বিদ্যাসাগর বিশ্ববিদ্যালয়

VIDYASAGAR UNIVERSITY

Question Paper

B.Sc. Honours Examinations 2021

(Under CBCS Pattern)

Semester - III

Subject : MATHEMATICS

Paper : C 6 - T

Full Marks : 60

Time : 3 Hours

Candidates are required to give their answers in their own words as far as practicable.
The figures in the margin indicate full marks.

[GROUP THEORY-I]

(Theory)

Group - A

1. Answer any ***four*** of the following questions : 12×4=48

(i) (a) Let G be the set consisting of the six functions $f_1, f_2, \ldots\ldots, f_6$ defined on $R \setminus \{0,1\}$ by $f_1(x) = x,\ f_2(x) = 1 - x,\ f_3(x) = \frac{1}{x},\ f_4(x) = \frac{1}{1-x},$ $f_5(x) = \frac{x-1}{x},\ f_6(x) = \frac{x}{x-1}$ and let o be the composition of functions. Then show thag $(G, 0)$ is a non-abelian group. 7

(b) Let G be a group and $a \in G$ is of order n. Then $\left(a^{p}\right)=\frac{n}{gcd\left(n,p\right)}$. Hence determine $O\left(a^{3}\right)$ and $O\left(a^{8}\right)$, when $O\left(a\right)=12$. 5

(ii) (a) Let T be the group of 2 × 2 invertible matrices over R under usual matrix multiplication. Let G be the subgroup of T generated by the matrices $A=\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}$ and $A=\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$.

Prove that $G \cong D_4$ (i.e., G is a dihedral group of degree 4.) 7

(b) Let $G=\left\{\begin{pmatrix}a & a\\a & a\end{pmatrix} \middle| a \in Q^{*}\right\}$, where $Q^{*}=Q-\{0\}$. Then prove that G is an abelian group with respect to multiplication of matrices. 5

(iii) (a) Define center of a group. Find center of S_3. 6

(b) Let H be a subgroup of a group G and $N\left(H\right)=\left\{a \in G \mid aHa^{-1}=H\right\}$. Prove that $N(H)$ is a subgroup of G which contains H. 6

(iv) (a) Find the order of (1, 2, 3, 4) (5, 6, 7) in S_7. 2

(b) Find all elements of order 8 in $\left(Z_{24},+\right)$. 3

(c) Let (G, *) be a group and H be a non-empty finite subset of G. Then show that (H, *) is a subgroup of (G, *) if and only if $a \in H,\ b \in H \Rightarrow a * b \in H$.
5

(d) Prove that every transposition is its own inverse. 2

(v) (a) Let H and K be subgroups of a group G. Then show that HK is a subgroup of G if and only if $HK = KH$. 4

(b) Let (G, *) be a group and (H, *) is a subgroup of (G, *). Let $a,b \in G$ and a relation ρ is defined on G by "$a \rho b$ *iff* $x * y^{-1} \in H$". Prove that ρ is an equivalence relation. 4

(c) Show that every subgroup of a cyclic group is cyclic. 4

(vi) (a) Suppose G_1 and G_2 are two groups. Then show that the subsets $G_1 \times \{e\}$ and $\{e\} \times G_2$ of $G_1 \times G_2$ are normal subgroups of $G_1 \times G_2$. 6

(b) Let G and G' be two a groups and $\theta : G \to G'$ be a homomorphism of G onto G'. Prove that

(i) If G is abelian, then G' is abelian.

(ii) If G is cyclic, then G' is cyclic.

(iii) If H is a normal subgroup of G, then $\theta(H)$ is also a normal subgroup of G'. 6

(vii) (a) If H and K both are normal subgroups of a group G such that $H \subseteq K$, then prove that $G/K \cong (G/H)/(K/H)$. 6

(b) Prove that any infinite cyclic group is isomorphic to the additive group Z of all integers. 6

(viii) (a) Prove that there are only two (up to isomorphism) groups of order 6. 4

(b) Let G be a finite cyclic group of order m. Then for every positive divisor d of m, there exists a unique subgroup of G of order d. 4

(c) Show that every proper subgroup of S_3 is cyclic. 4

Group - B

2. Answer any ***six*** of the following questions : 2×6=12

(i) If in a group G, $a^5 = e, aba^{-1} = b^2$ for $a, b \in G$, then show that $O(b) = 31$.

(ii) If G be a group of order 8 and G' be a group of order 3. Prove that there does not exist a homomorphism from G onto G'.

(iii) Show that $(Q, +)$ is not finitely generated.

(iv) Let G be an abelian group and n be a fixed positive integer. Let $H = \{a^n \mid a \in G\}$. Prove that H is a subgroup of G.

(v) Find order of $((1\ 2), 4)$ in $S_3 \times Z_6$.

(vi) Give an example of a noncommutative group in which every subgroup is normal.

(vii) Prove that a non-commutative group of order 10 must have a subgroup of order 5.

(viii) Let G be a group and $a, b \in G$. Prove that $O(ab) = O(ba)$.

(ix) Prove that the sub set $H = \{e, (1,2), (3,4), (1,2)(3,4)\}$ of S_4 forms a non-cyclic subgroup of S_4. Is the group S_4 cyclic?

(x) Let G be a group and $a \in G$. Prove that $N(a) = G$ iff $a \in Z(G)$.