\theta - \frac{3}{2}\cdot\left[\frac{r^3}{3}\right]_0^1 \left[-\frac{1}{2}\cos 2\theta\right]_0^{2\pi}.$

$= \frac{1}{2}\cdot\frac{1}{4}\left[\frac{1}{3}\sin 3\theta + 3\sin\theta - \frac{1}{3}\cos 3\theta + 3\cos\theta\right]_0^{2\pi} - \frac{3}{2}\cdot\frac{1}{3}\left[-\frac{1}{2}+\frac{1}{2}\right].$

$= \frac{1}{8}\left[-\frac{1}{3} + 3 + \frac{1}{3} - 3\right] - 0$

$= \frac{1}{8}\times 0 - 0 = 0$, which is the required prove.

—o—

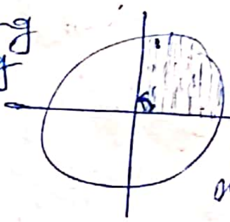
Ex III $\iint x^2y^2\,dx\,dy$ over

i) The region bounded by $x \geq 0,\ y \geq 0,\ x^2+y^2 \leq 1$.

ii) The circle $x^2+y^2 \leq 1$, Evaluate. [2011'89]

$\Rightarrow$ The domain of the integral being the shaddow in the figure, putting $x = r\cos\theta,\ y = r\sin\theta$.

$\therefore\ dx\,dy = r\,dr\,d\theta$.



Hence the given integral becomes

$$\int_{r=0}^{1}\int_{\theta=0}^{\pi/2} r^4 \sin^2\theta \cos^2\theta \cdot r\,dr\,d\theta.$$

$= \int_{r=0}^{1} r^5\,dr \cdot \int_0^{\pi/2} \frac{1}{8}(2\sin^2 2\theta)\, d\theta$

$= \left[\frac{r^6}{6}\right]_0^1 \cdot \frac{1}{8}\int_0^{\pi/2} (1-\cos 4\theta)\, d\theta.$

$= \frac{1}{48}\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\pi/2} = \frac{1}{48}\left[\frac{\pi}{2} - 0\right] = \frac{\pi}{96}$ Ans

—o—

ii) Here we have $\frac{1}{48}[2\pi - 0] = \frac{\pi}{24}$ Ans

—o—

Ex I prove that $\iint \{2a^2 - 2a(x+y) - (x^2+y^2)\}\,dx\,dy = 8a^4\pi$, the region of integration being, the circle $x^2+y^2+2a(x+y) = 2a^2$. [CU'91]

$\Rightarrow$ Let R being the region of integration being the circle

$x^2+y^2+2a(x+y) = 2a^2$

i.e., $(x+a)^2+(y+a)^2 = (2a)^2$.

or, $X^2+Y^2 = 4a^2$

where $X = x+a,\ Y = y+a$.

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putting $x = r\cos\theta$, $y = r\sin\theta$.

$\therefore dx\,dy = r\,dr\,d\theta$.

Then the given integral becomes,

$$\iint_R \{2a^2 - 2a(x+y) - (x^2+y^2)\}\,dx\,dy.$$

$$= \iint_R \{4a^2 - (x+a)^2 - (y+a)^2\}\,dx\,dy.$$

$$= \int_{r=0}^{2a}\int_{\theta=0}^{2\pi} (4a^2 - X^2 - Y^2)\,dX\,dY$$

$$= \int_{r=0}^{2a}\int_{\theta=0}^{2\pi} (4a^2 - r^2)\,r\,dr\,d\theta.$$

$$= \int_0^{2a} r(4a^2 - r^2)\,dr \int_0^{2\pi} d\theta$$

$$= \left[4a^2\cdot\frac{r^2}{2} - \frac{r^4}{4}\right]_0^{2a}\cdot[2\pi - 0]$$

$$= (8a^4 - 4a^4)\cdot 2\pi = 8\pi a^4 \quad \text{ANS}$$

Ex! Show that $\iint y\,dx\,dy = \frac{4}{3}ab^2$ over the region bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

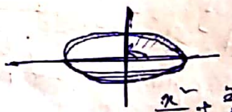
⇒ putting $x = ar\cos\theta$, $r = 0$ to 1

$y = br\sin\theta$, $\theta = 0$ to $\pi/2$

$\therefore dx\,dy = ab\,r\,dr\,d\theta$.

