GEOMETRY (2D)

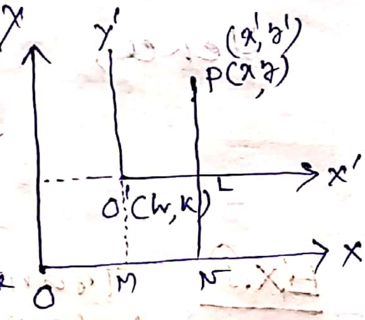
22/02/2020 (S.D)

Transformation of Co-ordinates

Introduction:- The equation of a curve or the co-ordinates of a point are always given w.r.to a fixed origin and a set of fixed co-ordinate axes. The equation will be different when either the origin or the co-ordinate axes are changed. This will happen when either the origin is changed to a point without altering the direction of axes or when the co-ordinate axes are rotated through a certain angle without shifting the origin. The shifting of origin without altering direction of axes is called translation while the rotation of co-ordinate axes without altering the origin is called rotation. All these transformations, when both the systems of axes are rectangular, are called orthogonal transformations.

Translation of axes :-

Changing the origin without changing the directions of the axes.



Let $P(x,y)$ be the position of a point referred to a set of rectangular cartesian co-ordinate axes OX and OY. Let $O'(h,k)$ be the new origin and $O'X'$ and $O'Y'$ be the new co-ordinate axes parallel to OX and OY respectively. Let the coordinates of P referred to $O'X'$ and $O'Y'$ axes be (x', y').

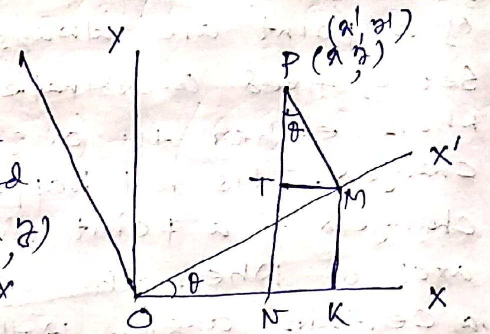
$$\therefore x = ON = OM + MN \qquad y = PN = NL + PL$$
$$= OM + O'L \qquad\qquad = O'M + PL$$
$$= h + x' \qquad\qquad = k + y'$$

Hence, we have, $x = x' + h$,
and $y = y' + k$.

These are the transformation formulae for the translation of axes.

Rotation :-

Transformation from one pair of rectangular axes to another with the same origin.



Let the axes OX and OY be turned about O through an angle θ to the position OX' and OY'. Let P be any point (x, y) referred to the system OX, OY and (x', y') referred to the new set of axes OX', OY'.

PN and PM are drawn perpendiculars to OX and OX' resp. Drawn MK and MT perpendiculars to OX and PN resp.

We have, $x = ON = OK - NK$ $\qquad y = PN = TN + PT$

$= OK - MT \qquad = MK + PT$

$= x'\cos\theta - y'\sin\theta \qquad = x'\sin\theta + y'\cos\theta.$

Hence, $x = x'\cos\theta - y'\sin\theta$
$y = x'\sin\theta + y'\cos\theta.$ are the transformation formulae for the rotation of axes.

Ex ①. Transform to parallel axes through the point (0,1), the equation $y^2 - y = 0$.

Soln:- Let the origin be shifted to (0,1) then, we have $x = x' + 0$ and $y = y' + 1$, where (x', y') are new co-ordinates.

Put these values of x and y in the given equation, we get-

$$(y'+1)^2 - (y'+1) = 0$$

$$y'^2 + y' = 0.$$

This is the required transformed equation.

Ex: ② Find the angle through which axes must be turned so that $ax^2 + 2hxy + by^2$ becomes an expression of the form $AX^2 + BY^2$.

Soln: Let the axes be rotated through an angle θ. Let (x, y) be the co-ordinates of any point P w.r.to old axes and (x', y') be the co-ordinates of the same point P w.r.to new axes. Then we have $x = x'\cos\theta - y'\sin\theta$ and

$y = x'\sin\theta + y'\cos\theta$

Put these values of x and y in the given expression $ax^2 + 2hxy + by^2$

$$a(x'\cos\theta - y'\sin\theta)^2 + 2h(x'\cos\theta - y'\sin\theta)(x'\sin\theta + y'\cos\theta) + b(x'\sin\theta + y'\cos\theta)^2$$

$$= (a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta)x'^2 + (a\sin^2\theta - 2h\sin\theta\cos\theta + b\cos^2\theta)y'^2 + 2\{(\cos^2\theta - \sin^2\theta)h - (a-b)\sin\theta\cos\theta\}x'y'.$$

To remove $x'y'$-term, we must have

$$(\cos^2\theta - \sin^2\theta)h - (a-b)\sin\theta\cos\theta = 0.$$

or, $$\tan 2\theta = \frac{2h}{a-b}$$

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2h}{a-b}\right).$$

Ex: ③ If by a rotation of co-ordinate axes the expression $ax^2 + 2hxy + by^2$ change to $a'x'^2 + 2h'x'y' + b'y'^2$, show that $a+b = a'+b'$, and $ab - h^2 = a'b' - h'^2$.

