MTMHCC :: C8T

CC-8 : Riemann Integration and Series of functions.

Unit 2 : Improper Integrals. Convergence of Beta and Gamma functions. Properties of Beta & Gamma functions.

Unit 3 : Pointwise and Uniform Convergence of Sequence of functions. Theorems on Continuity, derivability and integrability of the limit function of a sequence of functions. Series of functions.
Theorems on the Continuity and derivability of the sum function of a series of functions. Cauchy criterion for uniform convergence and Weierstrass M-Test.

Unit 4 : Fourier Series: Definition of Fourier Coefficients and Series, Reimann Lebesgue lemma, Bessel's inequality, Parseval's identity, Dirichlet's Condition. Examples of Fourier expansions and Summation results for series.

Unit 5 : Power Series, radius of Convergence, Cauchy Hadamard Theorem. Differentiation and Integration of power series; Abel's theorem; Weierstrass approximation theorem.

V.V.I.

Theorem: If f and g are positive and $f(x) \leq g(x)$, for all x in (a, X), then

(i) $\int_a^\infty f\,dx$ converges, if $\int_a^\infty g\,dx$ converges

and (ii) $\int_a^\infty g\,dx$ diverges, if $\int_a^\infty f\,dx$ diverges.

Proof. Let f and g be both bounded and integrable in (a, X), $X \geq a$.

Since f and g are both positive, and

$$f(x) \leq g(x), \ \forall\ x \in [a, X]$$

$$\therefore \int_a^X f(x)\,dx \leq \int_a^X g(x)\,dx \quad \text{——— (1)}$$

(i) Let $\int_a^\infty g\,dx$ be convergent so that there exists a positive number M such that

$$\int_a^X g\,dx < M, \quad X \geq a$$

Hence from equn. (1), we have

$$\int_a^X f\,dx < M, \quad X \geq a$$

Hence $\int_a^\infty f\,dx$ is convergent.

(ii) If $\int_a^\infty f\,dx$ is divergent then the positive function $\int_a^X f\,dx$ is not bounded above and therefore in view of (1), $\int_a^X g\,dx$ is also not bounded above.

Hence $\int_a^\infty g\,dx$ diverges.

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Improper Integrals:

The integral $\int_a^b f(x)dx$ as defined by Riemann's has a definite meaning only when both a and b are finite and $f(x)$ is bounded in (a,b).

So we require new definitions if

(1) The range of integration is infinite i.e. either a or b or both are infinite.

(2) The function $f(x)$ is not bounded in (a,b) although a and b are finite.

These two types of integrals are called improper integrals. Usually the term infinite is used for integrals of the type(1) and improper integral for integrals of type(2).

The two types of integrals were defined by cauchy as follows:

First Type:

(1) $\int_a^{\infty} f(x)dx = \underset{x\to\infty}{Lt} \int_a^{x} f(x)dx$, provided the limit exists and $f(x)$ is bounded and integrable in (a,x).

(2) $\int_{-\infty}^{b} f(x)dx = \underset{x\to-\infty}{Lt} \int_x^{b} f(x)dx$, provided the limit exists and $f(x)$ is bounded and integrable in (x,b).

(3) $\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{a} f(x)dx + \int_a^{\infty} f(x)dx$

$$= \underset{x\to-\infty}{Lt} \int_x^a f(x)dx + \underset{x\to\infty}{Lt} \int_a^x f(x)dx,$$

provided both the limits exist and $f(x)$ is bounded and integrable in (x,a) and (a,x).

Note: If the limits exists in the above two cases then the integral is said to Converge. It diverges if the limit tends to infinity. In case (3) both the limits must exist in order to Converge.

Second Type:

(1) If $f(x)$ has an infinite discontinuity only at the left hand end point a, then

$$\int_a^b f(x)dx = \underset{\epsilon\to0+}{Lt} \int_{a+\epsilon}^{b} f(x)dx.$$

(2) If $f(x)$ has an infinite discontinuity only at the right hand end point b, then

$$\int_a^b f(x)dx = \underset{\epsilon\to0+}{Lt} \int_a^{b-\epsilon} f(x)dx.$$

(3) If $f(x)$ has an infinite discontinuity at $x=c$ where $a<c<b$, then $\int_a^b f(x)\,dx = \lim_{\epsilon_1\to 0+}\int_a^{c-\epsilon_1} f(x)\,dx + \lim_{\epsilon_2\to 0+}\int_{c+\epsilon_2}^{b} f(x)\,dx$

If either of them fails to exist, we say that the integral does not exist.

Note: If the limit exists in (1) & (2), then the integral is said to converge. It diverges if the limit tends to infinity. In case (3) both the limit must exist in order to converge.

Problem: $\int_{-\infty}^{\infty} x e^{-x^2}\,dx$

Solution: It is an integral of first type, from (3)

$$\therefore \int_{-\infty}^{\infty} x e^{-x^2}dx = \int_{-\infty}^{0} x e^{-x^2}dx + \int_{0}^{\infty} x e^{-x^2}dx$$

$$= \lim_{x\to-\infty}\int_{x}^{0} x e^{-x^2}dx + \lim_{x\to\infty}\int_{0}^{x} x e^{-x^2}dx$$

$$= \lim_{x\to-\infty}\left[-\frac{1}{2}e^{-x^2}\right]_x^0 + \lim_{x\to\infty}\left[-\frac{1}{2}e^{-x^2}\right]_0^x$$

$$= \lim_{x\to-\infty}\left[-\frac{1}{2}+\frac{1}{2e^{x^2}}\right] + \lim_{x\to\infty}\left[+\frac{1}{2}-\frac{1}{2e^{x^2}}\right]$$

$$= -\frac{1}{2}+\frac{1}{2} = 0 \text{ (exist)}$$

$\therefore$ The given integral is convergent.

Problem: $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

Solution: This is an example of type two, from (1) the integrand has an infinite discontinuity at $x=1$.

$$\therefore \int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{\epsilon\to 0+}\int_0^{1-\epsilon}\frac{dx}{\sqrt{1-x^2}} = \lim_{\epsilon\to 0+}\left[\sin^{-1}x\right]_0^{1-\epsilon}$$

$$= \lim_{\epsilon\to 0+}\left[\sin^{-1}(1-\epsilon) - \sin^{-1}0\right]$$

$$= \sin^{-1}1 = \sin^{-1}(\sin \pi/2) = \pi/2$$

$\therefore$ The given integral is convergent.

Case of infinite region of integration:

* To establish a necessary and Sufficient Condition for The Convergence of $\int_a^\infty f(x)dx$

Proof: The improper integral

$$\int_a^\infty f(x)\,dx$$

is said to be Converge at ∞ if

$$\varphi(x) = \int_a^x f(x)\,dx$$

tends to a finite limit as $x\to\infty$ and the limit is said to be the value of the integral $\int_a^\infty f(x)\,dx$.

If for every $x>a$, $f(x)>0$, then

$\int_a^x f(x)\,dx$, which represents geometrically certain area, increases monotonically with x.

$\therefore$ $\varphi(x)$ is a monotonically increasing function of (x).

If over and above this monotonic property, $\varphi(x)$ is bounded above ie. there exists a positive number K such that

$$\varphi(x) \le K \ \forall x$$

then the improper integral $\int_a^\infty f(x)\,dx$ must be Convergent at ∞.

Hence the necessary and sufficient Condition for

$$\int_a^\infty f(x)\,dx$$

to be Convergent at ∞ is

$$\varphi(x) \le K \ \forall x$$

$$\text{ie; } \left|\int_{x_1}^{x_2} f(x)\,dx\right| \le K \text{ for } a \le x_1 < x_2 \text{ where } x_2$$

is not bounded.

* Evaluate $\int_1^\infty \frac{1}{x^2}\,dx$, if it Converges ②

Solution: Here first of all, we note that no attention need be paid to the singularity at $x=0$, since it does not lie within the range of integration. The singularity is only at the upper limit.

$$\text{Now } \int_1^\infty \frac{dx}{x^2} = \lim_{x\to\infty}\int_1^x \frac{dx}{x^2} = \lim_{x\to\infty}\left[-\frac{1}{x}\right]_1^x$$

$$= \lim_{x\to\infty}\left[-\frac{1}{x}+1\right] = 1$$

Hence $\int_1^\infty \frac{dx}{x^2}$ converges and its value is 1.

*Problem: Show that $\int_2^\infty \frac{dx}{x\log x}$ diverges.

Solution: $\lim\limits_{x\to\infty} \int_2^x \frac{dx}{x\log x}$

$$= \lim_{x\to\infty} \left[\log(\log x)\right]_2^x$$

$$= \lim_{x\to\infty} \left[\log(\log x) - \log(\log 2)\right]$$

Since $\log(\log x)$ increases beyond all bounds as $x\to\infty$ and $\log(\log 2)$ is finite, this integral diverges.

* Definition of the integral $\int_a^b f(x)dx$

If $f(x)$ is continuous in an interval $a \le x \le b$ with the single exception of the end point a then to obtain a necessary and sufficient condition for its convergence.

Since a is the only point of infinite discontinuity of $f(x)$ in the finite interval (a, b), the integral

$$\int_{a+\epsilon}^b f(x)dx, \quad 0 < \epsilon < b-a$$

exists and is a function of ϵ say $\phi(\epsilon)$.

$$\therefore \phi(\epsilon) = \int_{a+\epsilon}^b f(x)dx.$$

If when $\epsilon \to 0+0$, $\phi(\epsilon)$ tends to a finite limit, we say that the improper integral

$$\int_a^b f(x)dx$$

exists or converges at a.

Thus $\int_a^b f(x)dx = \lim\limits_{\epsilon\to 0} \int_{a+\epsilon}^b f(x)dx$, provided this limit exists.

If the limit does not exist, we say that $\int_a^b f(x)dx$ does not converge.

* To prove a criterion for convergence of $\int_a^b f(x)dx$ where a is the only point of infinite discontinuity of $f(x)$ in (a, b).

Let $f(x)$ and $\phi(x)$ be two functions of x and they are both positive in (a,b) and $f(x) \le \phi(x)$.
Then $\int_a^b f(x)\,dx$ converges
if $\int_a^b \phi(x)\,dx$ converges.

Proof: Let us assume that both $f(x)$ and $\phi(x)$ are integrable in $(a+\epsilon, b)$ where $0<\epsilon<b-a$.

Then $\because f(x) \le \phi(x)$

$$\int_{a+\epsilon}^{b} f(x)\,dx \le \int_{a+\epsilon}^{b} \phi(x)\,dx \quad \text{——— (1)}$$

Let $\int_a^b \phi(x)\,dx$ converges. Then there exists a positive number M such that

$$\int_{a+\epsilon}^{b} \phi(x)\,dx < M \text{ for } 0<\epsilon\le b-a$$

So from (1), we have

$$\int_{a+\epsilon}^{b} f(x)\,dx < M \text{ for } 0<\epsilon\le b-a$$

Hence $\int_a^b f(x)\,dx$ converges at a.

**Problem: Prove that $\int_0^\infty e^{-x^2}\,dx$ is convergent and its value is less than $\pi/2$.

Solution: We have

$$\int_0^\infty e^{-x^2}\,dx = \int_0^1 e^{-x^2}\,dx + \int_1^\infty e^{-x^2}\,dx \quad \text{——— (1)}$$

Now $\int_0^1 e^{-x^2}\,dx$ is bounded and the range of integration is finite. So it is a proper integral and therefore convergent.

Now consider $\int_1^\infty e^{-x^2}\,dx$.

For $x>1$, $e^{-x^2} < x e^{-x^2}$

$$\text{i.e. } \int_1^\infty e^{-x^2}\,dx < \int_1^\infty x e^{-x^2}\,dx.$$

$$\text{i.e. } < \lim_{x\to\infty} \int_1^x x e^{-x^2}\,dx$$

$$\text{i.e. } < \lim_{x\to\infty} \left[-\frac{1}{2}e^{-x^2}\right]_1^x$$

$$\text{i.e. } < -\frac{1}{2}\lim_{x\to\infty}\left[\frac{1}{e^{x^2}}\right]_1^\infty$$

$$\text{i.e. } < -\frac{1}{2}\lim_{x\to\infty}\left[\frac{1}{e^{x^2}}-\frac{1}{e}\right] \quad \text{i.e. } < \frac{1}{2e} = \text{finite}$$

$\therefore \int_1^{\infty} e^{-x^2} dx$ is Convergent.

Hence from (1) $\int_0^{\infty} e^{-x^2} dx$ is Convergent.

2nd part: Now $e^{-x^2} = \frac{1}{e^{x^2}} =$ positive

and $\frac{1}{e^{x^2}} < \frac{1}{1+x^2}$

$\therefore \int_0^{\infty} \frac{1}{e^{x^2}} dx < \int_0^{\infty} \frac{1}{1+x^2} dx$

ie. $< \lim_{x\to\infty} \int_0^{x} \frac{dx}{1+x^2}$

ie. $< \lim_{x\to\infty} \left[\tan^{-1} x\right]_0^x$

ie. $< \lim_{x\to\infty} \left[\tan^{-1} x - \tan^{-1} 0\right]$

ie. $< \lim_{x\to\infty} \tan^{-1} x$

ie. $< \tan^{-1} \infty$

ie. $< \tan^{-1}(\tan \pi/2)$ ie. $< \pi/2$

$\therefore \int_0^{\infty} e^{-x^2} dx < \pi/2$

Problem: Show that $\int_1^{\infty} \frac{\sqrt{x}}{(1+x)^2} dx$ converges to $\frac{1}{2} + \pi/4$

Solution: Now $\int_1^{\infty} \frac{\sqrt{x}}{(1+x)^2} dx = \lim_{x\to\infty} \int_1^{x} \frac{\sqrt{x}}{(1+x)^2} dx$

Now putting $x = \tan^2\theta$ So that $dx = 2\tan\theta \sec^2\theta \, d\theta$

$\therefore \lim_{x\to\infty} \int_1^{x} \frac{\sqrt{x}}{(1+x)^2} dx = \lim_{x\to\infty} \int_{\pi/4}^{\tan^{-1}\sqrt{x}} \frac{2\tan^2\theta \sec^2\theta \, d\theta}{\sec^4\theta}$

$= \lim_{x\to\infty} \int_{\pi/4}^{\tan^{-1}\sqrt{x}} 2\sin^2\theta \, d\theta$

$= \lim_{x\to\infty} \int_{\pi/4}^{\tan^{-1}\sqrt{x}} (1-\cos 2\theta) d\theta$

$= \lim_{x\to\infty} \left[\theta - \frac{\sin 2\theta}{2}\right]_{\pi/4}^{\tan^{-1}\sqrt{x}}$

$$= \lim_{x\to\infty} \left[\theta - \frac{\tan\theta}{1+\tan^2\theta}\right]_{\pi/4}^{\tan^{-1}\sqrt{x}}$$

$$= \lim_{x\to\infty} \left[\tan^{-1}\sqrt{x} - \frac{\sqrt{x}}{1+x} - \left(\frac{\pi}{4} - \frac{1}{2}\right)\right]$$

$$= \lim_{x\to\infty} \left[\tan^{-1}\sqrt{x} - \frac{1}{\frac{1}{\sqrt{x}}+\sqrt{x}} - \frac{\pi}{4} + \frac{1}{2}\right]$$

$$= \frac{\pi}{2} - \frac{\pi}{4} + \frac{1}{2}$$

$$= \frac{\pi}{4} + \frac{1}{2} = \text{finite}$$

Hence, the given integral is Convergent and Converges to $\frac{\pi}{4}+\frac{1}{2}$

<u>Problem</u>: Show that $\int^{\infty} \frac{x^2\,dx}{(a^2+x^2)^2}$ is Convergent.

