Ex: In the MVT applied to $f(x)$ in $[0,h]$ ie, in $f(h)=f(0)+hf'(\theta h)$, $0<\theta<1$. Prove that $\lim_{h\to 0^+}\theta=\frac{1}{2}$ when $f(x)=\cos x$.

Sol^n:- Applying MVT $f(x)=\cos x$ is continuous and derivable at all points in $[0,h]$

$$f(h)=f(0)+hf'(\theta h),\quad 0<\theta<1.$$

$$\cos h=\cos 0+h(-\sin\theta h)$$

$$\cos h=1-h\sin\theta h$$

$$h\sin\theta h=1-\cos h$$

$$\therefore \frac{\sin\theta h}{\theta h}\cdot\frac{\theta h}{h}=\frac{1-\cos h}{h^2}$$

$$\frac{\sin\theta h}{\theta h}\cdot\theta=\frac{2\sin^2 h/2}{h^2}$$

Taking limit as $h\to 0^+$, we have

$$\lim_{h\to 0^+}\left(\frac{\sin\theta h}{\theta h}\cdot\theta\right)=\frac{1}{2}\lim_{h\to 0^+}\left(\frac{\sin h/2}{h/2}\cdot\frac{\sin h/2}{h/2}\right)$$

$$\lim_{h\to 0^+}\theta=\frac{1}{2}.$$

Ex: Prove that $0<\frac{1}{\log(1+x)}-\frac{1}{x}<1$, $\forall\ x>0$

Sol^n: Let $f(x)=\log(1+x)$, $\forall\ x>0$ [where $f(x)$ satisfy all the conditions of ~~Rolle's~~ Lagrange's MVT.]

$$f'(x)=\frac{1}{1+x}$$

We have by Lagrange MVT of first order

$$f(x)=f(0)+xf'(\theta x),\quad 0<\theta<1.$$

$$\log(1+x)=0+x\cdot\frac{1}{1+\theta x}$$

$$1+\theta x=\frac{x}{\log(1+x)}$$

$$\theta=\frac{1}{\log(1+x)}-\frac{1}{x}$$

Since, $0<\theta<1$

$$0<\frac{1}{\log(1+x)}-\frac{1}{x}<1.\quad\forall\ x>0.$$

Ex! Prove that by MVT $0 < \frac{1}{x}\log\frac{e^x-1}{x} < 1$.

Sol^n: Let $f(x) = e^x$, $\forall x \in \mathbb{R}$ [where $f(x)$ satisfy all the conditions of Lagrange's MVT.]

$f'(x) = e^x$

We have by Lagrange MVT of first order

$$f(x) = f(0) + x f'(\theta x), \quad 0<\theta<1.$$

$$\therefore\ e^x = 1 + x e^{\theta x}$$

$$\Rightarrow\ \theta = \frac{1}{x}\log\frac{e^x-1}{x}$$

Since $0<\theta<1$

$$0 < \frac{1}{x}\log\frac{e^x-1}{x} < 1.$$

Ex! Show that $\frac{v-u}{1+v^2} < \tan^{-1}v - \tan^{-1}u < \frac{v-u}{1+u^2}$ and deduce that $\frac{\pi}{4}+\frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4}+\frac{1}{6}$.

Sol^n: Let $f(x) = \tan^{-1}x$

$$f'(x) = \frac{1}{1+x^2}$$

Applying Lagrange MVT to $f(x)$, we get

$$\frac{f(v)-f(u)}{v-u} = f'(\xi), \quad u<\xi<v.$$

$$\therefore\ \frac{\tan^{-1}v - \tan^{-1}u}{v-u} = \frac{1}{1+\xi^2} \quad \text{--- (1)}$$

But $\xi > u \Rightarrow \frac{1}{1+\xi^2} < \frac{1}{1+u^2}$

and $\xi < v \Rightarrow \frac{1}{1+\xi^2} > \frac{1}{1+v^2}$.

$$\therefore\ \frac{1}{1+v^2} < \frac{1}{1+\xi^2} < \frac{1}{1+u^2}$$

$$\Rightarrow\ \frac{1}{1+v^2} < \frac{\tan^{-1}v - \tan^{-1}u}{v-u} < \frac{1}{1+u^2}$$

$$\therefore\ \frac{v-u}{1+v^2} < \tan^{-1}v - \tan^{-1}u < \frac{v-u}{1+u^2}.$$

2nd part! put $v = \frac{4}{3}$ and $u = 1$, we have,

$$\frac{\frac{4}{3}-1}{1+\frac{16}{9}} < \tan^{-1}\frac{4}{3} - \tan^{-1}1 < \frac{\frac{4}{3}-1}{1+1}$$

$$\Rightarrow\ \frac{3}{25} < \tan^{-1}\frac{4}{3} - \frac{\pi}{4} < \frac{1}{6}$$

$$\Rightarrow\ \frac{\pi}{4}+\frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4}+\frac{1}{6}.$$