Now the given integral becomes

$$\int_{r=0}^{1}\int_{\theta=0}^{\pi/2} br\sin\theta \cdot ab\,r\,dr\,d\theta$$



$$= ab^2\int_0^1 r^2\,dr \int_0^{\pi/2} \sin\theta\,d\theta$$

$$= \frac{ab^2}{3}\cdot[-\cos\theta]_0^{\pi/2}$$

$$= \frac{ab^2}{3}[0+1] \quad \text{Ans}$$

$$= \frac{ab^2}{3}$$

Therefore $\iint y\,dx\,dy = 4\times\frac{ab^2}{3}$

$$= \frac{4ab^2}{3}.$$

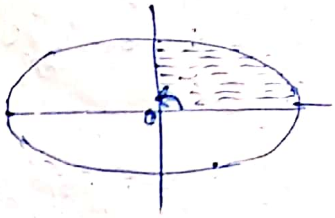
Ex! 14 Show that $\iint x^2 y \, dx dy = \frac{1}{15} a^3 b^2$ taken over the positive quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

$\Rightarrow$ We first integrating on the first quadrant.

putting $x = ar\cos\theta$

$y = br\sin\theta$

$\therefore dxdy = abr \, dr d\theta$.



$\therefore$ The given integral becomes

$$\int_{r=0}^{1} \int_{\theta=0}^{\pi/2} a^2 r^2 \cos^2\theta \cdot br\sin\theta \cdot abr \, dr d\theta .$$

$$= a^3 b^2 \int_0^1 r^4 dr \cdot \int_0^{\pi/2} \sin\theta \cos^2\theta \cdot d\theta .$$

$$= a^3 b^2 \left[\frac{r^5}{5}\right]_0^1 \cdot \frac{1}{4} \int_0^{\pi/2} 2\sin 2\theta \cos\theta \, d\theta .$$

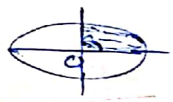
$$= \frac{a^3 b^2}{20} \int_0^{\pi/2} (\sin 3\theta + \sin\theta) \, d\theta .$$

$$= \frac{a^3 b^2}{20} \left[-\frac{\cos 3\theta}{3} - \cos\theta\right]_0^{\pi/2}$$

$$= \frac{a^3 b^2}{20} \left[\frac{1}{3} + 1\right] = \frac{a^3 b^2}{15} \quad \underline{\text{Ans}}$$

Ex! 15 Show that $\iint \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) dxdy = \frac{\pi ab}{8}$ over the +ve quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. [DH-11]

$\Rightarrow$ ~~Here~~ the region of the integration is the shaddow area.



The given integral becomes

putting $x = ar\cos\theta$

$y = br\sin\theta$.

$$\therefore \int_{r=0}^{1} \int_{\theta=0}^{\pi/2} (1 - r^2) \, ab \, r \, dr d\theta$$

$$= ab \int_0^1 (r - r^3) \, dr \cdot \int_0^{\pi/2} d\theta .$$

$$= ab \left[\frac{r^2}{2} - \frac{r^4}{4}\right]_0^1 \cdot [\theta]_0^{\pi/2}$$

$$= ab \left[\frac{1}{2} - \frac{1}{4}\right] \cdot \left[\frac{\pi}{2} - 0\right]$$

$$= \frac{ab}{4} \cdot \frac{\pi}{2} = \frac{\pi ab}{8} ,$$

which is the required prove.

Ex. 16 Show that $\iiint (x+y+z+1)^2\,dx\,dy\,dz = \frac{31}{60}$ taking through out the tetrahedron bounded by the plane $x=0, y=0, z=0, x+y+z=1$.

$\Rightarrow$ putting $x+y+z=u$,

$x+y = uv$, $\therefore$ $z = u(1-v)$

$x = uvw$ $y = uv(1-w)$

Now, $\frac{\partial(x,y,z)}{\partial(u,v,w)} = -u^2v$.

$\therefore dx\,dy\,dz = u^2v\,du\,dv\,dw$.

The new region becomes, $u: 0$ to 1, $v: 0$ to 1, $w: 0$ to 1.

$\int_0^1\int_0^1\int_0^1 (u+1)^2 u^2 v\,du\,dv\,dw$.

$= \int_0^1 (u^4+2u^3+u^2)\,du \cdot \int_0^1 v\,dv \cdot \int_0^1 dw$.

