Chapter 10

Potentials and Fields

10.1 The Potential Formulation

10.1.1 Scalar and Vector Potentials

In this chapter we ask how the sources (ρ and $\mathbf{J}$) generate electric and magnetic fields; in other words, we seek the *general* solution to Maxwell's equations,

$$\left.\begin{array}{llll} \text{(i)} & \nabla \cdot \mathbf{E} = \dfrac{1}{\epsilon_0}\rho, & \text{(iii)} & \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}, \\ \text{(ii)} & \nabla \cdot \mathbf{B} = 0, & \text{(iv)} & \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \dfrac{\partial \mathbf{E}}{\partial t}. \end{array}\right\} \tag{10.1}$$

Given $\rho(\mathbf{r}, t)$ and $\mathbf{J}(\mathbf{r}, t)$, what are the fields $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$? In the static case Coulomb's law and the Biot-Savart law provide the answer. What we're looking for, then, is the generalization of those laws to time-dependent configurations.

This is not an easy problem, and it pays to begin by representing the fields in terms of potentials. In electrostatics $\nabla \times \mathbf{E} = 0$ allowed us to write $\mathbf{E}$ as the gradient of a scalar potential: $\mathbf{E} = -\nabla V$. In electro*dynamics* this is no longer possible, because the curl of $\mathbf{E}$ is nonzero. But $\mathbf{B}$ remains divergenceless, so we can still write

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A},} \tag{10.2}$$

as in magnetostatics. Putting this into Faraday's law (iii) yields

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}(\nabla \times \mathbf{A}),$$

or

$$\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right) = 0.$$

Here is a quantity, unlike **E** alone, whose curl *does* vanish; it can therefore be written as the gradient of a scalar:

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla V.$$

In terms of V and $\mathbf{A}$, then,

$$\boxed{\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}.} \tag{10.3}$$

This reduces to the old form, of course, when **A** is constant.

The potential representation (Eqs. 10.2 and 10.3) automatically fulfills the two homogeneous Maxwell equations, (ii) and (iii). How about Gauss's law (i) and the Ampère/Maxwell law (iv)? Putting Eq. 10.3 into (i), we find that

$$\nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\frac{1}{\epsilon_0}\rho; \tag{10.4}$$

this replaces Poisson's equation (to which it reduces in the static case). Putting Eqs. 10.2 and 10.3 into (iv) yields

$$\nabla \times (\nabla \times \mathbf{A}) = \mu_0 \mathbf{J} - \mu_0\epsilon_0 \nabla\left(\frac{\partial V}{\partial t}\right) - \mu_0\epsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2},$$

or, using the vector identity $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$, and rearranging the terms a bit:

$$\left(\nabla^2\mathbf{A} - \mu_0\epsilon_0\frac{\partial^2 \mathbf{A}}{\partial t^2}\right) - \nabla\left(\nabla \cdot \mathbf{A} + \mu_0\epsilon_0\frac{\partial V}{\partial t}\right) = -\mu_0\mathbf{J}. \tag{10.5}$$

Equations 10.4 and 10.5 contain all the information in Maxwell's equations.

Example 10.1

Find the charge and current distributions that would give rise to the potentials

$$V = 0, \quad \mathbf{A} = \begin{cases} \dfrac{\mu_0 k}{4c}(ct - |x|)^2\,\hat{\mathbf{z}}, & \text{for } |x| < ct, \\ 0, & \text{for } |x| > ct, \end{cases}$$

where k is a constant, and $c = 1/\sqrt{\epsilon_0\mu_0}$.

Solution: First we'll determine the electric and magnetic fields, using Eqs. 10.2 and 10.3:

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} = -\frac{\mu_0 k}{2}(ct - |x|)\,\hat{\mathbf{z}},$$

$$\mathbf{B} = \nabla \times \mathbf{A} = -\frac{\mu_0 k}{4c}\frac{\partial}{\partial x}(ct - |x|)^2\,\hat{\mathbf{y}} = \pm\frac{\mu_0 k}{2c}(ct - |x|)\,\hat{\mathbf{y}},$$