<u>Solution</u>: Now $\int_0^{\infty} \frac{x^2\,dx}{(a^2+x^2)^2}$

$$= \lim_{x\to\infty} \int_0^{x} \frac{x^2\,dx}{(a^2+x^2)^2} = \lim_{x\to\infty} \int_0^{x} \frac{a^2+x^2-a^2}{(a^2+x^2)^2}\,dx$$

$$= \lim_{x\to\infty} \int_0^{x} \frac{dx}{a^2+x^2} - \lim_{x\to\infty} \int_0^{x} \frac{a^2}{(a^2+x^2)^2}\,dx$$

$$= \lim_{x\to\infty} \left[\frac{1}{a}\tan^{-1}\frac{x}{a}\right]_0^{x} - \lim_{x\to\infty} a^2 \int_0^{x} \frac{dx}{(a^2+x^2)^2}$$

$$= \frac{\pi}{2a} - \lim_{x\to\infty} a^2 \int_0^{x} \frac{dx}{(a^2+x^2)^2}$$

Now putting $x = a\tan\theta \quad \therefore dx = a\sec^2\theta\,d\theta.$

$$= \frac{\pi}{2a} - \lim_{x\to\infty} a^2 \int_0^{\tan^{-1}x/a} \frac{a\sec^2\theta\,d\theta}{a^4\sec^4\theta}$$

$$= \frac{\pi}{2a} - \lim_{x\to\infty} \frac{1}{a} \int_0^{\tan^{-1}x/a} \cos^2\theta\,d\theta$$

$$= \frac{\pi}{2a} - \lim_{x\to\infty} \frac{1}{2a} \int_0^{\tan^{-1}x/a} (1+\cos 2\theta)\,d\theta$$

$$= \frac{\pi}{2a} - \lim_{x\to\infty} \frac{1}{2a} \left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\tan^{-1}x/a}$$

$$= \frac{\pi}{2a} - \lim_{x\to\infty} \frac{1}{2a} \left[\theta + \frac{\tan\theta}{1+\tan^2\theta}\right]_0^{\tan^{-1}x/a}$$

$$= \frac{\pi}{2a} - \lim_{x\to\infty} \frac{1}{2a}\left[\tan^{-1}\frac{x}{a} + \frac{x/a}{1+x^2/a^2}\right]$$

$$= \frac{\pi}{2a} - \lim_{x\to\infty} \frac{1}{2a}\left[\tan^{-1}\frac{x}{a} + \frac{ax}{a^2+x^2}\right]$$

$$= \frac{\pi}{2a} - \lim_{x\to\infty} \frac{1}{2a}\left[\tan^{-1} x/a + \frac{a/x}{a^2/x^2+1}\right]$$

$$= \frac{\pi}{2a} - \lim_{x\to\infty} \frac{1}{2a}\left[\tan^{-1} x/a + \frac{a/x}{a^2/x^2+1}\right]$$

$$= \frac{\pi}{2a} - \frac{\pi}{4a} = \frac{\pi}{4a} \text{ (Definite value)}$$

Hence, the given integral is convergent and converges to $\frac{\pi}{4a}$.

<u>Problem</u>: Discuss the Convergence of $\int_0^\infty \frac{x\,dx}{1+x^2\sin^2 x}$

<u>Solution</u>: Now $\int_0^\infty \frac{x\,dx}{1+x^2\sin^2 x} = \lim_{x\to\infty}\int_0^x \frac{x\,dx}{1+x^2\sin^2 x}$

$$\text{i.e.} > \lim_{x\to\infty}\int_0^x \frac{x\,dx}{1+x^2} \qquad \because \sin^2 x \le 1$$

$$\therefore 1+x^2\sin^2 x \le 1+x^2$$

$$\text{i.e.} > \lim_{x\to\infty}\frac{1}{2}\int_0^x \frac{2x\,dx}{1+x^2} \qquad \text{i.e. } \frac{1}{1+x^2\sin^2 x} > \frac{1}{1+x^2}$$

$$\text{i.e.} > \lim_{x\to\infty}\frac{1}{2}\left[\log(1+x^2)\right]_0^x$$

$$\text{i.e.} > \lim_{x\to\infty}\frac{1}{2}\left[\log(1+x^2)\right]$$

$\therefore$ As $x\to\infty$, the given integral is divergent.

<u>Problem</u>: Test the convergence of the auxiliary integral $\int_a^\infty \frac{dx}{x^n}$ when $n>0$

<u>Solution</u>: We have $\int_a^\infty \frac{dx}{x^n} = \lim_{X\to\infty}\int_a^X \frac{dx}{x^n}$

$$= \lim_{X\to\infty}\left[\frac{x^{-n+1}}{-n+1}\right]_a^X \text{ if } n\ne 1$$

$$= \lim_{X\to\infty}\frac{1}{1-n}\left[\frac{1}{X^{n-1}} - \frac{1}{a^{n-1}}\right] \quad \text{——— (1)}$$

$= \frac{1}{1-n}\left[-\frac{1}{a^{n-1}}\right]$ if $n>1$

$= \frac{1}{(n-1)a^{n-1}}$ if $n>1$

Also from ① the given integral $= \infty$ if $n<1$

Again if $n=1$

$$\int_a^{\infty}\frac{dx}{x^n} = \lim_{X\to\infty}\frac{dx}{x} = \lim_{X\to\infty}\left[\log x\right]_a^X$$

$$= \lim_{X\to\infty}\left[\log X - \log a\right]$$

$$= \lim_{X\to\infty}\log\frac{X}{a} = \infty$$

$\therefore \int_a^{\infty}\frac{dx}{x^n}$ Converges if $n>1$ and diverges if $n\le 1$

provided $a>0$

<u>Problem</u>: Test the Convergence of the auxiliary integral

$$\int_a^b\frac{dx}{(b-x)^n}$$

<u>Solution</u>: Here $x=b$ is the only point of infinite discontinuity.

$$\therefore \int_a^b\frac{dx}{(b-x)^n} = \lim_{\epsilon\to 0}\int_a^{b-\epsilon}\frac{dx}{(b-x)^n} = \lim_{\epsilon\to 0}\left[\frac{1}{(n-1)(b-x)^{n-1}}\right]_a^{b-\epsilon}$$

$$= \lim_{\epsilon\to 0}\frac{1}{n-1}\left[\frac{1}{\epsilon^{n-1}} - \frac{1}{(b-a)^{n-1}}\right]$$

which tends to $\frac{1}{(1-n)(b-a)^{n-1}}$ or ∞ according as $n<1$ or $n>1$.

If $n=1$ we have

$$\int_a^b\frac{dx}{(b-x)^n} = \int_a^b\frac{dx}{(b-x)} = \lim_{\epsilon\to 0}\int_a^{b-\epsilon}\frac{dx}{b-x}$$

$$= \lim_{\epsilon\to 0}-\left[\log(b-x)\right]_a^{b-\epsilon}$$

$$= \lim_{\epsilon\to 0}-\left[\log\epsilon - \log(b-a)\right]$$

$$= \lim_{\epsilon\to 0}\left[\log(b-a) - \log\epsilon\right] = \infty$$

Hence, the given integral is Convergent if $n<1$ and divergent if $n\geq 1$.

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Problem: Test the convergence of the auxiliary integral $\int_a^b \frac{dx}{(x-a)^n}$

Solution: Here $x=a$ is the only point of infinite discontinuity.

Thus $\int_a^b \frac{dx}{(x-a)^n} = \lim_{\epsilon\to 0} \int_{a+\epsilon}^b \frac{dx}{(x-a)^n}$.

$$= \lim_{\epsilon\to 0}\left[\frac{(x-a)^{-n+1}}{-n+1}\right]_{a+\epsilon}^{b}$$

$$= \lim_{\epsilon\to 0} \frac{1}{1-n}\left[(b-a)^{1-n} - \epsilon^{1-n}\right]$$

$= \frac{1}{1-n}(b-a)^{1-n}$ when $n<1$ i.e. when $1-n$ is positive and $=\infty$ when $n>1$ i.e. when $n-1$ is positive i.e, $1-n$ is negative.

When $n=1$, we have

$$\int_a^b \frac{dx}{(x-a)^n} = \int_a^b \frac{dx}{(x-a)}$$

$$= \lim_{\epsilon\to 0} \int_{a+\epsilon}^b \frac{dx}{x-a}$$

$$= \lim_{\epsilon\to 0} \left[\log(x-a)\right]_{a+\epsilon}^{b}$$

$$= \lim_{\epsilon\to 0} \left[\log(b-a) - \log\epsilon\right]$$

$$= \infty$$

So, we find that

$\lim_{\epsilon\to 0} \int_{a+\epsilon}^b \frac{dx}{(x-a)^n}$ exists only when $n<1$

Hence, the given integral is convergent only when $n<1$ and divergent when $n\geq 1$.

Test of ~~conv~~ Convergence of infinite integral $\int_a^\infty f(x)\,dx$.

Comparison Test:

(a) Let $f(x)$ and $\varphi(x)$ be two functions both bounded and integrable in (a,∞) and $f(x)$ is positive

(i) if $|\varphi(x)| \leq f(x)$ for $x\geq a$

then $\int_a^\infty \varphi(x)\,dx$ is convergent if $\int_a^\infty f(x)\,dx$ is known to be convergent;

(ii) If $\varphi(x) \geq f(x)$ for $x \geq a$

then $\int_a^\infty \varphi(x)\,dx$ is divergent if $\int_a^\infty f(x)\,dx$ is known to be divergent.

(b) If $f(x)$ and $\varphi(x)$ are two functions of x such that both are positive for $x \geq a$ and if

$$\lim_{x\to\infty} \frac{f(x)}{\varphi(x)} = l$$

where l is neither zero nor infinite, then the two integrals $\int_a^\infty f(x)\,dx$ and $\int_a^\infty \varphi(x)\,dx$ either both converge or both donot converge.

Problem: Test the convergence of the integral

$$\int_0^\infty \frac{\sin^2 x}{x^2}\,dx$$

Solution: Let $\varphi(x) = \dfrac{\sin^2 x}{x^2}$ and $f(x) = \dfrac{1}{x^2}$

then $\left|\dfrac{\sin^2 x}{x^2}\right| \leq \dfrac{1}{x^2}$ $\quad\because |\sin^2 x| \leq 1$

$$\therefore \int_1^\infty \left|\frac{\sin^2 x}{x^2}\right| dx \leq \int_1^\infty \frac{dx}{x^2}$$

but $\int_1^\infty \dfrac{dx}{x^2}$ is convergent $\because n > 1$

$\therefore \int_1^\infty \dfrac{\sin^2 x}{x^2}\,dx$ is also convergent.

Since $\lim_{x\to 0} \dfrac{\sin^2 x}{x^2} = 1$

$\therefore$ then $x=0$ is not a point of infinite discontinuity.

$\therefore \int_0^1 \dfrac{\sin^2 x}{x^2}\,dx$ is proper integral and so convergent.

$$\therefore \int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \int_0^1 \frac{\sin^2 x}{x^2}\,dx + \int_1^\infty \frac{\sin^2 x}{x^2}\,dx$$

$=$ Convergent + Convergent

$=$ Convergent.

Problem: Test the convergence of the integral

$$\int_1^\infty \frac{dx}{(1+x)\sqrt{x}}$$

Solution: Let $\varphi(x) = \dfrac{1}{(1+x)\sqrt{x}}$ and $f(x) = \dfrac{1}{x\sqrt{x}} = \dfrac{1}{x^{3/2}}$

Now $\dfrac{1}{(1+x)\sqrt{x}} < \dfrac{1}{x\sqrt{x}}$ when $x \geq 1$

$$\therefore \int_1^\infty \frac{dx}{(1+x)\sqrt{x}} < \int_1^\infty \frac{dx}{x\sqrt{x}} \text{ i.e. } < \int_1^\infty \frac{dx}{x^{3/2}}$$

But $\int_1^\infty \frac{dx}{x^{3/2}}$ is convergent as $n>1$.

$\therefore \int_1^\infty \frac{dx}{(1+x)\sqrt{x}}$ is convergent.

Problem: Test the convergence of the integral

$$\int_0^\infty \frac{\sin x}{1+x^2}\,dx$$

Solution: Let $\varphi(x) = \frac{1}{1+x^2}$ and $f(x) = \frac{\sin x}{1+x^2}$

Since $\sin x \leq 1$.

$$\therefore \frac{\sin x}{1+x^2} \leq \frac{1}{1+x^2}$$

$$\therefore \int_0^\infty \frac{\sin x}{1+x^2}\,dx \leq \int_0^\infty \frac{dx}{1+x^2}$$

But
$$\int_0^\infty \frac{dx}{1+x^2} = \lim_{X\to\infty} \int_0^X \frac{dx}{1+x^2}$$

$$= \lim_{X\to\infty} \left[\tan^{-1} x\right]_0^X$$

$$= \lim_{X\to\infty} \tan^{-1} X = \tan^{-1}(\tan \pi/2)$$

$$= \pi/2$$

$\therefore \int_0^\infty \frac{dx}{1+x^2}$ Converges to $\pi/2$

Hence $\int_0^\infty \frac{\sin x}{1+x^2}\,dx$ is Convergent.