$= \left[\frac{u^5}{5}+\frac{2u^4}{4}+\frac{u^3}{3}\right]_0^1 \left[\frac{v^2}{2}\right]_0^1 \cdot [w]_0^1$

$= \left(\frac{1}{5}+\frac{1}{2}+\frac{1}{3}\right)\cdot\frac{1}{2}\cdot 1 = \frac{31}{60}$.

—o—

Ex. 17 Show that $\iiint (x+y+z)\,x^2y^2z^2\,dx\,dy\,dz = \frac{1}{50400}$ taken throughout the tetrahedron bounded by the planes $x=0, y=0, z=0, x+y+z=1$.

$\Rightarrow$ putting $x+y+z = u$

$x+y = uv$

$x = uvw$ $\therefore dx\,dy\,dz = u^2v\,du\,dv\,dw$

$z = u(1-v)$.

$\therefore$ The given integral becomes,

$\iiint_R (x+y+z)\,x^2y^2z^2\,dx\,dy\,dz$,

$= \int_0^1\int_0^1\int_0^1 u\,(uvw)^2\{uv(1-w)\}^2\{u(1-v)\}^2 u^2v\,du\,dv\,dw$

$= \int_0^1 u^9\,du \cdot \int_0^1 v^5(1-v)^2\,dv \cdot \int_0^1 w^2(1-w)^2\,dw$

$= \frac{1}{10}\left[\frac{v^6}{6}+\frac{v^8}{8}-\frac{2v^7}{7}\right]_0^1 \left[\frac{w^3}{3}+\frac{w^5}{5}-2\frac{w^4}{4}\right]_0^1$

$= \frac{1}{10}\left(\frac{1}{6}+\frac{1}{8}-\frac{2}{7}\right)\left(\frac{1}{3}+\frac{1}{5}-\frac{2}{4}\right)$

$= \frac{1}{10}\cdot\frac{1}{168}\cdot\frac{1}{30} = \frac{1}{50400}$ Ans

—o—

Ex. 18 Prove that $\iiint (x^2+y^2+z^2)\, xyz\, dx\,dy\,dz$ taken throughout the sphere $x^2+y^2+z^2 \le 1$ is zero, CU'84

$\Rightarrow$ put $x = r\cos\phi \sin\theta$
$y = r\sin\phi \sin\theta$
$z = r\cos\theta$

$\therefore x^2+y^2+z^2 = r^2\sin^2\theta(\cos^2\phi+\sin^2\phi) + r^2\cos^2\theta$
$= r^2(\sin^2\theta+\cos^2\theta)$
$= r^2$

$\therefore \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta$ $\therefore dx\,dy\,dz = r^2\sin\theta\, dr\, d\theta\, d\phi$.

Now the given integral becomes,

$$\int_{r=0}^{1}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} r^2 \cdot r^3 \cos\phi\sin\phi \cdot \sin^2\theta \cdot \cos\theta \cdot r^2\sin\theta\, dr\, d\theta\, d\phi.$$

$$= \int_{r=0}^{1} r^7 dr \int_0^{\pi} \sin^3\theta\cos\theta\, d\theta \cdot \int_0^{2\pi} \sin\phi\cos\phi\, d\phi.$$

$$= \frac{1}{8}\int_0^{\pi} \sin^3\theta\cos\theta\, d\theta \cdot \int_0^{2\pi} \sin\phi\, d(\sin\phi)$$

$$= \frac{1}{8}\int_0^{\pi} \sin^3\theta\, d(\sin\theta) \int_0^{2\pi} \sin\phi\, d(\sin\phi)$$

$$= \frac{1}{8}\left[\frac{\sin^4\theta}{4}\right]_0^{\pi}\left[\frac{\sin^2\phi}{2}\right]_0^{2\pi}$$

$$= \frac{1}{8}\cdot\frac{1}{8}\left[\sin^4\pi - \sin^4 0\right]\left[\sin^2 2\pi - \sin^2 0\right] = 0 \quad \underline{\text{Ans}}.$$

Hence the result.