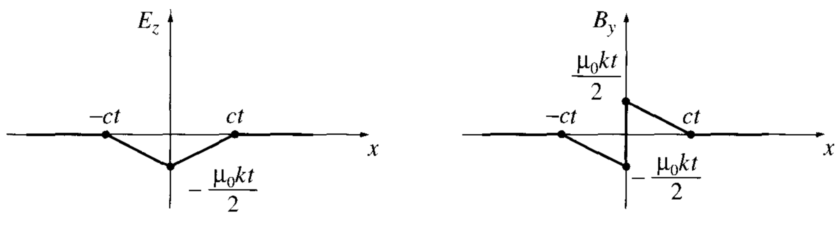

Figure 10.1

(plus for $x > 0$, minus for $x < 0$). These are for $|x| < ct$; when $|x| > ct$, $\mathbf{E} = \mathbf{B} = 0$ (Fig.10.1). Calculating every derivative in sight, I find

$$\nabla \cdot \mathbf{E} = 0; \quad \nabla \cdot \mathbf{B} = 0; \quad \nabla \times \mathbf{E} = \mp \frac{\mu_0 k}{2} \hat{\mathbf{y}}; \quad \nabla \times \mathbf{B} = -\frac{\mu_0 k}{2c} \hat{\mathbf{z}};$$

$$\frac{\partial \mathbf{E}}{\partial t} = -\frac{\mu_0 kc}{2} \hat{\mathbf{z}}; \quad \frac{\partial \mathbf{B}}{\partial t} = \pm \frac{\mu_0 k}{2} \hat{\mathbf{y}}.$$

As you can easily check, Maxwell's equations are all satisfied, with ρ and $\mathbf{J}$ both *zero*. Notice, however, that $\mathbf{B}$ has a discontinuity at $x = 0$, and this signals the presence of a surface current $\mathbf{K}$ in the yz plane; boundary condition (iv) in Eq. 7.63 gives

$$kt\,\hat{\mathbf{y}} = \mathbf{K} \times \hat{\mathbf{x}},$$

and hence

$$\mathbf{K} = kt\,\hat{\mathbf{z}}.$$

Evidently we have here a uniform surface current flowing in the z direction over the plane $x = 0$, which starts up at $t = 0$, and increases in proportion to t. Notice that the news travels out (in both directions) at the speed of light: for points $|x| > ct$ the message (that current is now flowing) has not yet arrived, so the fields are zero.

Problem 10.1 Show that the differential equations for V and $\mathbf{A}$ (Eqs. 10.4 and 10.5) can be written in the more symmetrical form

$$\left.\begin{aligned} \Box^2 V + \frac{\partial L}{\partial t} &= -\frac{1}{\epsilon_0}\rho, \\ \Box^2 \mathbf{A} - \nabla L &= -\mu_0 \mathbf{J}. \end{aligned}\right\} \tag{10.6}$$

where

$$\Box^2 \equiv \nabla^2 - \mu_0\epsilon_0 \frac{\partial^2}{\partial t^2} \quad \text{and} \quad L \equiv \nabla \cdot \mathbf{A} + \mu_0\epsilon_0 \frac{\partial V}{\partial t}.$$

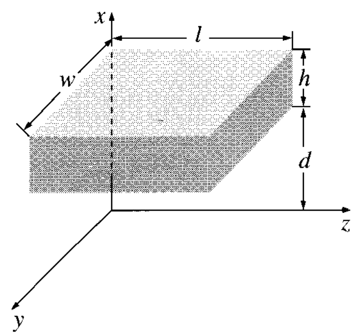

Figure 10.2

Problem 10.2 For the configuration in Ex. 10.1, consider a rectangular box of length l, width w, and height h, situated a distance d above the yz plane (Fig. 10.2).

(a) Find the energy in the box at time $t_1 = d/c$, and at $t_2 = (d+h)/c$.

(b) Find the Poynting vector, and determine the energy per unit time flowing into the box during the interval $t_1 < t < t_2$.

(c) Integrate the result in (b) from t_1 to t_2 and confirm that the increase in energy (part (a)) equals the net influx.

