বিদ্যাসাগর বিশ্ববিদ্যালয়

VIDYASAGAR UNIVERSITY

Question Paper

B.Sc. Honours Examinations 2021

(Under CBCS Pattern)

Semester - V

Subject : MATHEMATICS

Paper : C 12 - T

Full Marks : 60

Time : 3 Hours

Candidates are required to give their answers in their own words as far as practicable.
The figures in the margin indicate full marks.

[GROUP THEORY-II]

1. Answer any ***four*** questions : 12×4=48

(a) (i) Prove that *Aut* $(\mathbb{Z}_n, +)$ is isomorphic with U_n (the group of units modulo *n*).

(ii) Is the statement "S3×ℤ/12ℤ is isomorphic with ℤ/6ℤ×ℤ/12ℤ".—True or False? Justiy your answer.

(iii) Let G be a group acting on a non-empty set S and $|G| = p^n$ where p is a prime and n is a natural number. Show that $|S| \equiv_p |S_0|$ where $S_0 = \{a \in S \mid ga = a\ f\ or\ all\, g \in G\}$. 6+2+4=12

(b) (i) Let H and K be two characteristic subgroups of a group G. Prove that $H \cap K$ and HK are also characteristic subgroups of G.

(ii) Prove that the direct product $\mathbb{Z}_m \times \mathbb{Z}_n$ is isomorphic with $\mathbb{Z}_{mn}$ if and only if m and n are co-prime.

(iii) Let G be a group acting on a non-empty set S and G_x denote the stabilizer of $x \in G$. Then prove that $G_{bx} = bG_x b^{-1}$ for any $b \in G$. (2+2)+4+4=12

(c) (i) For any group $(G;)$, prove that $Inn(G)$ is a normal subgroup of $(Aut(G),o)$ where $Inn(G)$ denotes the set of all inner automorphisms of G.

(ii) Let $\alpha \in S_4$ be a k-cycle. Then prove that $\beta \in S_4$ is conjugate with α if and only if β is also a k-cycle.

(iii) Using (*ii*) deduce class equation of S_4. 2+6+4=12

(d) (i) Let a be non-zero rational number. Prove that $\phi \in Aut(\mathbb{Q}, +)$ where $\phi : (\mathbb{Q}, +) \to (\mathbb{Q}, +)$ is defined by $\phi(q) = qa$ for all $q \in \mathbb{Q}$. Hence prove that the only characteristic subgroups of $(\mathbb{Q}, +)$ are $\{0\}$ and $\mathbb{Q}$.

(ii) Prove that if G is a finite p-group then $Z(G)$ must be non-trivial.

(iii) Let G be a group of order pq where p, q are both primes with $p > q$. If q does not divide $(p - 1)$ then prove that G is cyclic. (3+2)+3+4=12

(e) (i) For any group $(G;)$ prove that $G/Z(G)$ is isomorphic with $Inn(G)$ where $Inn(G)$ denotes the set of all inner automorphisms of G.

(ii) Find the number of elements o order 5 in the direct product $\mathbb{Z}15 \times \mathbb{Z}5$.

(iii) Let G be a group and H be a subgroup of G. Let $S = \{aH \mid a \in G\}$. Prove that there exists a homomorphism $\psi : G \to A(S)$ such that $\ker \psi \subseteq H$.

4+4+4=12

(f) (i) Let G be a finite group and H be a subgroup of G of index p, where p is the smallest prime dividing $|G|$. Show that H is a normal subgroup of G.

(ii) Show that every non-cyclic group o order 21 contains exactly 14 elements of order 3. 6+6=12

(g) (i) Describe all the abelian groups of order 1200 up to isomorphism.

(ii) Using Sylow's theorem, show that every group of order 45 has a normal subgroup of order 9.

(iii) Prove that any group of order p^2 is abelian where p is a prime.

(iv) Can $4 = 1 + 1 + 2$ be a class equation for a group? Justify your answer.

5+2+3+2=12

(h) (i) Define commutator subgroup of a group G. Find the commutator subgroup of S_3.

(ii) Give example (with explanation) of a non-cyclic commutative group of order 28.

(iii) Using Sylow's theorem, show that no group of order 108 is a simple group. (2+3+2+5=12

Group-B

2. Answer any ***six*** questions : 2×6=12

(a) Prove that the set of all automorphisms of a group $(G;)$ form a group with respect to the composition of mappings.

(b) Define characteristic subgroup with an example.

(c) Consider two subgroups $H_1 = \{e, (1\ 2)\}$ and $H_2 = \{e, (1\ 2\ 3), (1\ 3\ 2)\}$ of S_3. Can S_3 be expressed as an internal direct product of the subgropups H_1 and H_2? Justify your answer with proper reasons.

(d) Consider the direct product $S_3 \times S_3$. Does it contain an element of order 9? Give reasons in support of your answer.

(e) Using the Fundamental theorem of finite abelian groups, describe all abelian groups o order 2^4 up to isomorphism.

(f) Consider the conjugation action of the group $(\mathbb{Z}_3, +)$ on itself. With respect to this group action, find all distinct orbits.

(g) Give an example of an infinite p-group.

(h) State Sylow's third theorem.

(i) Define a simple group with an example.

(j) Let *G* be a group of order 69. Prove that *Z(G)* is isomorphic with $\mathbb{Z}_{69}$.