—o—

Ex. 19 Show that $\iiint x^{\alpha} y^{\beta} z^{\gamma} (1-x-y-z)^{\lambda}\, dx\,dy\,dz$

$$= \frac{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\gamma+1)\Gamma(\lambda+1)}{\Gamma(\alpha+\beta+\gamma+\lambda+4)}$$

over the tetrahedron formed by the coordinate plane and the plane $x+y+z=1$, [CU'90, CU'87]

$\Rightarrow$ putting $x+y+z = u$ $\quad\therefore y = uv(1-w)$
$x+y = uv$ $\quad z = u(1-v)$.
$x = uvw$

$dx\,dy\,dz = +u^2 v\, du\, dv\, dw$.

So the given integral becomes

$$\int_0^1\int_0^1\int_0^1 (uvw)^{\alpha}\{uv(1-w)\}^{\beta}\{u(1-v)\}^{\gamma}(1-u)^{\lambda} u^2 v\, du\, dv\, dw$$

$$= \int_0^1 u^{\alpha+\beta+\gamma+2}(1-u)^{\lambda} du \cdot \int_0^1 v^{\alpha+\beta+1}(1-v)^{\gamma} dv \cdot \int_0^1 w^{\alpha}(1-w)^{\beta} dw.$$

$$= \int_0^1 u^{(\alpha+\beta+\gamma+3)-1}(1-u)^{\delta+1-1}\,du \int_0^1 v^{\alpha+\beta+2-1}\cdot(1-v)^{\gamma+1-1}\,dv \int_0^1 w^{\alpha+1-1}(1-w)^{\beta+1-1}\,dw$$

$$= \beta(\alpha+\beta+\gamma+3, \delta+1)\,\beta(\alpha+\beta+2, \gamma+1)\cdot\beta(\alpha+1,\beta+1).$$

$$= \frac{\Gamma(\alpha+\beta+\gamma+3)\Gamma(\delta+1)}{\Gamma(\alpha+\beta+\gamma+\delta+4)}\cdot\frac{\Gamma(\alpha+\beta+2)\Gamma(\gamma+1)}{\Gamma(\alpha+\beta+\gamma+3)}\cdot\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}$$

$$= \frac{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\gamma+1)\Gamma(\delta+1)}{\Gamma(\alpha+\beta+\gamma+\delta+4)}$$ proof.

Ex! 20 Show that $\iiint z^2\,dx\,dy\,dz = \frac{2}{15}\pi a^5$, where the reagion integration is expanded over the hemisphere $z\geq 0$, $x^2+y^2+z^2\leq a^2$, CH-86.

$\Rightarrow$ putting $x = r\cos\phi\sin\theta$
$y = r\sin\phi\sin\theta$
$z = r\cos\theta$ $\quad\therefore dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\phi.$

on the reagion $r: 0$ to a, $\theta: 0$ to $\pi/2$, $\phi: 0$ to 2π.

So the given integral becomes

$$\int_{r=0}^{a}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} r^4\cos^2\theta\sin\theta\,dr\,d\theta\,d\phi.$$

Note!- For the +ve quadrant of the sphere - $x^2+y^2+z^2\leq 1$
$r: 0$ to a
$\theta: 0$ to $\pi/2$
$\phi: 0$ to $\pi/2$

$$= \int_{r=0}^{a} r^4\,dr \int_{\theta=0}^{\pi/2} -\cos^2\theta\,d(\cos\theta)\int_{\phi=0}^{2\pi} d\phi$$

$$= \frac{a^5}{5}\left[-\frac{\cos^3\theta}{3}\right]_0^{\pi/2}\cdot 2\pi = \frac{2a^5\pi}{15}$$ ANS.

—o—

Ex! 21 Show that $\int_0^1 dx\int_0^1 dy\,\frac{x-y}{(x+y)^3} \neq \int_0^1 dy\int_0^1 \frac{x-y}{(x+y)^3}\,dx$ [CH-92].

Does the double integral exist over $R:[0,1;0,1]$.