10.1.2 Gauge Transformations

Equations 10.4 and 10.5 are *ugly*, and you might be inclined at this stage to abandon the potential formulation altogether. However, we *have* succeeded in reducing six problems—finding $\mathbf{E}$ and $\mathbf{B}$ (three components each)—down to four: V (one component) and $\mathbf{A}$ (three more). Moreover, Eqs. 10.2 and 10.3 do not uniquely define the potentials; we are free to impose extra conditions on V and $\mathbf{A}$, as long as nothing happens to $\mathbf{E}$ and $\mathbf{B}$. Let's work out precisely what this **gauge freedom** entails. Suppose we have two sets of potentials, $(V, \mathbf{A})$ and $(V', \mathbf{A}')$, which correspond to the *same* electric and magnetic fields. By how much can they differ? Write

$$\mathbf{A}' = \mathbf{A} + \boldsymbol{\alpha} \quad \text{and} \quad V' = V + \beta.$$

Since the two $\mathbf{A}$'s give the same $\mathbf{B}$, their curls must be equal, and hence

$$\nabla \times \boldsymbol{\alpha} = 0.$$

We can therefore write $\boldsymbol{\alpha}$ as the gradient of some scalar:

$$\boldsymbol{\alpha} = \nabla \lambda.$$

The two potentials also give the same **E**, so

$$\nabla \beta + \frac{\partial \boldsymbol{\alpha}}{\partial t} = 0,$$

or

$$\nabla \left(\beta + \frac{\partial \lambda}{\partial t} \right) = 0.$$

The term in parentheses is therefore independent of position (it could, however, depend on time); call it $k(t)$:

$$\beta = -\frac{\partial \lambda}{\partial t} + k(t).$$

Actually, we might as well absorb $k(t)$ into λ, defining a new λ by adding $\int_0^t k(t')dt'$ to the old one. This will not affect the gradient of λ; it just adds $k(t)$ to $\partial\lambda/\partial t$. It follows that

$$\left.\begin{aligned} \mathbf{A}' &= \mathbf{A} + \nabla \lambda, \\ V' &= V - \frac{\partial \lambda}{\partial t}. \end{aligned}\right\} \tag{10.7}$$

Conclusion: For any old scalar function λ, we can with impunity add $\nabla\lambda$ to **A**, provided we simultaneously subtract $\partial\lambda/\partial t$ from V. None of this will affect the physical quantities **E** and **B**. Such changes in V and **A** are called **gauge transformations**. They can be exploited to adjust the divergence of **A**, with a view to simplifying the "ugly" equations 10.4 and 10.5. In magnetostatics, it was best to choose $\nabla \cdot \mathbf{A} = 0$ (Eq. 5.61); in electrodynamics the situation is not so clear cut, and the most convenient gauge depends to some extent on the problem at hand. There are many famous gauges in the literature; I'll show you the two most popular ones.

Problem 10.3 Find the fields, and the charge and current distributions, corresponding to

$$V(\mathbf{r}, t) = 0, \quad \mathbf{A}(\mathbf{r}, t) = -\frac{1}{4\pi\epsilon_0}\frac{qt}{r^2}\,\hat{\mathbf{r}}.$$

Problem 10.4 Suppose $V = 0$ and $\mathbf{A} = A_0 \sin(kx - \omega t)\,\hat{\mathbf{y}}$, where A_0, ω, and k are constants. Find **E** and **B**, and check that they satisfy Maxwell's equations in vacuum. What condition must you impose on ω and k?

Problem 10.5 Use the gauge function $\lambda = -(1/4\pi\epsilon_0)(qt/r)$ to transform the potentials in Prob. 10.3, and comment on the result.

10.1.3 Coulomb Gauge and Lorentz* Gauge

The Coulomb Gauge. As in magnetostatics, we pick

$$\nabla \cdot \mathbf{A} = 0. \tag{10.8}$$

With this, Eq. 10.4 becomes

$$\nabla^2 V = -\frac{1}{\epsilon_0}\rho. \tag{10.9}$$

This is Poisson's equation, and we already know how to solve it: setting $V = 0$ at infinity,

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}', t)}{\mathscr{r}}\, d\tau'. \tag{10.10}$$

Don't be fooled, though—unlike electrostatics, V by itself doesn't tell you $\mathbf{E}$; you have to know $\mathbf{A}$ as well (Eq. 10.3).