⊛ **The μ-Test for the Convergence of infinite integrals.**

Let $f(x)$ be bounded and integrable in (a,∞) where $a>0$

(i) If there is a positive number $\mu > 1$ such that $\lim_{x\to\infty} x^\mu f(x)$ exists finitely, the limit being neither zero nor infinite, then the integral
$$\int_a^\infty f(x)\,dx \text{ Converges.}$$

(ii) If there is a positive number $\mu \leq 1$ such that $\lim_{x\to\infty} x^\mu f(x)$ exists and is not zero, then the integral
$$\int_a^\infty f(x)\,dx \text{ is divergent.}$$

Note: Take μ = (highest power of x in denominator) − (highest power of x in numerator)

Problem: Test the Convergence of
$$\int_0^\infty \frac{x\,dx}{(1+x)^3}$$
Solution: Take $\mu = 3-1 = 2$ and $\phi(x) = \frac{x}{(1+x)^3}$

Then $\lim_{x\to\infty}\left[x^\mu \phi(x)\right] = \lim_{x\to\infty}\left[x^2 \cdot \frac{x}{(1+x)^3}\right]$

$$= \lim_{x\to\infty} \frac{1}{\left(1+\frac{1}{x}\right)^3} = 1$$

Since $\mu > 1$ and $\lim_{x\to\infty}\left[x^\mu \phi(x)\right]$ exists and is neither zero nor infinite, by μ test the given integral is convergent.

Problem: Show that $\int_0^\infty \frac{x^{3/2}}{\sqrt{x^4-a^4}}dx$ is divergent, $x>a$

Solution: Let $\phi(x) = \frac{x^{3/2}}{\sqrt{x^4-a^4}}$

$\therefore \lim_{x\to\infty}\left\{x^\mu \cdot \phi(x)\right\}$

$$= \lim_{x\to\infty}\left\{x^\mu \cdot \frac{x^{3/2}}{\sqrt{x^4-a^4}}\right\}$$

$$= \lim_{x\to\infty}\left\{\frac{x^{\mu+3/2}}{x^2\sqrt{1-\frac{a^4}{x^4}}}\right\}$$

$$= 1 \text{ if } \mu + 3/2 = 2$$

i.e., if $\mu = 2 - 3/2 = \frac{1}{2} < 1$

Since $\mu<1$ and $\lim_{x\to\infty}\left\{x^\mu \cdot \phi(x)\right\}$ exists and not zero, the given integral is divergent.

⊛ Abel's test for the convergence of the integral of the product of two functions.

If $\int_0^\infty f(x)\,dx$ converges and $\phi(x)$ is bounded and monotonic for $x>0$ then

$$\int_0^\infty f(x)\phi(x)\,dx \text{ is Convergent}$$

Problem: Test the Convergence of $\int_0^\infty e^{-x}\sin\frac{1}{x}\,dx$.

Solution: Let $f(x) = \sin\frac{1}{x}$ and $\phi(x) = e^{-x}$

Now $\phi(x) = e^{-x}$ is bounded and monotonic decreasing function for $x>0$.

Also $\int_0^\infty \sin\frac{1}{x}\,dx = \int_\infty^0 \sin t\left(-\frac{1}{t^2}\right)dt$ putting $\frac{1}{x}=t$

$$= \int_0^\infty \frac{\sin t}{t^2}\,dt = \int_0^\infty \frac{\sin x}{x^2}\,dx$$

$$= \int_0^2 \frac{\sin x}{x^2}\,dx + \int_2^\infty \frac{\sin x}{x^2}\,dx \quad \text{--- (1)}$$

Now $\int_0^2 \frac{\sin x}{x^2}\,dx$ being a proper integral is convergent.

and $\int_2^\infty \frac{\sin x}{x^2}\,dx \le \int_2^\infty \frac{dx}{x^2}$ $\because \sin x \le 1$

But $\int_2^\infty \frac{dx}{x^2}$ is convergent as $n \geq 1$

$\therefore \int_2^\infty \frac{\sin x}{x^2}\,dx$ is convergent.

So from (1) $\int_0^\infty \sin\frac{1}{x}\,dx$ is convergent.

Hence, by Abel's Test

$\int_0^\infty e^{-x}\sin\frac{1}{x}\,dx$ is convergent.

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(*) Derichlet's test for the convergence of the integral of a product.

If $\phi(x)$ be bounded and monotic for $x \geq a$ and $\lim_{x\to\infty} \phi(x) = 0$ and $\int_a^x f(x)\,dx$ is bounded for all values of $x \geq a$, then $\int_a^\infty f(x)\phi(x)\,dx$ is convergent.

** Problem: Show that $\int_0^\infty \frac{\sin x}{x}\,dx$ is convergent.

Solution: Since $\lim_{x\to 0} \frac{\sin x}{x} = 1$

$x = 0$ is not a point of infinite discontinuity of the integrand.

Now $\int_0^\infty \frac{\sin x}{x}\,dx = \int_0^1 \frac{\sin x}{x}\,dx + \int_1^\infty \frac{\sin x}{x}\,dx$ --- (1)

As $\int_0^1 \frac{\sin x}{x}\,dx$ is a proper integral and it is convergent.

For the convergency of $\int_1^\infty \frac{\sin x}{x}\,dx$, let $\phi(x) = \frac{1}{x}$ which is bounded in $(1, \infty)$

and $\lim_{x\to\infty} \phi(x) = \lim_{x\to\infty} \frac{1}{x} = 0$ and

also let $f(x) = \sin x$

Then $\left|\int_1^x f(x)\,dx\right| = \left|\int_1^x \sin x\,dx\right| = \left|[-\cos x]_1^x\right|$

$= |\cos 1 - \cos x|$

$\leq |\cos 1| + |\cos x|$

< 2 for all finite values of x.

$\therefore \int_1^x \sin x\,dx$ bounded above for $x \geq 1$

Hence by Derichlet's Test $\int_0^\infty \frac{\sin x}{x}\,dx$ is convergent.

Convergent of improper integrals of second kind when the integrand is infinite and the range of integration finite.

Comparison Test:

If $\underset{x\to a}{Lt} \dfrac{f(x)}{\varphi(x)}$ = finite i.e. neither zero nor infinite, then two integrals $\int_a^b f(x)\,dx$ and $\int_a^b \varphi(x)\,dx$ are either both convergent or both divergent.

In this comparison test we shall use the auxiliary integral $\int_a^b \dfrac{dx}{(x-a)^n}$

which is convergent when $n<1$ and divergent when $n\geq 1$.

Problem: Test the convergence of $\int_0^1 \dfrac{dx}{x^3(1+x)^2}$.

Solution: Here $x=0$ is the only point of infinite discontinuity.

Let $\varphi(x) = \dfrac{1}{x^3} = \dfrac{1}{(x-0)^3}$

As $n=3$ i.e. >1 therefore $\int_0^1 \dfrac{dx}{(x-0)^3}$ is divergent.

Now $f(x) = \dfrac{1}{x^3(1+x)^2}$

$$\therefore \underset{x\to 0}{Lt} \frac{f(x)}{\varphi(x)} = \underset{x\to 0}{Lt} \frac{1}{x^3(1+x)^2}\cdot x^3 = \underset{x\to 0}{Lt} \frac{1}{(1+x)^2}$$

$= 1$ i.e. finite

So by comparison test both the integrals either convergent or divergent together.

But $\int_0^1 \dfrac{dx}{(x-0)^2}$ is divergent.

Hence the given integral $\int_0^1 \dfrac{dx}{x^3(1+x^2)}$ is divergent.

Problem: Test the convergence of $\int_0^1 \dfrac{dx}{x^{1/2}(1+x^2)^2}$

Solution: Here $x=0$ is the only point of infinite discontinuity.

Let $\varphi(x) = \dfrac{1}{x^{1/2}} = \dfrac{1}{(x-0)^{1/2}}$

As $n=\frac{1}{2}$ i.e. <1 then $\int_0^1 \dfrac{dx}{(x-0)^{1/2}}$ is convergent.

Now $\underset{x\to 0}{Lt} \dfrac{f(x)}{\varphi(x)} = \underset{x\to 0}{Lt} \dfrac{1}{x^{1/2}(1+x^2)^2}\cdot x^{1/2} = \underset{x\to 0}{Lt} \dfrac{1}{(1+x^2)^2} = 1$ i.e. finite

So by comparison test both the integrals either convergent or divergent. But $\int_0^1 \dfrac{dx}{(x-0)^{1/2}}$ is convergent

Hence the given integral $\int_0^1 \dfrac{dx}{x^{1/2}(1+x^2)^2}$ is convergent.

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*Problem: Test the Convergence of $\int_0^{\pi/2} \frac{x^p}{(\sin x)^q}\,dx$.

Solution: Here $\int_0^{\pi/2} \frac{x^p}{(\sin x)^q}\,dx = \int_0^{\pi/2} \frac{x^{p-q}}{\left(\frac{\sin x}{x}\right)^q}\,dx$

$$= \int_0^{\pi/2} \frac{dx}{x^{q-p}\left(\frac{\sin x}{x}\right)^q}$$

$\therefore$ $x=0$ is the only point of infinite discontinuity.

Let $\phi(x) = \frac{1}{x^{q-p}} = \frac{1}{(x-0)^{q-p}}$

Then $\int_0^{\pi/2} \frac{dx}{(x-0)^{q-p}}$ is convergent if $q-p<1$ and divergent if $q-p\geq 1$.

Now $\lim_{x\to 0} \frac{f(x)}{\phi(x)} = \lim_{x\to 0} \frac{1}{x^{q-p}\left(\frac{\sin x}{x}\right)^q}\cdot x^{q-p}$

$= \lim_{x\to 0} \frac{1}{\left(\frac{\sin x}{x}\right)^q} = 1$ i.e. finite

So both the integrals either converge or diverge together.

But $\int_0^{\pi/2} \frac{dx}{(x-0)^{q-p}}$ converges if $q-p<1$ and diverges if $q-p\geq 1$.

Hence the given integral

$$\int_0^{\pi/2} \frac{x^p}{(\sin x)^q}\,dx$$

is convergent if $q-p<1$ and divergent if $q-p\geq 1$.

*Problem: Test the convergence of $\int_0^{\pi/2} \frac{\sin x}{x^p}\,dx$.

Solution: Here $\int_0^{\pi/2} \frac{\sin x}{x^p}\,dx = \int_0^{\pi/2} \left(\frac{\sin x}{x}\right)\cdot\frac{1}{x^{p-1}}\,dx$

Let $\phi(x) = \frac{1}{x^{p-1}} = \frac{1}{(x-0)^{p-1}}$

Then $\int_0^{\pi/2} \frac{dx}{(x-0)^{p-1}}$ is convergent if $p-1<1$ i.e. if $p<2$ and divergent if $p-1\geq 1$ i.e. if $p\geq 2$

Now $\lim_{x\to 0} \frac{f(x)}{\phi(x)} = \lim_{x\to 0} \frac{\sin x}{x^p}\cdot x^{p-1} = \lim_{x\to 0} \frac{\sin x}{x}$
$= 1$ i.e. finite

$\therefore$ Both the integrals either converge or diverge together.

But $\int_0^{\pi/2} \frac{dx}{(x-0)^{p-1}}$ is convergent if $p<2$ and divergent if $p \geq 2$.

Hence the given integral

$$\int_0^{\pi/2} \frac{\sin x}{x^p}\,dx \text{ is convergent if } p<2 \text{ and divergent if } p \geq 2.$$

Problem: Show that the elliptic integral

$$\int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}, \quad k^2<1 \text{ is convergent.}$$

Solution: Let $f(x) = \dfrac{1}{\sqrt{(1-x^2)(1-k^2x^2)}}$

Here $x=1$ is the only point of infinite discontinuity.

Let $\phi(x) = \dfrac{1}{\sqrt{1-x^2}}$

Then $\displaystyle\int_0^1 \phi(x)\,dx = \lim_{\epsilon \to 0} \int_{0+\epsilon}^{1} \frac{dx}{\sqrt{1-x^2}}$

$$= \lim_{\epsilon \to 0+\epsilon} \left[\sin^{-1}x\right]_{0+\epsilon}^{1}$$

$$= \lim_{\epsilon \to 0} \left[\sin^{-1}(1) - \sin^{-1}\epsilon\right] = \pi/2$$

$\therefore \displaystyle\int_0^1 \phi(x)\,dx$ is convergent.

Now $\displaystyle\lim_{x \to 0} \frac{f(x)}{\phi(x)} = \lim_{x \to 0} \frac{1}{\sqrt{(1-x^2)(1-k^2x^2)}} \cdot \sqrt{1-x^2}$

$$= \lim_{x \to 0} \frac{1}{\sqrt{1-k^2x^2}} = 1 \text{ i.e. finite}$$

$\therefore \displaystyle\int_0^1 f(x)\,dx$ and $\displaystyle\int_0^1 \phi(x)\,dx$ either converge or diverge together.

But $\displaystyle\int_0^1 \phi(x)\,dx$ converges.

Hence $\displaystyle\int_0^1 f(x)\,dx$ converges

i.e. $\displaystyle\int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$ converges.

(*) The μ-test of Convergency of $\int_a^b f(x)\,dx$ where $x=a$ is a point of infinite discontinuity and $f(x)$ is bounded and integrable in $(a+\epsilon, b)$ where $0<\epsilon<b-a$.

(i) If there is a positive number $\mu<1$ and $\lim_{x\to a+0}(x-a)^{\mu}f(x)$ exists and is finite, then $\int_a^b f(x)dx$ is absolutely Convergent.

(ii) If $\mu \geq 1$ and $\lim_{x\to a+0}(x-a)^{\mu}f(x)$ is finite then $\int_a^b f(x)dx$ diverges.