$$\Rightarrow \text{L.H.S} = \int_0^1 dx\int_0^1 \frac{x-y}{(x+y)^3}\,dy$$

$$= \int_0^1 dx\int_0^1 \frac{2x-(x+y)}{(x+y)^3}\,dy$$

$$= \int_0^1 dx\int_0^1\left\{\frac{2x}{(x+y)^3} - \frac{1}{(x+y)^2}\right\}dy$$

$$= \int_0^1 dx\left[-\frac{2x}{2(x+y)^2} + \frac{1}{x+y}\right]_0^1$$

$$= \int_0^1\left[-\frac{x}{(x+1)^2} + \frac{1}{x+1}\right]dx$$

$$= \int_0^1 \frac{-x+x+1}{(x+1)^2}\,dx = \left[-\frac{1}{x+1}\right]_0^1 = 1/2.$$

$$R.H.S = \int_0^1 dy \int_0^1 \frac{x-y}{(x+y)^3}\,dx$$

$$= \int_0^1 dy \int_0^1 \frac{(x+y)-2y}{(x+y)^3}\,dx = \int_0^1 dy \int_0^1 \left\{ \frac{1}{(x+y)^2} - \frac{2y}{(x+y)^3} \right\} dx$$

$$= \int_0^1 dy \left[-\frac{1}{x+y} + \frac{y}{(x+y)^2} \right]_0^1$$

$$= \int_0^1 \left[-\frac{1}{1+y} + \frac{y}{(1+y)^2} + \frac{1}{y} - \frac{y}{y^2} \right] dy$$

$$= \int_0^1 \frac{y-1-y}{(1+y)^2}\,dy = -\int_0^1 \frac{1}{(1+y)^2}\,dy = \left[\frac{1}{1+y}\right]_0^1 = \frac{1}{2} - 1 = -1/2.$$

$\therefore$ L.H.S $\neq$ R.H.S

Ex! 22 P-611 (MPA). Show that, $\int_0^1 dx \int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}\,dy \neq \int_0^1 dy \int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}\,dx$. Does the double integral $\iint_R \frac{x^2-y^2}{(x^2+y^2)^2}\,dx\,dy$ exist over $R:[0,1;0,1]$

$$\Rightarrow L.H.S = \int_0^1 dx \int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}\,dy$$

$$= \int_0^1 dx \int_0^1 \frac{2x^2-(x^2+y^2)}{(x^2+y^2)^2}\,dy = \int_0^1 dx \int_0^1 \left\{ \frac{2x^2}{(x^2+y^2)^2} - \frac{1}{x^2+y^2} \right\} dy$$

$$= \int_0^1 dx \left[2x^2 \left\{ \frac{1}{2x^3}\left(\tan^{-1} y/x + \frac{xy}{x^2+y^2} \right) \right\} - \frac{1}{x}\tan^{-1} y/x \right]_0^1$$

By put $y = x\tan\theta$

$$= \int_0^1 \left[\frac{1}{x}\tan^{-1} y/x + \frac{y}{x^2+y^2} - \frac{1}{x}\tan^{-1} y/x \right]_0^1 dx$$

$$= \int_0^1 \frac{1}{x^2+1}\,dx = \left[\tan^{-1} x\right]_0^1 = \pi/4.$$

$$R.H.S = \int_0^1 dy \int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}\,dx$$

$$= \int_0^1 dy \int_0^1 \frac{(x^2+y^2)-2y^2}{(x^2+y^2)^2}\,dx = \int_0^1 dy \int_0^1 \left\{ \frac{1}{x^2+y^2} - \frac{2y^2}{(x^2+y^2)^2} \right\} dx.$$

$$= \int_0^1 dy \left[\frac{1}{y}\tan^{-1} x/y - 2y^2 \left\{ \frac{1}{2y^3}\left(\tan^{-1} x/y + \frac{xy}{x^2+y^2} \right) \right\} \right]_0^1$$

$$= \int_0^1 \left[\frac{1}{y}\tan^{-1} x/y - \frac{1}{y}\tan^{-1} x/y - \frac{x}{x^2+y^2} \right]_0^1 dy$$

$$= \int_0^1 -\frac{1}{1+y^2}\,dy$$

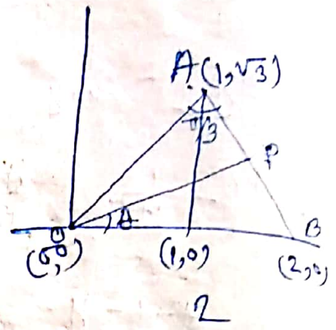
$$= -\left[\tan^{-1} y\right]_0^1 = -\pi/4..$$

$\therefore$ L.H.S $\neq$ R.H.S

2nd part: Since two iterated integrals exist but are unequal and so the double integral cannot exist over R.

Ex 2B Evaluate $\iint \frac{dx\,dy}{(1+x^2+y^2)^2}$ over the triangle whose vertices are $(0,0), (2,0), (1,\sqrt{3})$.

