ROLLE'S THEOREM :-

St:- If a function $f(x)$ defined on $[a,b]$ is
(i) Continuous on $[a,b]$
(ii) derivable on (a,b)
(iii) $f(a) = f(b)$

then there exists at least one real number c between a and b $(a<c<b)$ such that $f'(c)=0$.

Proof:- Since the function is continuous on $[a,b]$ it is bounded and attains its bounds. Thus m and M are the infimum (g.l.b) and the supremum (l.u.b) respectively of the function $f(x)$ then $\exists$ points c and d of $[a,b]$ such that $f(c)=m$ and $f(d)=M$.

There are two possibilities: either $m=M$ or $m\neq M$. If $m=M$, then f is constant over $[a,b]$ and therefore its derivatives $f'(x)=0$ $\forall x\in[a,b]$.

When $m\neq M$, both of these cannot be equal to the same quantity $f(a)$. At least one of these, say, m is different from $f(a)$ or $f(b)$. So that

$f(c)=m\neq f(a) \Rightarrow c\neq a$
$f(c)=m\neq f(b) \Rightarrow c\neq b$.

This means that c lies in (a,b).

We shall now show that c is the point where $f'(c)=0$. If $f'(c)<0$, then $\exists$ an interval $(c, c+\delta_1)$, $\delta_1>0$ for every point x of which $f(x)<f(c)=m$; which contradicts the fact that m is the infimum.

If $f'(c)>0$, $\exists$ an interval $(c-\delta_2, c)$, $\delta_2>0$ for every point x of which $f(x)<f(c)=m$, which is also a contradiction.

Hence the only possibility, $f'(c)=0$.

(*) Converse of Rolle's theorem may not be true :-

Ex:- Let us consider the function $f(x)=\frac{1}{x}+\frac{1}{1-x}$ in $[0,1]$.

Here (i) $f(x)$ is continuous in $0<x<1$, (not in $0\le x\le 1$).
(ii) $f'(x)=\frac{1}{(1-x)^2}-\frac{1}{x^2}$ exists in $0<x<1$.
(iii) $f(0)\neq f(1)$, both being undefined.

Thus all the conditions of Rolle's theorem do not hold. But yet $\exists$ a c where $f'(c)=0$, (namely $c=\frac{1}{2}$) where $0<c<1$.

So the conditions of Rolle's theorem are sufficient but not necessary.

Ex: Are the conditions of Rolle's theorem satisfied in the case of the following functions defined over given intervals:-

ⓐ $f(x)=|x|$ in $[-1,1]$
ⓑ $g(x)=x^2$ in $2\le x\le 3$
ⓒ $h(x)=\cos\frac{1}{x}$ in $[-1,1]$
ⓓ $\phi(x)=\tan x$ in $0\le x\le\pi$.

Solⁿ:- ⓐ (i) Here $f(x)=|x|$ is continuous throughout the interval $[-1,1]$

(ii) But $f(x)$ is not derivable at $x=0$, so $f(x)$ is not derivable on $(-1,1)$.

(iii) $f(-1)=f(1)=1$

Here $f(x)$ does not satisfy all the conditions of Rolle's theorem.

ⓑ Here $g(x)=x^2$ is a polynomial function in x, so (i) $g(x)$ is continuous on $[2,3]$

(ii) derivable on ~~[2,3]~~ $(2,3)$

(iii) $f(2)\neq f(3)$.

Here $g(x)$ does not satisfy all conditions of Rolle's theorem.

ⓒ (i) Here $h(x)=\cos\frac{1}{x}$ is discontinuous at $x=0$

So $h(x)$ is not continuous on $[-1,1]$.

(ii) $h(x)$ cannot be derivable at $x=0$ as $\lim_{x\to 0}\cos\frac{1}{x}$ does not exist.

So $h(x)$ is not derivable on $(-1,1)$.

(iii) $h(-1)=h(1)=\cos 1$.

Thus $h(x)$ does not satisfy all conditions of Rolle's theorem.

Ex:- Verify Rolle's theorem for $f(x) = 2x^3 + x^2 - 4x - 2$.

Solⁿ: Since $f(x) = 2x^3 + x^2 - 4x - 2$ is a poly. in x of degree 3, then $f(x)$ is
(i) Continuous for all real values of x
(ii) derivable for every real x.

Now $f(x) = 0$ gives $2x^3 + x^2 - 4x - 2 = 0$

$\Rightarrow (x^2 - 2)(2x + 1) = 0$

$\Rightarrow x = -\frac{1}{2}, -\sqrt{2}, \sqrt{2}$.

We take the interval $[-\sqrt{2}, \sqrt{2}]$ so that in this interval all the three conditions of Rolle's theorem are satisfied.

Now, $f'(x) = 6x^2 + 2x - 4 = 0$

$\Rightarrow (3x - 2)(x + 1) = 0$

$\Rightarrow x = -1$ or $x = 2/3$

i.e, $f'(-1) = f'(2/3) = 0$.

Since both the points $x = -1$ and $x = 2/3$ lie in the open interval $[-\sqrt{2}, \sqrt{2}]$, Rolle's theorem is verified.

First Mean value theorem of differential calculus :-

Lagrange's Mean value theorem :-

St:- If a function $f(x)$ defined on $[a, b]$ is
(i) Continuous on $[a, b]$
(ii) derivable on (a, b)

then $\exists$ at least one real number c between a and b, $(a < c < b)$ such that $\frac{f(b) - f(a)}{b - a} = f'(c)$.