(iii) Also if $\lim_{x\to a+0}(x-a)^{\mu}f(x)$ is $\pm\infty$, then also $\int_a^b f(x)dx$ diverges.

(iv) In case $x=b$ is a point of infinite discontinuity, we should evaluate $\lim_{x\to b-0}(b-x)^{\mu}f(x)$ and other conditions remaining the same.

Problem: Test the Convergence of $\int_0^1 \frac{dx}{\sqrt{x(1-x)}}$

Solution: Here $x=0$ and $x=1$ are points of infinite discontinuity so we write

$$\int_0^1 \frac{dx}{\sqrt{x(1-x)}} = \int_0^a \frac{dx}{\sqrt{x(1-x)}} + \int_a^1 \frac{dx}{\sqrt{x(1-x)}} \quad \text{——— (1)}$$

where $0<a<1$.

Now consider $\int_0^a \frac{dx}{\sqrt{x(1-x)}}$

Here $x=0$ is a point of infinite discontinuity

$$\therefore \lim_{x\to 0+0} \frac{(x-0)^{1/2}}{x^{1/2}(1-x)^{1/2}} = \lim_{x\to 0+0}\frac{1}{(1-x)^{1/2}} = 1 \text{ i.e. finite}$$

and $\mu = \frac{1}{2} < 1$.

So by μ-test $\int_0^a \frac{dx}{\sqrt{x(1-x)}}$ is Convergent.

Next consider $\int_a^1 \frac{dx}{\sqrt{x(1-x)}}$

Here $x=1$ is a point of infinite discontinuity.

$$\therefore \lim_{x\to 1-0} (1-x)^{\frac{1}{2}}\cdot\frac{1}{x^{1/2}(1-x)^{1/2}} = \lim_{x\to 1-0}\frac{1}{x^{1/2}} = \lim_{h\to 0}\frac{1}{(1-h)^{1/2}}$$

$= 1$ i.e. finite

and $\mu = \frac{1}{2} < 1$

So by μ-Test $\int_a^1 \frac{dx}{\sqrt{x(1-x)}}$ is Convergent.

Hence from (1), the given integral is Convergent.

Problem: Test the Convergence of $\int_0^2 \frac{\log x}{\sqrt{2-x}}dx$

Solution: Here $x=0$ and 2 are points of infinite discontinuity So we write

$$\int_0^2 \frac{\log x}{\sqrt{2-x}}dx = \int_0^a \frac{\log x}{\sqrt{2-x}}dx + \int_a^2 \frac{\log x}{\sqrt{2-x}}dx \quad \text{——— (1)}$$

where $0<a<2$

Now Consider $\int_0^2 \frac{\log x}{\sqrt{2-x}}\,dx$.

Here $x=0$ is a point of infinite discontinuity.

$\therefore \underset{x\to 0+0}{Lt}\ x^{\mu}\frac{\log x}{\sqrt{2-x}} = 0$ if $\mu>0$

So if we take $\mu<1$ and positive

then $\int_0^2 \frac{\log x}{\sqrt{2-x}}\,dx$ is Convergent.

Next consider $\int_2^2 \frac{\log x}{\sqrt{2-x}}\,dx$

Here $x=2$ is a point of infinite discontinuity.

$\therefore \underset{x\to 2-0}{Lt}\ (2-x)^{\mu}\frac{\log x}{\sqrt{2-x}} = \underset{h\to 0}{Lt}\ h^{\mu}\frac{\log(2-h)}{\sqrt{h}} = \log 2$ if $\mu=\frac{1}{2}$

So by μ-test $\int_2^2 \frac{\log x}{\sqrt{2-x}}\,dx$ is Convergent.

Hence from (1), the given integral i.e. $\int_0^2 \frac{\log x}{\sqrt{2-x}}\,dx$ is Convergent.

<u>Problem</u>: Examine the Convergence of $\int_0^1 x^{n-1}\log x\,dx$

<u>Solution</u>: <u>Case-I</u>: If $n-1>0$ i.e., positive.

Let $n-1=m$ where $m>0$.

Then $\int_0^1 x^{n-1}\log x\,dx = \int_0^1 x^{m}\log x\,dx$ which is a proper integral and so Convergent.

<u>Case-II</u>: If $n=1$, the given integral

$$\int_0^1 \log x\,dx$$

$$= \underset{\epsilon\to 0}{Lt}\int_{\epsilon}^1 \log x\,dx$$

$$= \underset{\epsilon\to 0}{Lt}\left[x\log x - x\right]_{\epsilon}^1$$

$$= \underset{\epsilon\to 0}{Lt}\left[-1-\epsilon\log\epsilon+\epsilon\right] = -1$$

$\therefore$ The given integral is Convergent if $n=1$

<u>Case-III</u>: If $n<1$, Let $f(x)=x^{n-1}\log x$.

$$\therefore \underset{x\to 0}{Lt}\ x^{\mu}f(x) = \underset{x\to 0}{Lt}\ x^{\mu}.x^{n-1}\log x$$

$$= \underset{x\to 0}{Lt}\ x^{\mu+n-1}\log x$$

$= 0$ if $\mu+n-1>0$ i.e., if $\mu>1-n$ —— (1)

and $=\infty$ if $\mu+n-1\le 0$ i.e., if $\mu\le 1-n$ —— (2)

So from (1) we get integral is Convergent.

If $1-n<\mu<1$, the given

Now $1-n<1$ means $-n<0$ i.e. $n>0$.

But we have $n<1$.

$\therefore$ for $0<n<1$ the given integral is Convergent.

From (2) we get $1-n\geq\mu\geq 1$

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i.e., $1-n \geq 1$ i.e., $-n \geq 0$ i.e., $n \leq 0$

$\therefore$ If $n \leq 0$ the given integral is divergent.

Hence, the given integral is convergent if $n > 0$ and divergent if $n \leq 0$.

** **Problem:** Show that $\int_0^1 \frac{\log x}{\sqrt{x}}\, dx$ is convergent.

Solution: By the above problem i.e.

$$\int_0^1 x^{n-1} \log x \, dx \text{ is convergent if } n>0$$

For this problem $n-1 = -\frac{1}{2}$ i.e. $n = 1-\frac{1}{2} = \frac{1}{2} > 0$

$$\therefore \int_0^1 \frac{\log x}{\sqrt{x}}\, dx = \int_0^1 x^{-\frac{1}{2}} \log x \, dx$$

$$= \int_0^1 x^{n-1} \log x \, dx \text{ which is convergent.}$$

Hence $\int_0^1 \frac{\log x}{\sqrt{x}}\, dx$ is convergent. Since $n = \frac{1}{2} > 0$

Problem: Test the convergence of $\int_0^1 x^{\alpha-1} e^{-x}\, dx$

Solution: If $\alpha \geq 1$, the given integral is a proper integral and so convergent.

If $\alpha \leq 1$, then $x=0$ is a point of infinite discontinuity.

Let $f(x) = x^{\alpha-1} e^{-x}$

then

$$\lim_{x\to 0} x^{\mu} f(x) = \lim_{x\to 0} x^{\mu} \cdot x^{\alpha-1} e^{-x}$$

$$= \lim_{x\to 0} x^{\mu+\alpha-1} e^{-x}$$

$$= 1 \text{ if } \mu+\alpha-1 = 0$$

i.e. if $\mu = 1-\alpha$

Then $\mu < 1$ if $1-\alpha < 1$ i.e. if $-\alpha < 0$ i.e. if $\alpha > 0$

also $\mu > 0$ if $1-\alpha > 0$ i.e. if $\alpha < 1$

$\therefore$ $0 < \mu < 1$ if $0 < \alpha < 1$

Again $\mu \geq 1$ if $1-\alpha \geq 1$ i.e. if $\alpha \leq 0$

So by μ-test, the given integral is convergent if $0<\alpha$ and divergent if $\alpha \leq 0$

Again it was also proved that the integral is converge if $\alpha \geq 1$.

Hence the given integral is convergent if $\alpha > 0$ and divergent if $\alpha \leq 0$.

Problem: Discuss the convergence of $\int_0^{\infty} \frac{x^{\alpha-1}}{1+x}\, dx$

Solution: We write

$$\int_0^{\infty} \frac{x^{\alpha-1}}{1+x}\, dx = \int_0^{a} \frac{x^{\alpha-1}}{1+x}\, dx + \int_a^{\infty} \frac{x^{\alpha-1}}{1+x}\, dx \quad \text{——} (1)$$

where $0 < a < \alpha$

Now if $\alpha \geq 1$, $\int_0^a \frac{x^{\alpha-1}}{1+x}dx$ is a proper integral and so Convergent.

Again Consider $\int_a^\infty \frac{x^{\alpha-1}}{1+x}dx$ if $\alpha \geq 1$

Let $f(x) = \frac{x^{\alpha-1}}{1+x}$ if $\alpha \geq 1$

Then $\lim_{x\to\infty} x^\mu f(x) = \lim_{x\to\infty} x^\mu \cdot \frac{x^{\alpha-1}}{1+x} = \lim_{x\to\infty} \frac{x^{\mu+\alpha-1}}{1+x}$

$= \lim_{x\to\infty} \frac{x^{\mu+\alpha-2}}{1+\frac{1}{x}}$

$= 1$ if $\mu+\alpha-2=0$

i.e., if $\mu = 2-\alpha$ $\because \alpha \geq 1$

i.e., if $\mu \leq 1$

So by μ-test $\int_a^\infty \frac{x^{\alpha-1}}{1+x}dx$ is divergent if $\alpha \geq 1$

Hence from (1), the given integral is divergent if $\alpha \geq 1$

Again if $\alpha < 1$, $x=0$ is a point of infinite discontinuity.

So we write

$$\int_0^\infty \frac{x^{\alpha-1}}{1+x}dx = \int_0^a \frac{x^{\alpha-1}}{1+x}dx + \int_a^\infty \frac{x^{\alpha-1}}{1+x}dx \quad \text{—— (2)}$$

& Consider $\int_0^a \frac{x^{\alpha-1}}{1+x}dx$

$\therefore \lim_{x\to 0} x^\mu f(x) = \lim_{x\to 0} x^\mu \cdot \frac{x^{\alpha-1}}{1+x} = \lim_{x\to 0} \frac{x^{\mu+\alpha-1}}{1+x}$

$= 1$ if $\mu+\alpha-1=0$

i.e. if $\mu = 1-\alpha < 1$ $\because \alpha < 1$

$\therefore 0 < \mu < 1$ if $0<\alpha<1$

Hence by μ-test, $\int_0^a \frac{x^{\alpha-1}}{1+x}dx$ is Convergent.

Next Consider $\int_a^\infty \frac{x^{\alpha-1}}{1+x}dx$ if $\alpha<1$

$\therefore \lim_{x\to\infty} x^\mu \cdot f(x) = \lim_{x\to\infty} x^\mu \cdot \frac{x^{\alpha-1}}{1+x} = \lim_{x\to\infty} \frac{x^{\mu+\alpha-1}}{1+x}$

$= \lim_{x\to\infty} \frac{x^{\mu+\alpha-2}}{1+\frac{1}{x}} = 1$ if $\mu+\alpha-2=0$

i.e, if $\mu = 2-\alpha > 1$ $\because \alpha<1$

So by μ-test $\int_a^\infty \frac{x^{\alpha-1}}{1+x}dx$ is convergent if $\alpha<1$

Hence from (2) we get

$\int_0^\infty \frac{x^{\alpha-1}}{1+x}dx$ is Convergent if $\alpha<1$

Thus the given integral is Convergent if $\alpha<1$ and divergent if $\alpha \geq 1$.

GAMMA AND BETA FUNCTIONS AND THEIR CONVERGE[NCE]

The integral $\int_0^\infty x^{n-1} e^{-x} dx, n>1$ is called a Gamma function and is denoted by Γn

$$\therefore \Gamma n = \int_0^\infty x^{n-1} e^{-x} dx, n>1$$

Integrating by parts, keeping x^{n-1} as the first function

$$\Gamma n = \left[x^{n-1} \cdot \frac{e^{-x}}{-1}\right]_0^\infty + (n-1)\int_0^\infty x^{n-2} e^{-x} dx$$

$$= \left[-\frac{x^{n-1}}{e^x}\right]_0^\infty + (n-1)\int_0^\infty x^{n-2} e^{-x} dx \quad \text{--- (1)}$$

Now $\lim_{x\to 0} \frac{x^{n-1}}{e^x} = \frac{0}{e} = 0$

and $\lim_{x\to\infty} \frac{x^{n-1}}{e^x} = \lim_{x\to\infty} \frac{x^{n-1}}{1+x+x^2/2!+\cdots}$

$$= \lim_{x\to\infty} \frac{1}{\frac{1}{x^{n-1}} + \frac{1}{x^{n-2}} + \cdots} = \frac{1}{\infty} = 0$$

$\therefore$ from (1), we get

$$\Gamma n = (n-1)\int_0^\infty x^{n-2} e^{-x} dx$$

$$= (n-1)\Gamma(n-1)$$

Proceeding in this way, we get

$$\Gamma(n-1) = (n-2)\Gamma(n-2)$$

$$\Gamma(n-2) = (n-3)\Gamma(n-3)$$

....

....

$$\Gamma 3 = 2\Gamma 2$$

$$\Gamma 2 = 1\Gamma 1$$

But $\Gamma 1 = \int_0^\infty x^{1-1} e^{-x} dx = \int_0^\infty e^{-x} dx$

$$= -\left[e^{-x}\right]_0^\infty = -\left[\frac{1}{e^\infty} - \frac{1}{e^0}\right] = -[0-1] = 1$$

Hence $\Gamma n = (n-1)(n-2)\cdots\cdots 2.1$

$$= \underline{|n-1}$$

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(*) Discuss the convergence of

$$\int_0^\infty x^{n-1} e^{-x} dx$$

Solution: Case-1: Let $n \geq 1$ and 'a' be a positive number such that $0 < a < \infty$.

Then $\int_0^\infty x^{n-1} e^{-x} dx = \int_0^a x^{n-1} e^{-x} dx + \int_a^\infty x^{n-1} e^{-x} dx$ —— (1)

Since $n \geq 1$, $\int_0^a x^{n-1} e^{-x} dx$ is a proper integral and so Convergent.