$\Rightarrow$ The triangle which is the field of integration is unchanged by transformation to polars, on the triangle which is a equilateral triangle with side 2 vertices from 0 to $\pi/3$. and vertices from 0 to op from the triangle OAP



Now, $\frac{op}{\sin OAP} = \frac{OA}{\sin OPA}$ or, $\frac{op}{\sin \pi/3} = \frac{2}{\sin(\pi/3+\theta)}$ when $\left[\begin{array}{l}\theta \to 0 \text{ to } \pi/3 \\ \varphi \to \pi/3 \text{ to } 2\pi/3\end{array}\right.$

or, $op = \frac{\sqrt{3}}{\sin\varphi}$, where $\varphi = (\pi/3+\theta)$.

The required integral

$$= \frac{1}{2}\int_{\pi/3}^{2\pi/3} d\varphi \int_0^{\sqrt{3}/\sin\varphi} \frac{2r\,dr}{(1+r^2)^2}$$

where $x = r\cos\varphi$, $y = r\sin\varphi$.

$$= \frac{1}{2}\int_{\pi/3}^{2\pi/3} d\varphi \left[-\frac{1}{1+r^2}\right]_0^{\sqrt{3}/\sin\varphi}$$

$$= \frac{1}{2}\int_{\pi/3}^{2\pi/3} \left(-\frac{1}{1+\frac{3}{\sin^2\varphi}} + 1\right) d\varphi.$$

$$= \frac{1}{2}\int_{\pi/3}^{2\pi/3} \left(1 - \frac{\sin^2\varphi}{3+\sin^2\varphi}\right) d\varphi.$$

$$\frac{1}{2}\int_{\pi/3}^{2\pi/3} \frac{3\sec^2\varphi\,d\varphi}{3+4\tan^2\varphi}.$$

$$= \frac{1}{2}\int_{\pi/3}^{2\pi/3} \frac{3}{3+\sin^2\varphi}\,d\varphi. = \frac{3}{2}\int_{\pi/3}^{2\pi/3} \frac{d\varphi}{4-\cos^2\varphi}\,d\varphi.$$

$$= \frac{3}{2}\cdot\frac{1}{4}\int_{\pi/3}^{2\pi/3} \left(\frac{1}{2+\cos\varphi} + \frac{1}{2-\cos\varphi}\right) d\varphi.$$

$$= \frac{3}{8}\int_{\pi/3}^{2\pi/3} \left[\frac{1}{2(\cos^2\varphi/2+\sin^2\varphi/2)+(\cos^2\varphi/2-\sin^2\varphi/2)} + \frac{1}{2(\cos^2\varphi/2+\sin^2\varphi/2)-(\cos^2\varphi/2-\sin^2\varphi/2)}\right] d\varphi$$

$$= \frac{3}{8}\int_{\pi/3}^{2\pi/3} \left(\frac{1}{3\cos^2\varphi/2+\sin^2\varphi/2} + \frac{1}{\cos^2\varphi/2+3\sin^2\varphi/2}\right) d\varphi.$$

$$= \frac{3}{8}\int_{\pi/3}^{2\pi/3}\left(\frac{\sec^2\varphi/2}{3+\tan^2\varphi/2} + \frac{2\, d(\tan\varphi/2)}{1+3\tan^2\varphi/2}\right) d\varphi.$$

$$= \frac{3}{8}\left[\frac{2}{\sqrt3}\tan^{-1}\left(\frac{\tan(\varphi/2)}{\sqrt3}\right) + \frac{2\sqrt3}{3}\tan^{-1}\left(\frac{\tan\varphi/2}{1/\sqrt3}\right)\right]_{\pi/3}^{2\pi/3}$$

$$= \frac{3}{8}\left[\frac{2}{\sqrt3}\tan^{-1}\left(\frac{1}{\sqrt3}\tan\varphi/2\right) + \frac{2}{\sqrt3}\tan^{-1}(\sqrt3\tan\varphi/2)\right]_{\pi/3}^{2\pi/3}.$$

$$= \frac{\sqrt3}{4}\left(\tan^{-1}1 + \tan^{-1}3 - \tan^{-1}1/3 - \tan^{-1}1\right)$$

$$= \frac{\sqrt3}{4}\tan^{-1}\left(\frac{3-\frac{1}{3}}{1+3\cdot 1/3}\right) = \frac{\sqrt3}{4}\tan^{-1}\left(\frac{8}{6}\right)$$

$$= \frac{\sqrt3}{4}\tan^{-1}4/3 = \frac{\sqrt3}{2}\tan^{-1}\frac{1}{2}$$

— o —

Ex: 25 Show that $\iiint \frac{dx\,dy\,dz}{x^2+y^2+(z-2)^2} = \pi\left(2 - \frac{3}{2}\log 3\right)$.

extended over the sphere $x^2+y^2+z^2 \le 1$.