2nd MVT of differential calculus :-

Cauchy's MVT :-

St: If two function $f(x)$ and $g(x)$ defined on $[a,b]$ are (i) continuous on $[a,b]$

(ii) derivable on (a,b)

(iii) $g'(x) \neq 0$ for any $x \in (a,b)$

Then $\exists$ at least one real number $\xi \in (a,b)$ such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(\xi)}{g'(\xi)}.$$

Proof: Let us consider the function -

$$\phi(x) = f(x) + A\,g(x) \quad , \forall x \in (a,b)$$

where A is a constant to be determine such that

$$\phi(a) = \phi(b)$$

$$f(a) + A g(a) = f(b) + A g(b)$$

$$f(b) - f(a) = A\{g(a) - g(b)\}$$

$$A = \frac{f(b)-f(a)}{g(a)-g(b)}$$

Since $\phi(x)$ being the sum of two continuous and derivable function. then $\phi(x)$ is itself

(i) Continuous on $[a,b]$

(ii) derivable on (a,b)

(iii) $\phi(a) = \phi(b)$

$\therefore$ $\phi(x)$ satisfy all the conditions of Rolle's theorem then $\exists$ a number $\xi \in (a,b)$ such that

$$\phi'(\xi) = 0$$

$$f'(\xi) + A g'(\xi) = 0$$

$$-A = \frac{f'(\xi)}{g'(\xi)}$$

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(\xi)}{g'(\xi)}, \quad a<\xi<b.$$

(*) Deduce Lagrange's MVT from Cauchy's MVT. $\forall x \in (a,b)$.

$\Rightarrow$ put $\phi(x) = x$ in Cauchy M.V.T we have

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(\xi)}{g'(\xi)}, \quad a<\xi<b \qquad [\because \phi'(x)=1, \forall x \in (a,b)]$$

$$\frac{f(b)-f(a)}{b-a} = f'(\xi), \text{ Which is the Lagrange's MVT.}$$

Ex! Use MVT to prove the following inequality,

$$\frac{x}{1+x} < \log(1+x) < x, \quad x>0$$

Soln: Let $f(x) = \log(1+x)$ $x>0$
where $f(x)$ are continuous and differentiable $\forall x>0$.
We apply Lagrange MVT for $f(x)$, on $[0,x]$

$$f(x) = f(0) + x f'(\theta x), \quad 0<\theta<1$$

$$\log(1+x) = 0 + \frac{x}{1+x\theta}$$

$$\therefore \log(1+x) = \frac{x}{1+\theta x} \quad \text{--- (1)}$$

Since, $\theta > 0$

$\theta x > 0$

$1+\theta x > 1$

$\frac{1}{1+\theta x} < 1$

$\frac{x}{1+\theta x} < x$

$\log(1+x) < x$

Again, $\theta < 1$

$\theta x < x$

$1+\theta x < 1+x$

$\frac{x}{1+\theta x} > \frac{x}{1+x}$

$\log(1+x) > \frac{1}{1+x}$.

$$\therefore \frac{x}{1+x} < \log(1+x) < x, \quad x>0.$$

Ex! Show that $\frac{\sin\alpha - \sin\beta}{\cos\beta - \cos\alpha} = \cot\theta, \quad 0<\alpha<\theta<\beta<\frac{\pi}{2}.$

Soln: Let $f(x) = \sin x$ and $g(x) = \cos x$, for $x \in [\alpha, \beta]$.
$f'(x) = \cos x$ and $g'(x) = -\sin x$.

where $f(x)$ and $g(x)$ both continuous and differentiable therefore by Cauchy's MVT on $[\alpha,\beta]$.

$$\frac{\sin\beta - \sin\alpha}{\cos\beta - \cos\alpha} = \frac{\cos\theta}{-\sin\theta}; \quad \alpha<\theta<\beta$$

$$\therefore \frac{\sin\beta - \sin\alpha}{\cos\beta - \cos\alpha} = -\cot\theta$$

$$\therefore \frac{\sin\alpha - \sin\beta}{\cos\beta - \cos\alpha} = \cot\theta, \quad \alpha<\theta<\beta.$$

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