So, we are to test $\int_a^\infty x^{n-1} e^{-x} dx$ for convergence if $a > 0$ and $n \geq 1$.

Now $\lim_{x\to\infty} x^\mu f(x) = \lim_{x\to\infty} x^\mu \cdot \frac{x^{n-1}}{e^x}$

$= \lim_{x\to\infty} \frac{x^{\mu+n-1}}{e^x}$

$= \lim_{x\to\infty} \frac{x^m}{e^x}$ [Choosing $\mu > 1$ and putting $\mu+n-1 = m$ = positive]

$= 0$ for all values of m.

Thus choosing $\mu > 1$, we can make the above limit $= 0$ for all values of m.

$\therefore$ By μ-test $\int_0^\infty x^{n-1} e^{-x} dx$ is Convergent for all values of m.

Thus if $n \geq 1$, $\int_0^\infty x^{n-1} e^{-x} dx$ is Convergent.

Case-II: Let $0 < n < 1$ and we write

$$\int_0^\infty x^{n-1} e^{-x} dx = \int_0^a x^{n-1} e^{-x} dx + \int_a^\infty x^{n-1} e^{-x} dx \quad \text{—— (2)}$$

Now Consider $\int_0^a x^{n-1} e^{-x} dx$

Since $n < 1$, $x = 0$ is an infinite discontinuity.

$\therefore \lim_{x\to 0} x^\mu f(x) = \lim_{x\to 0} x^\mu . x^{n-1} . e^{-x}$

$= \lim_{x\to 0} \frac{x^{\mu+n-1}}{e^x} = 1$ if $\mu+n-1 = 0$
i.e. if $\mu = 1-n < 1 \because 0 < n < 1$

$\therefore$ By μ-Test $\int_0^a x^{n-1} e^{-x} dx$ is Convergent where $0 < n < 1$.

Also by case-I, $\int_0^\infty x^{n-1} e^{-x} dx$ is Convergent for all values of n.

Hence $\int_0^\infty x^{n-1} e^{-x} dx$ is Convergent for $0 < n < 1$.

Case-III: Let $n \leq 0$ and write

$$\int_0^\infty x^{n-1} e^{-x} dx = \int_0^a x^{n-1} e^{-x} dx + \int_a^\infty x^{n-1} e^{-x} dx \quad \text{—— (3)}$$

Consider $\int_0^a x^{n-1} e^{-x} dx$ for $n \leq 0$

Here $x=0$ is a point of infinite discontinuity

$$\therefore \lim_{x\to 0} x^{\mu} f(x) = \lim_{x\to 0} x^{\mu}\cdot x^{n-1} e^{-x}$$

$$= \lim_{x\to 0} \frac{x^{\mu+n-1}}{e^{x}}$$

$$= 1 \text{ if } \mu+n-1=0$$

i.e. if $\mu = 1-n \geq 1 \quad \because n\leq 0.$

$\therefore$ By μ-test $\int_0^a x^{n-1} e^{-x}dx$ is divergent if $n\leq 0$.

Also by Case-I, $\int_0^a x^{n-1}e^{-x}dx$ is convergent for all values of n.

$\therefore \int_0^\infty x^{n-1}e^{-x}dx$ is divergent if $n\leq 0$.

Hence, the given integral is convergent if and only if $n>0$ and divergent if $n\leq 0$.

Problem: Show that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

Solution: Since $\int_0^\infty e^{-x^2}dx$ is convergent, it can be shown that $\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$

Now in the gamma function,

$$\Gamma n = \int_0^\infty x^{n-1}e^{-x}dx \quad \text{---} \quad (1)$$

Put $x^n = y \quad \therefore n x^{n-1}dx = dy$

$\therefore x^{n-1}dx = \frac{1}{n}dy$

Also $x = y^{1/n}$

$\therefore -x = -y^{\frac{1}{n}}$

$\therefore$ From (1), we get

$$\Gamma n = \frac{1}{n}\int_0^\infty e^{-y^{1/n}}dy$$

$$\therefore \int_0^\infty e^{-y^{1/n}}dy = n\Gamma n$$

Putting $n=\frac{1}{2}$, we have

$$\int_0^\infty e^{-y^2}dy = \frac{1}{2}\Gamma(\tfrac{1}{2})$$

$$\therefore \int_0^\infty e^{-x^2}dx = \frac{1}{2}\Gamma(\tfrac{1}{2})$$

or $\frac{\sqrt{\pi}}{2} = \frac{1}{2}\Gamma(\frac{1}{2})$

or $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

N.H.'19

Problem: Assuming Convergence, Define the Gamma function Γx and establish the formula

$$\Gamma(x+1) = x\Gamma x, \quad 0<x<\infty$$

Solution: The Gamma function Γx is defined by the integral $\int_0^\infty y^{x-1} e^{-y} dy$, $x>1$ and denoted by Γx.

Thus $\Gamma x = \int_0^\infty y^{x-1} e^{-y} dy$, for $x>1$

Since the Gamma function is Convergent we get by integrating by parts

$$\Gamma x = \left[y^{x-1} \frac{e^{-y}}{-1}\right]_0^\infty + (x-1)\int_0^\infty y^{x-2} e^{-y} dy$$

$$= \left[-\frac{y^{x-1}}{e^y}\right]_0^\infty + (x-1)\Gamma(x-1)$$

$$= 0 + (x-1)\Gamma(x-1)$$

$$= (x-1)\Gamma(x-1)$$

Changing x by $x+1$ we get

$$\Gamma(x+1) = x\Gamma x$$

Problem: Prove that $\int_0^\infty y^{n-1} e^{-ky} dy = \frac{\Gamma n}{k^n}$

Solution: We have $\Gamma n = \int_0^\infty x^{n-1} e^{-x} dx$

Put $x = ky$ $\therefore dx = k\,dy$

$$\therefore \Gamma n = \int_0^\infty (ky)^{n-1} e^{-ky} . k\,dy$$

$$= \int_0^\infty k^{n-1} y^{n-1} e^{-ky} . k\,dy$$

$$= k^n \int_0^\infty y^{n-1} e^{-ky} dy$$

Hence $\int_0^\infty y^{n-1} e^{-ky} dy = \frac{\Gamma n}{k^n}$

Problem: Prove that $\Gamma n = \int_0^1 \left(\log\frac{1}{y}\right)^{n-1} dy$.

Solution: We have

$$\Gamma n = \int_0^\infty x^{n-1} e^{-x} dx \quad \text{--- (1)}$$

Put $e^{-x} = y$

$\therefore -e^{-x} dx = dy$ and $e^x = \frac{1}{y}$ or $\therefore x = \log\left(\frac{1}{y}\right)$

When $x=0$, $y=1$ and $x=\infty$, $y=0$

Then from (1), we get $\Gamma n = \int_1^0 \left\{\log\left(\frac{1}{y}\right)\right\}^{n-1} . (-dy) = \int_0^1 \left\{\log\left(\frac{1}{y}\right)\right\}^{n-1} dy$

*Problem: Discuss the Convergence of

$$\int_0^1 x^{m-1}(\log x)^n dx$$

Solution: We have $\int_0^1 x^{m-1}(\log x)^n dx$

Put $x = e^{-z}$ then $dx = -e^{-z}dz$

$$\therefore \int_0^1 x^{m-1}(\log x)^n dx = \int_\infty^0 (e^{-z})^{m-1}(-z)^n(-e^{-z}dz)$$

$$= (-1)^n \int_0^\infty e^{-mz+z} z^n e^{-z} dz$$

$$= (-1)^n \int_0^\infty z^n e^{-mz} dz$$

Now putting $mz = y$ $\therefore dz = \frac{dy}{m}$

$$= (-1)^n \int_0^\infty \left(\frac{y}{m}\right)^n e^{-y} \frac{dy}{m}$$

$$= \frac{(-1)^n}{m^{n+1}} \int_0^\infty y^n e^{-y} dy = \frac{(-1)^n}{m^{n+1}} \int_0^\infty y^{(n+1)-1} e^{-y} dy$$

$$= \frac{(-1)^n}{m^{n+1}} \int_0^\infty y^{(n+1)-1} e^{-y} dy$$

which is a Gamma function provided $m>0$ and so Convergent when $n+1>0$ i.e. when $n>-1$.

Hence the given integral is convergent if $m>0$ and $n>-1$

Beta Function and its Evolution:

Definition: The first Eulerian function is denoted by $B(m,n)$ and is defined by the integral

$$\int_0^1 x^{m-1}(1-x)^{n-1} dx, \quad m>0, n>0$$

i.e. $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx, \ m>0, n>0$

Thus $B(n,m) = \int_0^1 x^{n-1}(1-x)^{m-1} dx, \ n>0, m>0$

$= \int_0^1 (1-x)^{n-1}(1-1+x)^{m-1} dx$ Changing x by $1-x$ and adjusting the limit

$= \int_0^1 (1-x)^{n-1} x^{m-1} dx$

$= \int_0^1 x^{m-1}(1-x)^{n-1} dx$

$= B(m,n)$

*To prove that $B(m,n) = \frac{\Gamma m \Gamma n}{\Gamma(m+n)}$, m,n being positive integers.

Proof: We have $\Gamma n = \int_0^\infty x^{n-1} e^{-x} dx, \ n>0$

Now Consider

$$I = \int_0^\infty e^{-kx} x^{n-1}\, dx$$

Putting $kx = z$ so that $k\,dx = dz$

$\therefore dx = \frac{dz}{k}$

we get

$$I = \int_0^\infty e^{-z} \left(\frac{z}{k}\right)^{n-1} \cdot \frac{dz}{k}$$

$$= \int_0^\infty e^{-z} \cdot \frac{z^{n-1}}{k^{n-1}} \cdot \frac{dz}{k}$$

$$= \frac{1}{k^n} \int_0^\infty z^{n-1} \cdot e^{-z}\, dz$$

$$= \frac{\Gamma n}{k^n}$$

$$\therefore \Gamma n = k^n \cdot I$$

$$= k^n \cdot \int_0^\infty e^{-kx} x^{n-1}\, dx$$

$$= \int_0^\infty e^{-kx} k^n x^{n-1}\, dx$$

or $$\Gamma n\, e^{-k} k^{m-1} = \int_0^\infty e^{-k} k^{m-1} e^{-kx} k^n x^{n-1}\, dx$$

$$= \int_0^\infty e^{-k(1+x)} k^{m+n-1} x^{n-1}\, dx$$

Integrating with respect to k from 0 to ∞, we have

$$\Gamma n \int_0^\infty e^{-k} k^{m-1}\, dk = \int_0^\infty \left\{ \int_0^\infty e^{-k(1+x)} k^{m+n-1}\, dk \right\} x^{n-1}\, dx \quad \text{—— (1)}$$

Now to integrate $\int_0^\infty e^{-k(1+x)} k^{m+n-1}\, dk$

Put $k(1+x) = y \quad \therefore dk = \frac{dy}{1+x}$

$$\therefore \text{Integral} = \int_0^\infty e^{-y} \cdot \left(\frac{y}{1+x}\right)^{m+n-1} \cdot \frac{dy}{1+x}$$

$$= \frac{1}{(1+x)^{m+n}} \int_0^\infty e^{-y} y^{m+n-1}\, dy$$

$$= \frac{1}{(1+x)^{m+n}} \Gamma(m+n)$$

So from (1), we have

$$\Gamma n\, \Gamma m = \int_0^\infty \frac{\Gamma(m+n)}{(1+x)^{m+n}} x^{n-1}\, dx$$

or $$\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\, dx = \frac{\Gamma n\, \Gamma m}{\Gamma(m+n)} \quad \text{—— (2)}$$

Now from the definition of Beta function

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx, \quad m>0, n>0$$

Putting $x = \frac{y}{1+y}$ So that $\frac{-y\,dy + (1+y)dy}{(1+y)^2} = dx$

$$\therefore dx = \frac{dy}{(1+y)^2}$$

and adjusting the limit, we get.

$$B(m,n) = \int_0^\infty \left(\frac{y}{1+y}\right)^{m-1}\left(1 - \frac{y}{1+y}\right)^{n-1} \cdot \frac{dy}{(1+y)^2}$$

$$= \int_0^\infty \frac{y^{m-1}}{(1+y)^{m-1}} \cdot \frac{1}{(1+y)^{n-1}} \cdot \frac{dy}{(1+y)^2}$$

$$= \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}dy = \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}dy \text{ [interchanging } m \,\&\, n\text{]}$$

$$= \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}dx$$

Hence from (2), we have

$$B(m,n) = \frac{\Gamma n \, \Gamma m}{\Gamma(m+n)} \quad \text{——— (3)}$$

Finally we get

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx = \frac{\Gamma n \, \Gamma m}{\Gamma(m+n)}, \quad m>0, n>0.$$

which is the relation between Beta and Gamma function.

<u>Cor.1</u>: From (2) & (3)

$$B(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}dx, \quad m>0, n>0$$

$$= \int_0^\infty \frac{t^{n-1}}{(1+t)^{m+n}}dt, \quad m>0, n>0$$

$$* \quad = \int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}}dt \text{ (Interchanging } m \,\&\, n\text{)}$$

$$\therefore \boxed{B(x,y) = \int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}}dt \text{ where } x>0, y>0}$$

(*) Discuss the Convergence of the Beta function

$$\int_0^1 x^{m-1}(1-x)^{n-1}dx$$

<u>Proof</u>: <u>Case-I</u>: When $m-1 \geq 0$; $n-1 \geq 0$; the integral is finite for all values of x in $(0,1)$. So if $m \geq 1$ and $n \geq 1$, the given integral is convergent.

Case-II: When $m-1 \le 0$ and $n-1 < 0$

i.e. $m \le 1$ and $n < 1$, the integrand has infinite discontinuity at $x=0$ and $x=1$.