$\Rightarrow$ putting $x = r\sin\theta\cos\varphi$

$y = r\sin\theta\sin\varphi$ $\quad \therefore dx\,dy\,dz = r^2\sin\theta\, dr\,d\theta\,d\varphi.$

$z = r\cos\theta.$

$\therefore$ The reagion becomes, $\quad r = 0$ to 1

$\theta = 0$ to π

$\varphi = 0$ to 2π.

$\therefore$ The integral becomes,

$$\int_{r=0}^{1}\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi} \frac{r^2\sin\theta\cdot dr\,d\theta\,d\varphi}{r^2\sin^2\theta + (r\cos\theta-2)^2}.$$

$$= \int_{r=0}^{1}\int_{\theta=0}^{\pi} \frac{r^2\sin\theta\, dr\,d\theta}{r^2-4r\cos\theta+4}\cdot\int_0^{2\pi} d\varphi.$$

$$= \int_0^1\int_0^\pi \frac{r^2\sin\theta\, dr\,d\theta}{r^2-4r\cos\theta+4}\cdot 2\pi.$$

put $t = r^2 - 4r\cos\theta + 4$

$dt = 4r\sin\theta\, d\theta$

θ	0	π
t	$(2-r)^2$	$(2+r)^2$

$$= \frac{2\pi}{4}\int_0^1 dr\int_{(2-r)^2}^{(2+r)^2} \frac{r\,dt}{t}$$

$$= \frac{2\pi}{4}\int_0^1 dr\,\left[r\log|t|\right]_{(2-r)^2}^{(2+r)^2}$$

$$= \frac{2\pi}{4}\int_0^1 dr\,\left[r\log(2+r)^2 - r\log(2-r)^2\right]$$

$$= \frac{\pi}{2}\int_0^1 2r\log\left|\frac{2+r}{2-r}\right|\cdot dr.$$

$$= \pi\left[\frac{x^2}{2}\log\left(\frac{2+x}{2-x}\right)\right]_0^1 - \pi\int_0^1 \frac{2-x}{2+x}\cdot\frac{2-x+x+2}{(2-x)^2}\cdot\frac{x^2}{2}\,dx$$

$$= \pi\,\frac{1}{2}\log 3 - \pi/2\int_0^1 \frac{4x^2}{4-x^2}\,dx.$$

$$= \frac{\pi}{2}\log 3 + \frac{\pi}{2}\cdot 4\int_0^1 \frac{4-x^2-4}{4-x^2}\,dx.$$

$$= \frac{\pi}{2}\log 3 + 2\pi\int_0^1\left(1-\frac{4}{4-x^2}\right)dx.$$

$$= \frac{\pi}{2}\log 3 + 2\pi\,[x]_0^1 - 2\pi\int_0^1\left[\frac{1}{2+x}+\frac{1}{2-x}\right]dx.$$

$$= \frac{\pi}{2}\log 3 + 2\pi - 2\pi\left[\log\left(\frac{2+x}{2-x}\right)\right]_0^1$$

$$= \frac{\pi}{2}\log 3 + 2\pi - 2\pi\log 3$$

$$= 2\pi - \frac{3\pi}{2}\log 3$$

$$= \pi\left(2-\frac{3}{2}\log 3\right) \quad \text{Ans}.$$

Ex) 26. Show that if R be the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

(i) $\iiint_R [a^2b^2c^2 - b^2c^2x^2 - c^2a^2y^2 - a^2b^2z^2]^{1/2}\,dx\,dy\,dz = \frac{1}{4}\pi a^2b^2c^2.$

(ii) $\iiint_R xyz\,dx\,dy\,dz = \frac{1}{6}a^2b^2c^2.$ — C.H. 2001

$\Rightarrow$ $x = ar\sin\theta\cos\phi$

$y = br\sin\theta\sin\phi$

$z = cr\cos\theta.$ $\quad \therefore dx\,dy\,dz = r^2abc\sin\theta\,dr\,d\theta\,d\phi$

Then the new region be $r = 0$ to 1

$\theta = 0$ to π

$\phi = 0$ to 2π.