Soln: Let the original axes be rotated through an angle θ. Then we have

$$\left.\begin{aligned} x &= x'\cos\theta - y'\sin\theta \\ \text{and}\quad y &= x'\sin\theta + y'\cos\theta \end{aligned}\right\} \quad (1)$$

Using ①, the expression $ax^2 + 2hxy + by^2$ becomes –

$ax^2 + 2hxy + by^2$

$$= a(x'\cos\theta - y'\sin\theta)^2 + 2h(x'\cos\theta - y'\sin\theta)(x'\sin\theta + y'\cos\theta) + b(x'\sin\theta + y'\cos\theta)^2$$

$$= x'^2(a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta) + y'^2(a\sin^2\theta - 2h\sin\theta\cos\theta + b\cos^2\theta) + 2x'y'\{(\cos^2\theta - \sin^2\theta)h - (a-b)\sin\theta\cos\theta\} = a'x'^2 + 2h'x'y' + b'y'^2.$$

We have - $a' = a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta$

$$b' = a\sin^2\theta - 2h\sin\theta\cos\theta + b\cos^2\theta$$

$$h' = h(\cos^2\theta - \sin^2\theta) - (a-b)\sin\theta\cos\theta$$

Here $a'+b' = a(\cos^2\theta + \sin^2\theta) + b(\cos^2\theta + \sin^2\theta)$

$$= a+b.$$

Again, $2a' = a(1+\cos 2\theta) + 2h\sin 2\theta + b(1-\cos 2\theta)$

$$= (a+b) + (a-b)\cos 2\theta + 2h\sin 2\theta$$

Similarly,

$$2b' = (a+b) - \{(a-b)\cos 2\theta + 2h\sin 2\theta\}$$

$$\therefore 4a'b' = (a+b)^2 - \{(a-b)\cos 2\theta + 2h\sin 2\theta\}^2$$

and $4h'^2 = [2h\cos 2\theta - (a-b)\sin 2\theta]^2$

$$\therefore 4(a'b' - h'^2) = \{(a+b)^2 - [4h^2 + (a-b)^2]\}$$

$$= 4(ab - h^2).$$

$$\therefore a'b' - h'^2 = ab - h^2.$$

Ex. ④ Find the angle through which a set of rectangular axes must be turned without the change of origin so that xy-term is removed from the equation $7x^2 + 4xy + 3y^2 = 0$.

Soln: Let the axes be rotated through an angle θ. Then we have $x = x'\cos\theta - y'\sin\theta$ and

$$y = x'\sin\theta + y'\cos\theta.$$

Substituting these values of x and y in the given expression we have

$7x^2+4xy+3y^2$

$= 7(x'\cos\theta - y'\sin\theta)^2 + 4(x'\cos\theta - y'\sin\theta)(x'\sin\theta + y'\cos\theta)$
$\quad + 3(x'\sin\theta + y'\cos\theta)^2$

$= x'^2(7\cos^2\theta + 4\sin\theta\cos\theta + 3\sin^2\theta) + y'^2(7\sin^2\theta - 4\sin\theta\cos\theta$
$\quad + 3\cos^2\theta) - 4x'y'(\sin 2\theta - \cos 2\theta)$.

To remove the $x'y'$-term we ~~must~~ have –

$$-4(\sin 2\theta - \cos 2\theta) = 0$$

$$\therefore \sin 2\theta - \cos 2\theta = 0$$

$$\therefore \tan 2\theta = 1$$

$$\therefore 2\theta = 45^\circ$$

$$\therefore \theta = 22\tfrac{1}{2}^\circ$$

Ex! ⑤ To what points must the origin be moved in order to remove the terms of the first degree in the equations ⓘ $2x^2-3y^2-4x-12y=0$

ⓘⓘ $2x^2-3xy+4y^2+10x-19y+23=0$.

Solⁿ. ⓘ Let the point to which the origin is to be shifted be (h,k).

Substituting $x = x'+h$ and $y = y'+k$ in the given equation, we have

$$2(x'+h)^2 - 3(y'+k)^2 - 4(x'+h) - 12(y'+k) = 0$$

$$\therefore 2x'^2 - 3y'^2 + x'(4h-4) + y'(-6k-12) + 2h^2 - 3k^2 - 4h - 12k = 0$$

If we remove the first degree terms, then

$$4h-4=0 \quad \text{and} \quad -6k-12=0$$

$$\therefore h=1, \quad \therefore k=-2$$

Hence, the origin shifted ~~to the~~ point $(1,-2)$.

Ex. 6: Find the equation to the curve $9x^2+4y^2+18x-16y=11$ referred to parallel axes through the point $(-1, 2)$.

Ex. 7: Transform the equation (i) $x^2-y^2=a^2$

(ii) $x^2+2axy+y^2=a^2$

to axes inclined at 45° to the original axes.

Ex. 8: Choose a new origin (h,k) without changing the directions of the axes such that the equation $5x^2-2y^2-30x+8y=0$ may be reduced to the form $Ax'^2+By'^2=1$.

Ex. 9: Find the angle through which the axes must be turned so that the equation $lx+n-my=0$, $(m\neq 0)$ may be reduced to the form $ay+b=0$.

Ex. 10: Through what angle must the axes be turned to remove the term

(i) xy from $x^2+2\sqrt{3}\,xy-y^2=4$

(ii) x^2 from $x^2-4xy+3y^2=0$