So for $0 < a < 1$, we write

$$\int_0^1 x^{m-1}(1-x)^{n-1}dx = \int_0^a x^{m-1}(1-x)^{n-1}dx + \int_a^1 x^{m-1}(1-x)^{n-1}dx$$

Now Consider $\int_0^a x^{m-1}(1-x)^{n-1}dx$ ——(1)

Then $\lim_{x\to 0} x^{\mu} f(x)$

$= \lim_{x\to 0} x^{\mu}\cdot x^{m-1}(1-x)^{n-1}$

$= \lim_{x\to 0} x^{\mu+m-1}(1-x)^{n-1}$

$= 1$ if $\mu+m-1=0$

i.e. if $\mu = 1-m \geq 1$ if $m \le 0$

Also μ lies between 0 and 1 if $0<m<1$

$\therefore \int_0^a x^{m-1}(1-x)^{n-1}dx$ is Convergent if $0<m<1$ and divergent if $m \le 0$

Again Consider $\int_a^1 x^{m-1}(1-x)^{n-1}dx$

We have

$\lim_{x\to 1-0} (1-x)^{\mu} f(x) = \lim_{x\to 1-0} (1-x)^{\mu}\cdot x^{m-1}(1-x)^{n-1}$

$= \lim_{x\to 1-0} (1-x)^{\mu+n-1}\cdot x^{m-1}$

$= \lim_{h\to 0} h^{\mu+n-1}\cdot(1-h)^{m-1}$

$= 1$ if $\mu+n-1=0$

i.e. if $\mu = 1-n \geq 1$ if $n \le 0$

and μ lies between 0 and 1 if $0<n<1$.

$\therefore \int_a^1 x^{m-1}(1-x)^{n-1}dx$ is Convergent if $0<n<1$ and divergent if $n \le 0$.

Hence from (1), we Conclude that the given integral is Convergent if m and n lies between 0 and 1, and divergen if $m \le 0$ and $n \le 0$.

i.e. Convergent for positive values of m and n and divergent for zero and negative values of m and n

Problem: using the Beta function, prove that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

Solution: We have
$$B(m,n) = \frac{\Gamma n\, \Gamma m}{\Gamma(m+n)}$$
$$\therefore B(\tfrac{1}{2},\tfrac{1}{2}) = \frac{\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}+\frac{1}{2})} = \frac{(\Gamma\frac{1}{2})^2}{\Gamma 1} = (\Gamma\tfrac{1}{2})^2 \quad \text{--- (1)}$$
$[\because \Gamma 1 = 1]$

Again
$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$$
Putting $x = \sin^2\theta$
$$= \int_0^{\pi/2} (\sin^2\theta)^{m-1}(\cos^2\theta)^{n-1} \cdot 2\sin\theta\cos\theta\, d\theta$$
$$= 2\int_0^{\pi/2} (\sin\theta)^{2m-1}(\cos\theta)^{2n-1}d\theta$$
$$\therefore B(\tfrac{1}{2},\tfrac{1}{2}) = 2\int_0^{\pi/2}(\sin\theta)^0(\cos\theta)^0 d\theta = 2\int_0^{\pi/2} d\theta$$
$$= 2.[\theta]_0^{\pi/2} = 2[\tfrac{\pi}{2} - 0] = \pi \quad \text{--- (2)}$$

From (1) & (2), we get
$$(\Gamma\tfrac{1}{2})^2 = B(\tfrac{1}{2},\tfrac{1}{2}) = \pi \qquad \therefore \Gamma\tfrac{1}{2} = \sqrt{\pi}$$

***Problem**: Prove that $\sqrt{\pi}\,\Gamma(2m) = 2^{2m-1}\Gamma(m)\Gamma(m+\frac{1}{2})$ *

Solution: We have
$$B(m,n) = \frac{\Gamma n\, \Gamma m}{\Gamma(m+n)}$$
i.e.
$$\frac{\Gamma(n)\Gamma(m)}{\Gamma(m+n)} = B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$$
$$= 2\int_0^{\pi/2}(\sin\theta)^{2m-1}(\cos\theta)^{2n-1}d\theta \quad \text{Putting } x = \sin^2\theta$$
$$= 2\int_0^{\pi/2}\sin^{2m-1}\theta\,\cos^{2n-1}\theta\, d\theta \quad \text{--- (1)}$$

Let $n = m$. Then we get
$$\frac{\Gamma(m)\Gamma(m)}{\Gamma(2m)} = 2\int_0^{\pi/2}\sin^{2m-1}\theta\,\cos^{2m-1}\theta\,d\theta$$
$$= \frac{2}{2^{2m-1}}\int_0^{\pi/2}(2\sin\theta\cos\theta)^{2m-1}d\theta$$
$$= \frac{1}{2^{2m-2}}\int_0^{\pi/2}\sin^{2m-1}2\theta\, d\theta$$

$$= \frac{1}{2^{2m-2}} \int_0^{\pi} \sin^{2m-1}\varphi \cdot \frac{d\varphi}{2}$$ Put $2\theta = \varphi$ So that $2d\theta = d\varphi$

$$= \frac{1}{2^{2m-1}} \int_0^{\pi} \sin^{2m-1}\varphi \, d\varphi = \frac{1}{2^{2m-2}} \int_0^{\pi/2} \sin^{2m-1}\varphi \, d\varphi \quad \text{——} (2)$$

Putting $n = \frac{1}{2}$ in (1), we get

$$\frac{\Gamma(m)\Gamma(\frac{1}{2})}{\Gamma(m+\frac{1}{2})} = 2\int_0^{\pi/2} \sin^{2m-1}\theta \, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\varphi \, d\varphi \quad \text{——} (3)$$

Now from (2) and (3), we get

$$\frac{\Gamma(m)\Gamma(m)}{\Gamma(2m)} = \frac{1}{2^{2m-2}} \cdot \frac{\Gamma(m)\Gamma(\frac{1}{2})}{2\Gamma(m+\frac{1}{2})} = \frac{1}{2^{2m-1}} \cdot \frac{\Gamma(m)\Gamma(\frac{1}{2})}{\Gamma(m+\frac{1}{2})}$$

ie, $$\frac{\Gamma(m)}{\Gamma(2m)} = \frac{1}{2^{2m-1}} \cdot \frac{\Gamma(\frac{1}{2})}{\Gamma(m+\frac{1}{2})}$$

ie, $$\Gamma(\tfrac{1}{2})\Gamma(2m) = 2^{2m-1}\Gamma(m)\Gamma(m+\tfrac{1}{2})$$

ie. $$\sqrt{\pi}\,\Gamma(2m) = 2^{2m-1}\Gamma(m)\Gamma(m+\tfrac{1}{2})$$

* Problem: Prove by integration by parts that

$$(m+n+1)B(m+1,n+1) = n\,B(m+1,n)$$

if $B(m+1,n+1) = \int_0^1 x^m(1-x)^n dx$ where m, n are positive integers. Deduce that $B(m+1,n+1) = \frac{\lfloor m \lfloor n}{\lfloor m+n+1}$

Solution:

We have

$$(m+n+1)B(m+1,n+1) = (m+n+1)\int_0^1 x^m(1-x)^n dx$$

$$= (m+n+1)\int_0^1 x^m(1-x)^{n-1}(1-x)\,dx$$

$$= (m+n+1)\int_0^1 x^m(1-x)^{n-1}dx - (m+n+1)\int_0^1 x^{m+1}(1-x)^{n-1}dx$$

$$= (m+n+1)B(m+1,n) - (m+n+1)\int_0^1 x^{m+1}\cdot(1-x)^{n}\frac{(1-x)^{n-1}}{dx}dx$$

$$= (m+n+1)B(m+1,n) - (m+n+1)\left[\left\{x^{m+1}\cdot -\frac{(1-x)^n}{n}\right\}_0^1 + \frac{m+1}{n}\int_0^1 x^m(1-x)^n dx\right]$$

$$= (m+n+1)B(m+1,n) - \frac{(m+n+1)(m+1)}{n}B(m+1,n+1)$$

or $(m+n+1)\,B(m+1,n+1) = (m+n+1)\,B(m+1,n) - \frac{(m+n+1)(m+1)}{n}\,B(m+1,n+1)$

or $(m+n+1)\,B(m+1,n+1) + \frac{(m+n+1)(m+1)}{n}\,B(m+1,n+1) = (m+n+1)\,B(m+1,n)$

or, $(m+n+1)\left(1+\frac{m+1}{n}\right)B(m+1,n+1) = (m+n+1)\,B(m+1,n)$

or $\left(1+\frac{m+1}{n}\right)B(m+1,n+1) = B(m+1,n)$

or $\frac{(m+n+1)}{n}\,B(m+1,n+1) = B(m+1,n)$

or $(m+n+1)\,B(m+1,n+1) = nB(m+1,n)$

$\therefore (m+n+1)\,B(m+1,n+1) = n\,B(m+1,n)$ Proved

Deduction:

We have $\frac{B(m+1,n+1)}{B(m+1,n)} = \frac{n}{m+n+1}$.

$$\therefore \frac{B(m+1,n)}{B(m+1,n-1)} = \frac{n-1}{m+n}$$

$$\frac{B(m+1,n-1)}{B(m+1,n-2)} = \frac{n-2}{m+n-1}.$$

$$\cdots\cdots \qquad \cdots$$

$$\frac{B(m+1,2)}{B(m+1,1)} = \frac{1}{m+2}$$

Multiplying into vertical columns, we get

$$\frac{B(m+1,n+1)}{B(m+1,1)} = \frac{n(n-1)(n-2)\cdots 1}{(m+n+1)(m+n)\cdots(m+2)}$$

$$= \frac{n!}{(m+n+1)(m+n)\cdots(m+2)}$$

$$= \frac{n!\,m!\,(m+1)}{(m+n+1)(m+n)\cdots(m+2)(m+1)\,m!}$$

$$= \frac{n!\,m!\,(m+1)}{(m+n+1)!}$$

$$\therefore B(m+1,n+1) = \frac{n!\,m!\,(m+1)}{(m+n+1)!}\,B(m+1,1)$$

$$= \frac{n!\,m!\,(m+1)}{(m+n+1)!}\int_0^1 x^m\,dx$$

$$= \frac{n!\,m!\,(m+1)}{(m+n+1)!}\cdot\left[\frac{x^{m+1}}{m+1}\right]_0^1$$

$$= \frac{n!\,m!\,(m+1)}{(m+n+1)!}\cdot\frac{1}{m+1}$$

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$\therefore B(m+1, n+1) = \frac{\lceil m \lceil n}{\lceil m+n+1}$ Proved.

Problem: Show that

$$I_{m,n} = \int_0^\infty \frac{x^m\,dx}{(1+x)^{m+n}} \text{ is Convergent}$$

when m and $n-1$ are positive integers.

Solution:

We have $I_{m,n} = \int_0^\infty \frac{x^m\,dx}{(1+x)^{m+n}}$

Putting $\frac{x}{1+x} = z$

we get $\frac{dx}{(1+x)^2} = dz$ and $x = \frac{z}{1-z}$

$\therefore 1+x = 1+\frac{z}{1-z} = \frac{1-z+z}{1-z} = \frac{1}{1-z}$

Also as x changes from 0 to ∞, z changes from 0 to 1

$$\therefore I_{m,n} = \int_0^1 \left(\frac{x}{1+x}\right)^m \cdot \frac{1}{(1+x)^{n-2}} \cdot \frac{1}{(1+x)^2}\,dx$$

$$= \int_0^1 z^m \cdot \frac{1}{\frac{1}{(1-z)^{n-2}}} \cdot dz = \int_0^1 z^m \cdot (1-z)^{n-2}\,dz$$

$= B(m+1, n-1)$ which is a Beta function and so converges when $m+1$ and $n-1$ are positive integers i.e. when m and $n-1$ are positive integers.

Problem: Show that $\int_0^{\pi/2} (\cos x)^l (\sin x)^m\,dx$ is Convergent if and only if $l > -1$, $m > -1$.

Solution: We have $I = \int_0^{\pi/2} (\cos x)^l (\sin x)^m\,dx$

Putting $\sin^2 x = z$

So that $\cos^2 x = 1 - \sin^2 x = 1 - z$

$\therefore 2\sin x \cos x\,dx = dz$

Then

$$I = \int_0^{\pi/2} (\cos^2 x)^{l/2} (\sin^2 x)^{m/2}\,dx$$

$$= \int_0^1 (1-z)^{l/2} z^{m/2} \frac{dz}{2z^{1/2}(1-z)^{1/2}}$$

$$= \frac{1}{2}\int_0^1 z^{\frac{m-1}{2}} (1-z)^{\frac{l-1}{2}}\,dz$$

$$= \frac{1}{2}\int_0^1 z^{\frac{m+1}{2}-1} (1-z)^{\frac{l+1}{2}-1}\,dz$$

$= \frac{1}{2} B\left(\frac{m+1}{2}, \frac{l+1}{2}\right)$ which is a Beta function and so converges when $\frac{m+1}{2} > 0$ and $\frac{l+1}{2} > 0$ i.e. when $m > -1$ and $l > -1$

Problem : Show that

(i) $B(m,n) = B(m+1,n) + B(m,n+1)$

(ii) $\dfrac{B(m+1,n)}{m} = \dfrac{B(m,n+1)}{n} = \dfrac{B(m,n)}{m+n}$

Solution: (i) we have

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$$

$$= \int_0^1 x^{m-1}(1-x)^{n-1}(x+1-x)\,dx$$

$$= \int_0^1 x^{m-1}(1-x)^{n-1}\cdot x\,dx + \int_0^1 x^{m-1}(1-x)^{n-1}\cdot(1-x)\,dx$$

$$= \int_0^1 x^{m}(1-x)^{n-1}dx + \int_0^1 x^{m-1}(1-x)^{n}dx$$

$$= \int_0^1 x^{(m+1)-1}(1-x)^{n-1}dx + \int_0^1 x^{m-1}(1-x)^{n+1-1}dx$$

$$= B(m+1,n) + B(m,n+1)$$

(ii) we have

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$$

$$= \int_0^1 x^{m-1}(1-x)^{n-1}(x+1-x)\,dx$$

$$= \int_0^1 x^{m}(1-x)^{n-1}dx + \int_0^1 x^{m-1}(1-x)^{n}dx$$

$$= \int_0^1 x^{m}\cdot(1-x)^{n-1}dx + B(m,n+1)$$

$$= \left[x^{m}\cdot -\frac{(1-x)^{n}}{n}\right]_0^1 + \int_0^1 m\cdot x^{m-1}\cdot\frac{(1-x)^{n}}{n}dx + B(m,n+1)$$

$$= \frac{m}{n}\int_0^1 x^{m-1}(1-x)^{n}dx + B(m,n+1)$$

$$= \frac{m}{n}B(m,n+1) + B(m,n+1)$$

$$= \left(\frac{m}{n}+1\right)B(m,n+1) = \frac{m+n}{n}\cdot B(m,n+1)$$

$$\therefore \frac{B(m,n)}{m+n} = \frac{B(m,n+1)}{n}$$

Again

$$B(m,n) = \int_0^1 x^{m}(1-x)^{n-1}dx + \int_0^1 x^{m-1}(1-x)^{n}dx$$

$$= B(m+1,n) + \left[(1-x)^{n}\cdot\frac{x^{m}}{m}\right]_0^1 + \int_0^1 n(1-x)^{n-1}\frac{x^{m}}{m}dx$$

$$= B(m+1,n) + \frac{n}{m}\int_0^1 x^{m}(1-x)^{n-1}dx$$

$$= B(m+1,n) + \frac{n}{m} B(m+1,n)$$

$$= B(m+1,n)\left(1+\frac{n}{m}\right)$$

$$= B(m+1,n)\,\frac{m+n}{m}$$

$$\therefore \frac{B(m,n)}{m+n} = \frac{B(m+1,n)}{m}$$

Finally we get

$$\frac{B(m+1,n)}{m} = \frac{B(m,n+1)}{n} = \frac{B(m,n)}{m+n}$$

Problem: Show that $\int_0^{\pi/2} \sin^m\theta \cos^n\theta \, d\theta$

$$= \frac{1}{2} B\left(\frac{m+1}{2}, \frac{n+1}{2}\right) \text{ where } m>-1, n>-1$$

$$= \frac{1}{2} \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{m+n}{2}+1\right)}$$

Deduce the values of

$$\int_0^{\pi/2} \sin^m\theta \cos^n\theta \, d\theta, \quad \int_0^{\pi/2} \sin^m\theta \, d\theta$$

when m and n are positive integers.