The integral becomes,

$$\int_{r=0}^{1}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}[a^2b^2c^2 - r^2b^2c^2a^2)^{1/2}\,r^2abc\sin\theta\,dr\,d\theta\,d\phi.$$

$$= a^2b^2c^2\int_0^1 r^2\sqrt{1-r^2}\,dr\int_0^\pi \sin\theta\,d\theta\int_0^{2\pi}d\phi$$

$r = \cos\alpha$

$dr = -\sin\alpha\,d\alpha$

r	0	1
α	$\frac{\pi}{2}$	0

$$= a^2b^2c^2\int_{\pi/2}^{0}\cos^2\alpha\cdot\sin^2\alpha\,d\alpha\,[-\cos\theta]_0^\pi\cdot 2\pi$$

$$= \frac{1}{4}a^2b^2c^2\int_0^{\pi/2}\sin^2 2\alpha\,d\alpha\,(1+1)\cdot 2\pi$$

$= \frac{4\pi}{4} a^2 b^2 c^2 \cdot \frac{1}{2} \int_0^{\pi/2} (1-\cos 4\alpha) d\alpha$

$= \frac{\pi}{2} a^2 b^2 c^2 \left[\alpha - \frac{\sin 4\alpha}{4}\right]_0^{\pi/2}$

$= \frac{\pi}{2} a^2 b^2 c^2 (\pi/2 - 0) = \frac{\pi^2}{4} a^2 b^2 c^2.$

(ii) $\int_{r=0}^{1}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} a r^3 \sin^2\theta \cdot \cos\phi \cdot \cos\theta \sin\phi \cdot abc\, r^2 \sin\theta\, dr d\theta d\phi$

$= \frac{a^2 b^2 c^2}{2} \int_0^1 r^5 dr \int_0^{\pi} \sin^3\theta \cos\theta d\theta \int_0^{2\pi} \sin 2\phi d\phi.$

$= \frac{a^2 b^2 c^2}{2} \cdot \frac{1}{6} \left[\frac{1}{4} \sin^4\theta\right]_0^{\pi} \left[-\frac{\cos 2\phi}{2}\right]_0^{2\pi}$??

$= \frac{a^2 b^2 c^2}{6}$ ANS

Ex1: Show that $\iiint \frac{dx dy dz}{(x+y+z+1)^3} = \frac{1}{16} \log\frac{256}{e5}$, taken over the tetrahedron bounded by the plane $x=0$, $y=0$, $x+z+y=1$.

$\Rightarrow$ putting $x+y+z=u$, $x+y=uv$, $x=uvw$

$y=uv(1-w)$, $z=u(1-v)$.

$\therefore dx dy dz = u^2 v\, du dv dw.$

$\therefore \int_{u=0}^{1}\int_{v=0}^{1}\int_{w=0}^{1} \frac{u^2 v}{(1+u)^3} du dv dw$

$= \int_0^1 \frac{u^2}{(1+u)^3} du \cdot \int_0^1 v dv \int_0^1 dw.$

put $u+1=z$
$\therefore u=(z-1)$

$= \int_0^1 \left[\frac{1}{(1+u)^3} - \frac{2}{(1+u)^2} + \frac{1}{1+u}\right] du \quad \left[\frac{v^2}{2}\right]_0^1 \cdot [w]_0^1$

$= \cancel{\frac{1}{2}}\left[-\cancel{\frac{1}{2(1+u)^2}} - \frac{2}{(1+u)^2} + \cancel{\frac{1}{1+u}}\right]$

$= \frac{1}{2}\left[-\frac{1}{2(1+u)^2} + \frac{2}{1+u} + \log(1+u)\right]_0^1$

$= \frac{1}{2}\left[-\frac{1}{8} + 1 + \log 2 + \frac{1}{2} - 2\right] = \frac{1}{2}\left[\log 2 - \frac{5}{8}\right]$

$= \frac{1}{16}(8\log 2 - \log 5) = \frac{1}{16}(\log 256 - 5\log e)$

$= \frac{1}{16} \log\frac{256}{e^5}$ ANS.