Proof:- Let us consider a function

$$\phi(x) = f(x) + Ax, \quad x \in [a, b]$$

where A is a constant to be determined such that $\phi(a) = \phi(b)$

$\Rightarrow f(a) + Aa = f(b) + Ab$

$\therefore A = -\frac{f(b) - f(a)}{b - a}$.

Now the function $\phi(x)$, being the sum of two continuous and derivable functions, is itself

(i) continuous on $[a,b]$

(ii) derivable on (a,b) and

(iii) $\phi(a) = \phi(b)$

Therefore, $\phi(x)$ satisfy all the conditions of Rolle's theorem, then $\exists$ a real number $c \in (a,b)$ such that $\phi'(c) = 0$

$$\therefore f'(c) + A = 0$$

$$\therefore -A = f'(c)$$

$$\therefore \frac{f(b)-f(a)}{b-a} = f'(c), \quad a<c<b.$$

Another statement :-

If a function $f(x)$ defined on $[a, a+h]$ is

(i) continuous on $[a, a+h]$

(ii) derivable on $(a, a+h)$

then $\exists$ at least one real number $\theta \in (0,1)$ such that $f(a+h) = f(a) + h f'(a+\theta h)$, $0<\theta<1$.

Deductions :-

① If a function $f(x)$ satisfies the conditions of the M.V.T and $f'(x) = 0 \ \forall x \in (a,b)$, then $f(x)$ is constant on $[a,b]$.

$\Rightarrow$ Let x_1, x_2 $(x_1 < x_2)$ be any two distinct points of $[a,b]$.

Hence by Lagrange's M.V.T -

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} = f'(c) = 0, \quad (x_1<c<x_2)$$

$$\therefore f(x_2) - f(x_1) = 0$$

$$\therefore f(x_1) = f(x_2)$$

This prove that $f(x)$ is constant function on $[a,b]$.

② If a function $f(x)$ is (i) Continuous on $[a,b]$ (ii) derivable on (a,b) and (iii) $f'(x) > 0$, $\forall x \in (a,b)$ then $f(x)$ is strictly increasing on $[a,b]$.

$\Rightarrow$ Let x_1, x_2 (where $x_1 < x_2$) be any two distinct points of $[a,b]$, then by Lagrange's MVT

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) > 0, \quad \text{for } x_1 < c < x_2$$

or, $f(x_2) - f(x_1) > 0$

or, $f(x_2) > f(x_1)$ for $x_2 > x_1$

This prove that $f(x)$ is strictly increasing on $[a,b]$.

NOTE :- deduce ~~Lag~~ Rolle's theorem from Lagrange M.V.T.

$\Rightarrow$ put $f(a) = f(b)$ in Lagrange's M.V.T

Ex! Is mean-value theorem valid for $f(x) = x^2+3x+2$ in $1 \le x \le 2$? Find c, if the theorem applicable.

Soln :- Here $f'(x) = 2x+3$ exists in $(1,2)$

$f(x)$ is continuous in $[1,2]$.

Hence MVT is applicable. Using the theorem

$$f'(c) = \frac{f(2) - f(1)}{2-1} = 12 - 6 = 6,$$

i.e, $2c + 3 = 6$

, $c = 3/2$

Ex:- Let $f(x)$ be a real-valued function defined over $[-1,1]$ such that $f(x) = \begin{cases} x \sin\frac{1}{x} & , x \neq 0 \\ 0 & , x = 0 \end{cases}$

Does the MVT hold for $f(x)$ in $[-1,1]$?

Soln: The function $f(x)$ is continuous in $[-1,1]$ but not derivable in $(-1,1)$ (infact, f has no derivative at $x=0$).

$\therefore$ MVT is not applicable for f in $[-1,1]$.

Ex: If $f(x) = (x-a)^m (x-b)^n$ where m and n are positive integers, show that c in Rolle's theorem divides the segment $a \le x \le b$ in the ratio $m:n$.

Solⁿ: $f(x) = (x-a)^m (x-b)^n$ is continuous in $a \le x \le b$.

$f'(x) = (x-a)^{m-1}(x-b)^{n-1}\{m(x-b)+n(x-a)\}$ exists in $a<x<b$.

Also, $f(a) = f(b) = 0$, Hence by Rolle's theorem

$\exists$ c such that $f'(c) = 0$

or, $(c-a)^{m-1}(c-b)^{n-1}\{m(c-b)+n(c-a)\} = 0$

$\therefore$ $m(c-b) = n(a-c)$

$\therefore$ $c = \dfrac{bm+an}{m+n}$

i.e, c divides the segment $a \le x \le b$ in the ratio $m:n$.

Ex: Verify Rolle's theorem in each of the following cases:-

ⓐ $f(x) = x^2$ in $[-1,1]$

ⓑ $f(x) = 1 - x^{2/3}$ in $[-1,1]$

ⓒ $f(x) = x^3 - 6x^2 + 11x - 6$ in $[1,3]$

Ex: Examine the validity of the hypothesis and conclusion of Lagrange M.V.T in each of the following cases:

ⓐ $f(x) = x^2 + 3x + 2$ in $[1,2]$

ⓑ $f(x) = x(x-1)(x-2)$ in $[0,\frac{1}{2}]$

ⓒ $f(x) = 1 + x^{2/3}$ in $[-8,1]$