There is a peculiar thing about the scalar potential in the Coulomb gauge: it is determined by the distribution of charge *right now*. If I move an electron in my laboratory, the potential V on the moon immediately records this change. That sounds particularly odd in the light of special relativity, which allows no message to travel faster than the speed of light. The point is that V *by itself* is not a physically measurable quantity—all the man in the moon can measure is $\mathbf{E}$, and that involves $\mathbf{A}$ as well. Somehow it is built into the vector potential, in the Coulomb gauge, that whereas V instantaneously reflects all changes in ρ, the combination $-\nabla V - (\partial \mathbf{A}/\partial t)$ does *not*; $\mathbf{E}$ will change only after sufficient time has elapsed for the "news" to arrive.[1]

The *advantage* of the Coulomb gauge is that the *scalar* potential is particularly simple to calculate; the *disadvantage* (apart from the acausal appearance of V) is that $\mathbf{A}$ is particularly *difficult* to calculate. The differential equation for $\mathbf{A}$ (10.5) in the Coulomb gauge reads

$$\nabla^2 \mathbf{A} - \mu_0\epsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J} + \mu_0\epsilon_0 \nabla\left(\frac{\partial V}{\partial t}\right). \tag{10.11}$$

The Lorentz gauge. In the Lorentz gauge we pick

$$\boxed{\nabla \cdot \mathbf{A} = -\mu_0\epsilon_0 \frac{\partial V}{\partial t}.} \tag{10.12}$$

This is designed to eliminate the middle term in Eq. 10.5 (in the language of Prob. 10.1, it sets $L = 0$). With this

$$\nabla^2 \mathbf{A} - \mu_0\epsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J}. \tag{10.13}$$

Meanwhile, the differential equation for V, (10.4), becomes

$$\nabla^2 V - \mu_0\epsilon_0 \frac{\partial^2 V}{\partial t^2} = -\frac{1}{\epsilon_0}\rho. \tag{10.14}$$

*There is some question whether this should be attibuted to H. A. Lorentz or to L. V. Lorenz (see J. Van Bladel, *IEEE Antennas and Propagation Magazine* **33**(2), 69 (1991)). But all the standard textbooks include the t, and to avoid possible confusion I shall adhere to that practice.

[1] See O. L. Brill and B. Goodman. *Am. J. Phys.* **35**, 832 (1967).

The virtue of the Lorentz gauge is that it treats V and $\mathbf{A}$ on an equal footing: the same differential operator

$$\boxed{\nabla^2 - \mu_0\epsilon_0\frac{\partial^2}{\partial t^2} \equiv \Box^2,} \tag{10.15}$$

(called the **d'Alembertian**) occurs in both equations:

$$\boxed{\begin{aligned} &\text{(i)} \quad \Box^2 V = -\frac{1}{\epsilon_0}\rho, \\ &\text{(ii)} \quad \Box^2 \mathbf{A} = -\mu_0 \mathbf{J}. \end{aligned}} \tag{10.16}$$

This democratic treatment of V and $\mathbf{A}$ is particularly nice in the context of special relativity, where the d'Alembertian is the natural generalization of the Laplacian, and Eqs. 10.16 can be regarded as four-dimensional versions of Poisson's equation. (In this same spirit the wave equation, for propagation speed c, $\Box^2 f = 0$, might be regarded as the four-dimensional version of Laplace's equation.) In the Lorentz gauge V and $\mathbf{A}$ satisfy the **inhomogeneous wave equation**, with a "source" term (in place of zero) on the right. From now on I shall use the Lorentz gauge exclusively, and the whole of electrodynamics reduces to the problem of *solving the inhomogeneous wave equation for specified sources*. That's my project for the next section.

Problem 10.6 Which of the potentials in Ex. 10.1, Prob. 10.3, and Prob. 10.4 are in the Coulomb gauge? Which are in the Lorentz gauge? (Notice that these gauges are not mutually exclusive.)

Problem 10.7 In Chapter 5, I showed that it is always possible to pick a vector potential whose divergence is zero (Coulomb gauge). Show that it is always possible to choose $\nabla \cdot \mathbf{A} = -\mu_0\epsilon_0(\partial V/\partial t)$, as required for the Lorentz gauge, assuming you know how to solve equations of the form 10.16. Is it always possible to pick $V = 0$? How about $\mathbf{A} = 0$?

10.2 Continuous Distributions

10.2.1 Retarded Potentials

In the static case, Eqs. 10.16 reduce to (four copies of) Poisson's equation,

$$\nabla^2 V = -\frac{1}{\epsilon_0}\rho, \quad \nabla^2 \mathbf{A} = -\mu_0 \mathbf{J},$$

with the familiar solutions

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{\mathscr{r}}\,d\tau', \qquad \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')}{\mathscr{r}}\,d\tau', \tag{10.17}$$