Solution: we have

$$I = \int_0^{\pi/2} \sin^m\theta \cos^n\theta \, d\theta$$

Putting $\sin^2\theta = z$ so that $\cos^2\theta = 1-\sin^2\theta$
$= 1-z$

we get $2\sin\theta\cos\theta \, d\theta = dz$

Then $I = \int_0^{\pi/2} (\sin^2\theta)^{m/2} (\cos^2\theta)^{n/2} d\theta$

$$= \int_0^1 z^{m/2}(1-z)^{n/2} \cdot \frac{dz}{2z^{\frac{1}{2}}(1-z)^{1/2}}$$

$$= \frac{1}{2}\int_0^1 z^{\frac{m-1}{2}} (1-z)^{\frac{n-1}{2}} dz$$

$$= \frac{1}{2}\int_0^1 z^{\frac{m+1}{2}-1} (1-z)^{\frac{n+1}{2}-1} dz$$

$= \frac{1}{2} B\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$ which is a Beta function and so converges when $\frac{m+1}{2} > 0$ and $\frac{n+1}{2} > 0$ i.e. when $m>-1$ and $n>-1$

$$= \frac{1}{2} \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{m+n}{2}+1\right)}$$

Problem: Prove that $\int_0^1 x^{m-1}(1-x^p)^{n-1}dx = \frac{1}{p}B\left(\frac{m}{p},n\right)$

Solution: We have

$$I = \int_0^1 x^{m-1}(1-x^p)^{n-1}dx$$

Putting $x^p = z$ So that $px^{p-1}dx = dz$

$$\therefore I = \int_0^1 z^{\frac{m-1}{p}}(1-z)^{n-1}\cdot\frac{dz}{pz^{\frac{p-1}{p}}}$$

$$= \frac{1}{p}\int_0^1 z^{\frac{m-p}{p}}(1-z)^{n-1}dz$$

$$= \frac{1}{p}\int_0^1 z^{\frac{m}{p}-1}(1-z)^{n-1}dz$$

$$= \frac{1}{p}B\left(\frac{m}{p},n\right)$$

Problem: Show if a, b, m, n be all positive,

$$\int_0^a x^{m-1}(a^b-x^b)^{n-1}dx = \frac{a^{b(n-1)+m}}{b}\cdot B\left(\frac{m}{b},n\right)$$

Solution: We have

$$I = \int_0^a x^{m-1}(a^b-x^b)^{n-1}dx$$

$$= \int_0^a x^{m-1}\left[1-\left(\frac{x}{a}\right)^b\right]^{n-1}\cdot a^{b(n-1)}dx$$

$$= a^{b(n-1)}\int_0^a x^{m-1}\left[1-\left(\frac{x}{a}\right)^b\right]^{n-1}dx$$

Putting $\left(\frac{x}{a}\right)^b = z$ So that $b\cdot\left(\frac{x}{a}\right)^{b-1}\cdot\frac{1}{a}dx = dz$

$$\therefore I = a^{b(n-1)}\int_0^1 \left(a z^{\frac{1}{b}}\right)^{m-1}(1-z)^{n-1}\cdot\frac{dz}{\frac{b}{a}\left(z^{\frac{1}{b}}\right)^{b-1}}$$

$$= a^{b(n-1)+m}\int_0^1 z^{\frac{m-1}{b}}\cdot z^{\frac{b-1}{b}}(1-z)^{n-1}dz$$

$$= a^{b(n-1)+m}\int_0^1 z^{\frac{m-b}{b}}(1-z)^{n-1}dz$$

$$= a^{b(n-1)+m}\int_0^1 z^{\frac{m}{b}-1}(1-z)^{n-1}dz$$

$$= a^{b(n-1)+m}B\left(\frac{m}{b},n\right)$$

Problem: Show that $\int_0^n \left(1-\frac{x}{n}\right)^n x^{m-1}dx = n^m B(m,n+1)$

where n is a positive integer and $m>0$.

Solution: We have

$$I = \int_0^n \left(1-\frac{x}{n}\right)^n x^{m-1}dx$$

Putting $x/n = z$ So that $x = nz$

we get $dx = n\,dz$

Then $I = \int_0^1 (1-z)^n (nz)^{m-1} n\,dz$

$= n^m \int_0^1 z^{m-1}(1-z)^n \, dz$

$= n^m B(m, n+1)$ which is a Beta function and so converges when n is a positive integer and $m>0$.

Problem: Prove that $\displaystyle\int_0^{\pi/2} \frac{\sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta}{(a\sin^2\theta + b\cos^2\theta)^{m+n}} = \frac{B(m,n)}{2a^m b^n}$

$$= \frac{\Gamma(m)\Gamma(n)}{2a^m b^n \Gamma(m+n)}, \; m>0, n>0$$

Solution: We have

$$\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = B(m,n) = \int_0^1 y^{m-1}(1-y)^{n-1} dy$$

Putting $y = \frac{x}{1+x}$

so that $dy = \frac{dx}{(1+x)^2}$

$$= \int_0^\infty \frac{x^{m-1}}{(1+x)^{m-1}} \cdot \frac{1}{(1+x)^{n-1}} \frac{dx}{(1+x)^2}$$

$$= \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$$

Again put $x = \frac{ay}{b}$ so that $dx = \frac{a}{b} dy$

$$= \int_0^\infty \frac{\left(\frac{a}{b}y\right)^{m-1}}{\left(1+\frac{a}{b}y\right)^{m+n}} \cdot \frac{a}{b} dy$$

$$= a^m b^n \int_0^\infty \frac{y^{m-1}}{(ay+b)^{m+n}} dy$$

$$\therefore \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = a^m b^n \int_0^\infty \frac{y^{m-1}}{(ay+b)^{m+n}} dy$$

i.e. $$\frac{\Gamma(m)\Gamma(n)}{a^m b^n \Gamma(m+n)} = \int_0^\infty \frac{y^{m-1}}{(ay+b)^{m+n}} dy$$

Now we put $y = \tan^2\theta$ so that $dy = 2\tan\theta \sec^2\theta \, d\theta$

$$= \int_0^{\pi/2} \frac{(\tan^2\theta)^{m-1}}{\left(\frac{a\sin^2\theta + b\cos^2\theta}{\cos^2\theta}\right)^{m+n}} \cdot 2\tan\theta \cdot \sec^2\theta \, d\theta$$

$$= 2\int_0^{\pi/2} \frac{\tan^{2(m-1)}\theta}{(a\sin^2\theta + b\cos^2\theta)^{m+n}} \tan\theta \cdot \sec^2\theta \cos^{2(m+n)}\theta \, d\theta$$

$$= 2\int_0^{\pi/2} \frac{\tan^{2m-1}\theta}{(a\sin^2\theta + b\cos^2\theta)^{m+n}} \cdot \frac{1}{\cos^2\theta} \cdot \cos^{2(m+n)}\theta \, d\theta$$

$$= 2\int_0^{\pi/2} \frac{\frac{\sin^{2m-1}\theta}{\cos^{2m-1}\theta}\cos^{2m+2n-2}\theta}{(a\sin^2\theta + b\cos^2\theta)^{m+n}}\,d\theta$$

$$= 2\int_0^{\pi/2} \frac{\sin^{2m-1}\theta\,\cos^{-(2m-1)}\theta\,.\cos^{2m+2n-2}\theta}{(a\sin^2\theta + b\cos^2\theta)^{m+n}}\,d\theta$$

$$\therefore \frac{\Gamma(m)\Gamma(n)}{a^m b^n\,\Gamma(m+n)} = 2\int_0^{\pi/2} \frac{\sin^{2m-1}\theta\,\cos^{2n-1}\theta}{(a\sin^2\theta + b\cos^2\theta)^{m+n}}\,d\theta$$

$$\therefore \int_0^{\pi/2} \frac{\sin^{2m-1}\theta\,\cos^{2n-1}\theta}{(a\sin^2\theta + b\cos^2\theta)^{m+n}}\,d\theta = \frac{\Gamma(m)\Gamma(n)}{2a^m b^n\,\Gamma(m+n)} = \frac{B(m,n)}{2a^m b^n}$$

<u>Problem</u>: Evaluate $\int_0^\infty e^{-x^2}dx$

<u>Solution</u>: we have $\int_0^\infty e^{-x^2}dx$

Putting $x^2 = z$

So that $2x\,dx = dz \quad \therefore dx = \frac{dz}{2x} = \frac{dz}{2z^{1/2}}$

$$\therefore \int_0^\infty e^{-x^2}dx = \int_0^\infty e^{-z}\frac{dz}{2z^{1/2}}$$

$$= \frac{1}{2}\int_0^\infty e^{-z}z^{-\frac{1}{2}}\,dz = \frac{1}{2}\int_0^\infty z^{\frac{1}{2}-1}e^{-z}dz$$

$$= \frac{1}{2}\Gamma\left(\tfrac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$$

we have

$$\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = B(m,n) \quad \text{Then } B\left(\tfrac{1}{2},\tfrac{1}{2}\right) = \frac{\Gamma(1/2)\Gamma(1/2)}{\Gamma(1)}$$

$$\therefore B\left(\tfrac{1}{2},\tfrac{1}{2}\right) = \left[\Gamma\left(\tfrac{1}{2}\right)\right]^2$$

Again $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$

Put $x = \sin^2\theta$

So that $dx = 2\sin\theta\cos\theta\,d\theta$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta\,\cos^{2n-1}\theta\,d\theta$$

$$\therefore B\left(\tfrac{1}{2},\tfrac{1}{2}\right) = 2\int_0^{\pi/2} d\theta = 2[\theta]_0^{\pi/2} = 2[\pi/2 - 0] = \pi$$

finally we get $\left[\Gamma\left(\tfrac{1}{2}\right)\right]^2 = \pi \quad \therefore \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$

<u>Problem</u>: Show that $\Gamma\left(\tfrac{1}{3}\right)\Gamma\left(\tfrac{2}{3}\right) = \frac{2\pi}{\sqrt{3}}$

<u>Solution</u>: We know that $\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}$

Putting $n = 1/3$

$$\therefore \Gamma\left(\tfrac{1}{3}\right)\Gamma\left(1-\tfrac{1}{3}\right) = \frac{\pi}{\sin \pi/3}$$

$$\Rightarrow \Gamma\left(\tfrac{1}{3}\right)\Gamma\left(\tfrac{2}{3}\right) = \frac{\pi}{\sqrt{3}/2} = \frac{2\pi}{\sqrt{3}}$$

Problem: Prove that $\int_0^{\pi/2} \frac{dx}{\sqrt{\sin x}} \times \int_0^{\pi/2} \sqrt{\sin x}\, dx = \pi$

Solution: We have

$$I = \int_0^{\pi/2} \frac{dx}{\sqrt{\sin x}} \times \int_0^{\pi/2} \sqrt{\sin x}\, dx$$

$$= \int_0^{\pi/2} (\sin x)^{-\frac{1}{2}} dx \times \int_0^{\pi/2} (\sin x)^{1/2} dx$$

$$= \int_0^{\pi/2} (\sin x)^{-\frac{1}{2}} \cos^0 x\, dx \times \int_0^{\pi/2} (\sin x)^{1/2} \cos^0 x\, dx$$

$$= \frac{\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{-\frac{1}{2}+0+2}{2}\right)} \times \frac{\Gamma\left(\frac{\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{\frac{1}{2}+0+2}{2}\right)}$$

$$= \frac{\Gamma(\frac{1}{4})\Gamma(\frac{1}{2})}{2\Gamma(3/4)} \times \frac{\Gamma(3/4)\Gamma(1/2)}{2\Gamma(5/4)}$$

$$= \frac{\Gamma(1/4)\Gamma(1/2)}{2\Gamma(3/4)} \times \frac{\Gamma(3/4)\Gamma(1/2)}{2\cdot\Gamma(\frac{1}{4}+1)}$$

$$= \frac{\cancel{\Gamma(1/4)}\Gamma(1/2)}{\cancel{2\Gamma(3/4)}} \times \frac{\cancel{\Gamma(3/4)}\Gamma(1/2)}{\cancel{2}\cdot\cancel{\frac{1}{4}}\cancel{\Gamma(1/4)}} \quad \text{as } \Gamma(n+1) = n\Gamma(n)$$

$$= \left[\Gamma(\tfrac{1}{2})\right]^2 = (\sqrt{\pi})^2 = \pi$$

Problem: Show that $\int_0^1 \frac{dx}{(1-x^6)^{1/6}} = \pi/3$

Solution: Let $I = \int_0^1 \frac{dx}{(1-x^6)^{1/6}}$

Putting $x^6 = \sin^2\theta$

So that $6x^5\, dx = 2\sin\theta\cos\theta\, d\theta$

$$\therefore 3x^5\, dx = \sin\theta\cos\theta$$

$$\therefore dx = \frac{1}{3}\frac{\sin\theta\cos\theta}{x^5} d\theta$$

$$= \frac{1}{3}\frac{\sin\theta\cos\theta}{\sin^{5/3}\theta}$$

$$= \frac{1}{3}\sin^{-2/3}\theta\cos\theta\, d\theta$$

$$\therefore I = \frac{1}{3}\int_0^{\pi/2} \frac{\sin^{-2/3}\theta\cos\theta\, d\theta}{(1-\sin^2\theta)^{1/6}} = \int_0^{\pi/2} \frac{\sin^{-2/3}\theta\cos\theta\, d\theta}{\cos^{1/3}\theta}$$

$$= \frac{1}{3}\int_0^{\pi/2} \sin^{-2/3}\theta\cos^{2/3}\theta\, d\theta$$

$$= \frac{1}{3}\cdot\frac{\Gamma\left(\frac{-2/3+1}{2}\right)\Gamma\left(\frac{2/3+1}{2}\right)}{2\Gamma\left(\frac{-2/3+2/3+2}{2}\right)} = \frac{1}{3}\cdot\frac{1}{2}\frac{\Gamma(1/6)\Gamma(5/6)}{\Gamma(1)}$$

$$= \frac{1}{6}\Gamma(1/6)\Gamma(1-\tfrac{1}{6}) = \frac{1}{6}\frac{\pi}{\sin\pi/6} \quad \text{as } \Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}$$

$= \frac{1}{6}\cdot\frac{\pi}{\sin 30^\circ} = \frac{1}{6}\cdot\frac{\pi}{1/2} = \pi/3$

Problem: Prove that $\int_0^1 \frac{x^2\,dx}{(1-x^4)^{1/2}} \times \int_0^1 \frac{dx}{(1+x^4)^{1/2}} = \frac{\pi}{4\sqrt{2}}$

Solution: Let $I = \int_0^1 \frac{x^2\,dx}{(1-x^4)^{1/2}} \times \int_0^1 \frac{dx}{(1+x^4)^{1/2}}$

$= I_1 \times I_2$

where $I_1 = \int_0^1 \frac{x^2\,dx}{(1-x^4)^{1/2}}$ & $I_2 = \int_0^1 \frac{dx}{(1+x^4)^{1/2}}$

now $I = \int_0^1 \frac{x^2\,dx}{(1-x^4)^{1/2}}$ Putting $x^2 = \sin\theta$

So that $2x\,dx = \cos\theta\,d\theta$

$\therefore dx = \frac{\cos\theta\,d\theta}{2x} = \frac{\cos\theta\,d\theta}{2\sqrt{\sin\theta}}$

$= \int_0^{\pi/2} \frac{\sin\theta}{(1-\sin^2\theta)^{1/2}}\cdot\frac{\cos\theta}{2\sin^{1/2}\theta}\,d\theta$

$= \int_0^{\pi/2} \frac{\sin\theta}{\cancel{\cos\theta}}\cdot\frac{\cancel{\cos\theta}}{2\sin^{1/2}\theta}\,d\theta$

$= \frac{1}{2}\int_0^{\pi/2} \sin^{\frac{1}{2}}\theta\,d\theta = \frac{1}{2}\int_0^{\pi/2} \sin^{1/2}\theta\cos^0\theta\,d\theta$

$= \frac{1}{2}\cdot\frac{\Gamma\left(\frac{\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{\frac{1}{2}+0+2}{2}\right)} = \frac{1}{4}\frac{\Gamma(3/4)\,\Gamma(1/2)}{\Gamma(5/4)}$

$= \frac{1}{4}\frac{\Gamma(3/4)\,\Gamma(1/2)}{\Gamma(\frac{1}{4}+1)} = \frac{1}{4}\frac{\Gamma(3/4)\,\Gamma(1/2)}{\frac{1}{4}\Gamma(1/4)}$ as $\Gamma(n+1) = n\Gamma(n)$

$= \frac{\Gamma(3/4)\,\Gamma(1/2)}{\Gamma(1/4)}$

now Consider $I_2 = \int_0^1 \frac{dx}{(1+x^4)^{1/2}}$

Putting $x^2 = \tan\theta$ So that $2x\,dx = \sec^2\theta\,d\theta$

$\therefore dx = \frac{1}{2}\frac{\sec^2\theta}{\tan^{1/2}\theta}\,d\theta$

$= \frac{1}{2}\int_0^{\pi/4} \frac{\sec^2\theta}{(1+\tan^2\theta)^{1/2}\tan^{1/2}\theta}\,d\theta$

$= \frac{1}{2}\int_0^{\pi/4} \frac{\sec^2\theta}{\sec\theta\tan^{1/2}\theta}\,d\theta = \frac{1}{2}\int_0^{\pi/4} \frac{\sec\theta}{\sqrt{\tan\theta}}\,d\theta$

$= \frac{1}{2}\int_0^{\pi/4} \frac{d\theta}{\sqrt{\frac{\sin\theta}{\cos\theta}\cdot\cos^2\theta}} = \frac{1}{2}\int_0^{\pi/4} \frac{d\theta}{\sqrt{\sin\theta\cos\theta}}$

$= \frac{1}{\sqrt{2}}\int_0^{\pi/4} \frac{d\theta}{\sqrt{2\sin\theta\cos\theta}} = \frac{1}{\sqrt{2}}\int_0^{\pi/4} \frac{d\theta}{\sqrt{\sin 2\theta}}$

Again put $2\theta = t$

$\therefore 2\,d\theta = dt$

i.e. $d\theta = dt/2$

$$= \frac{1}{\sqrt{2}} \int_0^{\pi/2} \sin^{-1/2} t \cdot \frac{dt}{2}$$

$$= \frac{1}{2\sqrt{2}} \int_0^{\pi/2} \sin^{-1/2} t \cos^0 t \, dt$$

$$= \frac{1}{2\sqrt{2}} \frac{\Gamma\left(\frac{-1/2+1}{2}\right) \Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{-1/2+0+2}{2}\right)} = \frac{1}{4\sqrt{2}} \frac{\Gamma(1/4)\,\Gamma(1/2)}{\Gamma(3/4)}$$

Finally $I = I_1 \times I_2$

$$= \frac{\cancel{\Gamma(3/4)}\,\Gamma(1/2)}{\cancel{\Gamma(1/4)}} \times \frac{1}{4\sqrt{2}} \frac{\cancel{\Gamma(1/4)}\,\Gamma(1/2)}{\cancel{\Gamma(3/4)}}$$

$$= \frac{[\Gamma(1/2)]^2}{4\sqrt{2}} = \frac{(\sqrt{\pi})^2}{4\sqrt{2}} = \frac{\pi}{4\sqrt{2}}$$

Problem: Prove that $\displaystyle\int_0^1 \frac{dx}{\sqrt{(1-x^n)}} = \frac{\sqrt{\pi}}{n} \frac{\Gamma(1/n)}{\Gamma\left(\frac{1}{2}+\frac{1}{n}\right)}$

$$= \frac{2^{\frac{2}{n}-1}\left[\Gamma(1/n)\right]^2}{n\,\Gamma(2/n)}$$

Solution: Let $\displaystyle I = \int_0^1 \frac{dx}{\sqrt{(1-x^n)}}$

Putting $x^n = \sin^2\theta$ so that $n x^{n-1} dx = 2\sin\theta\cos\theta\, d\theta$

$$\therefore dx = \frac{2}{n} \frac{\sin\theta\cos\theta}{x^{n-1}} d\theta$$

$$= \frac{2}{n} \frac{\sin\theta\cos\theta}{\sin^{\frac{2}{n}(n-1)}\theta} d\theta$$

$$= \frac{2}{n} \sin\theta \cdot \sin^{-\frac{2}{n}(n-1)}\theta \cos\theta\, d\theta$$

$$= \frac{2}{n} \sin\theta \cdot \sin^{2/n - 2}\theta \cos\theta\, d\theta$$

$$= \frac{2}{n} \sin^{2/n - 1}\theta \cos\theta\, d\theta$$

$$\therefore I = \frac{2}{n} \int_0^{\pi/2} \frac{\sin^{\frac{2}{n}-1}\theta \cos\theta\, d\theta}{\sqrt{(1-\sin^2\theta)}}$$

$$= \frac{2}{n} \int_0^{\pi/2} \frac{\sin^{2/n-1}\theta\, \cancel{\cos\theta}\, d\theta}{\cancel{\cos\theta}}$$

$$= \frac{2}{n} \int_0^{\pi/2} \sin^{2/n-1}\theta\, d\theta = \frac{2}{n} \int_0^{\pi/2} \sin^{\frac{2}{n}-1}\theta \cos^0\theta\, d\theta$$

$$= \frac{2}{n} \cdot \frac{\Gamma\left(\frac{\frac{2}{n}-1+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{2/n-1+0+2}{2}\right)} = \frac{1}{n} \frac{\Gamma(1/n)\,\Gamma(1/2)}{\Gamma\left(\frac{2/n+1}{2}\right)}$$

$$= \frac{1}{n} \frac{\Gamma(1/n)\,\Gamma(1/2)}{\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)} = \frac{\sqrt{\pi}}{n} \frac{\Gamma(1/n)}{\Gamma\left(\frac{1}{2}+\frac{1}{n}\right)}$$

According to duplication formula, we

$$\frac{\Gamma(m)\,\Gamma(n)}{2\,\Gamma(m+n)} = \int_0^{\pi/2} \sin^{2m-1}\theta \,\cos^{2n-1}\theta \, d\theta$$

Putting $m=\frac{1}{n}$, $n=\frac{1}{n}$; we get

$$\frac{\Gamma(\frac{1}{n})\,\Gamma(1/n)}{2\,\Gamma(2/n)} = \int_0^{\pi/2} \sin^{2/n-1}\theta \,\cos^{2/n-1}\theta \, d\theta$$

i.e. $$\frac{[\Gamma(1/n)]^2}{\Gamma(2/n)} = 2\int_0^{\pi/2} (\sin\theta\cos\theta)^{2/n-1} d\theta$$

$$= \frac{2}{2^{2/n-1}} \int_0^{\pi/2} (2\sin\theta\cos\theta)^{2/n-1} d\theta$$

$$= 2^{2-2/n} \int_0^{\pi/2} (\sin 2\theta)^{2/n-1} d\theta$$

$$= 2^{2-\frac{2}{n}} \int_0^{\pi} (\sin t)^{2/n-1} \cdot \frac{dt}{2}$$

Again putting $2\theta = t$

$\therefore 2d\theta = dt$

Then $d\theta = \frac{dt}{2}$

$$= 2^{1-\frac{2}{n}} \int_0^{\pi} (\sin t)^{2/n-1} dt$$

$$= 2^{1-\frac{2}{n}} \cdot 2 \int_0^{\pi/2} (\sin t)^{\frac{2}{n}-1} dt$$

$$= 2^{2-\frac{2}{n}} \int_0^{\pi/2} (\sin t)^{\frac{2}{n}-1} \cos^0\theta \, dt$$

$$= 2^{2-\frac{2}{n}} \frac{\Gamma\left(\frac{\frac{2}{n}-1+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\,\Gamma\left(\frac{\frac{2}{n}-1+0+2}{2}\right)}$$

$$= 2^{1-\frac{2}{n}} \frac{\Gamma(1/n)\,\Gamma(1/2)}{\Gamma(1/n+\frac{1}{2})}$$

$$\therefore \frac{2^{\frac{2}{n}-1}[\Gamma(1/n)]^2}{\Gamma(2/n)} = \frac{\Gamma(1/n)\,\Gamma(1/2)}{\Gamma(1/2+\frac{1}{n})} = \frac{\sqrt{\pi}\;\Gamma(1/n)}{\Gamma(1/2+\frac{1}{n})}$$

$$\therefore \frac{2^{2/n-1}[\Gamma(1/n)]^2}{n\,\Gamma(2/n)} = \frac{\sqrt{\pi}}{n} \cdot \frac{\Gamma(1/n)}{\Gamma(\frac{1}{2}+\frac{1}{n})}$$

Problem: Show that $\displaystyle\int_0^1 \frac{dx}{(1-x^n)^{1/n}} = \frac{\pi/n}{\sin \pi/n}$

Solution: Let $\displaystyle I = \int_0^1 \frac{dx}{(1-x^n)^{1/n}}$

Putting $x^n = \sin^2\theta$

So that $n x^{n-1} dx = 2\sin\theta\cos\theta\, d\theta$

$$\therefore \quad I = \frac{2}{n}\int_0^{\pi/2} \frac{\sin\theta\cos\theta \, d\theta}{\sin^{\frac{2}{n}(n-1)}\theta \, \cos^{\frac{2}{n}}\theta}$$

$$= \frac{2}{n}\int_0^{\pi/2} \sin^{\frac{2}{n}-1}\theta \, \cos^{1-\frac{2}{n}}\theta \, d\theta$$

$$= \frac{2}{n} \frac{\Gamma\left(\frac{2/n-1+1}{2}\right)\Gamma\left(\frac{1-\frac{2}{n}+1}{2}\right)}{2\,\Gamma\left(\frac{\frac{2}{n}-1+1-2/n+2}{2}\right)}$$

$$= \frac{1}{n} \frac{\Gamma(1/n)\,\Gamma\left(1-\frac{1}{n}\right)}{\Gamma(1)}$$

$$= \frac{1}{n} \frac{\Gamma(1/n)\,\Gamma\left(1-\frac{1}{n}\right)}{1} \quad \text{as } \Gamma(1)=1$$

$$= \frac{1}{n}\Gamma\left(\frac{1}{n}\right)\Gamma\left(1-\frac{1}{n}\right)$$

$$= \frac{1}{n}\frac{\pi}{\sin \pi/n} \quad \text{as } \Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}$$

$$= \frac{\pi/n}{\sin \pi/n}$$

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