Linearly dependent:- (L.D)

A finite set of vectors $\{\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n\}$ of a vector space V over a field F is said to be linearly dependent in V, if $\exists$ scalars $c_1, c_2, \ldots, c_n$ not all zero in F such that

$$c_1\alpha_1 + c_2\alpha_2 + \ldots + c_n\alpha_n = \theta$$

Linearly independent :- (L. I)

A finite set of vectors $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of a vector space V over a field F is said to be linearly independent in V, if $\exists$ scalars $c_1, c_2, \ldots, c_n$ in F such that

$$c_1\alpha_1 + c_2\alpha_2 + \ldots + c_n\alpha_n = \theta \Rightarrow c_1 = c_2 = \ldots = c_n = 0.$$

Ex: Show that the set of vectors $\{(2,3,1), (2,1,3), (1,1,1)\}$ is linearly dependent in $\mathbb{R}^3$

Soln:- Let, $\alpha_1 = (2,3,1)$, $\alpha_2 = (2,1,3)$, $\alpha_3 = (1,1,1)$

Let us consider the relation

$$c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 = \theta. \; ; \text{ where } c_1, c_2, c_3 \in \mathbb{R}.$$

or, $c_1(2,3,1) + c_2(2,1,3) + c_3(1,1,1) = (0,0,0)$

or, $(2c_1 + 2c_2 + c_3, \; 3c_1 + c_2 + c_3, \; c_1 + 3c_2 + c_3) = (0,0,0)$.

$\therefore 2c_1 + 2c_2 + c_3 = 0 \;\text{---}(1)$, $3c_1 + c_2 + c_3 = 0 \;\text{---}(2)$, $c_1 + 3c_2 + c_3 = 0 \;\text{---}(3)$

Solving (1) and (2) we have

$$\frac{c_1}{2-1} = \frac{c_2}{3-2} = \frac{c_3}{2-6}$$

or, $\frac{c_1}{1} = \frac{c_2}{1} = \frac{c_3}{-4} = k$ (say), $k \in \mathbb{R}$.

$\therefore c_1 = k$, $c_2 = k$, $c_3 = -4k$.

~~putting these values in (3) we have~~

$\therefore$ The soln is $k(1,1,-4)$ which satisfy (3) for all $k \in \mathbb{R}$.

If we choose $k = 1$, then we have $c_1 = 1$,
$c_2 = 1$
$c_3 = -4$.

Since, c_1, c_2, c_3 are non zero for which $c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 = \theta$.

So, the set of vectors are linearly dependent.

Ex: Show that the set of vectors $\{(1,2,3), (2,3,1), (3,1,2)\}$ are linearly independent in $\mathbb{R}^3$

Soln: Let, $\alpha_1 = (1,2,3)$, $\alpha_2 = (2,3,1)$, $\alpha_3 = (3,1,2)$

Let us consider the relation

$$c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 = \theta \; ; \text{ where } c_1, c_2, c_3 \in \mathbb{R}^3$$

or $c_1(1,2,3) + c_2(2,3,1) + c_3(3,1,2) = (0,0,0)$

$\therefore (c_1+2c_2+3c_3,\ 2c_1+3c_2+c_3,\ 3c_1+c_2+2c_3)=(0,0,0)$

$\therefore c_1+2c_2+3c_3=0 \text{ — (1)},\quad 2c_1+3c_2+c_3=0 \text{ — (2)},\quad 3c_1+c_2+2c_3=0 \text{ — (3)}$.

These are the homogeneous system of equations

The co-efficient determinant is $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{vmatrix}$

$= 1(6-1)-2(4-3)+3(2-9)$

$= 5-2-21$

$= -18 \ (\neq 0)$.

$\therefore$ By cramer's rule, we have unique soln of c_1, c_2, c_3.
and the soln is $c_1=0,\ c_2=0,\ c_3=0$.

Thus $c_1\alpha_1+c_2\alpha_2+c_3\alpha_3=0 \Rightarrow c_1=c_2=c_3=0$.

Hence the set of vectors are linearly independent.

~~Ex~~ Q. Determine K, so that the set $S=\{(K,1,1)(1,K,1)(1,1,K)\}$ is L.D in $\mathbb{R}^3$.

Soln: let $\alpha_1=(K,1,1)$

$\alpha_2=(1,K,1)$

$\alpha_3=(1,1,K)$.

let us consider the realtn.

$c_1\alpha_1+c_2\alpha_2+c_3\alpha_3=0$, where $c_1,c_2,c_3\in\mathbb{R}$.

$\therefore c_1(K,1,1)+c_2(1,K,1)+c_3(1,1,K)=(0,0,0)$

$\therefore Kc_1+c_2+c_3=0 \text{ — (1)},\quad c_1+Kc_2+c_3=0 \text{ — (2)},\quad c_1+c_2+Kc_3=0 \text{ — (3)}$

Since set of vectors are L.D, the coeff. determinant of the homogeneous system of eqns must be zero

$\begin{vmatrix} K & 1 & 1 \\ 1 & K & 1 \\ 1 & 1 & K \end{vmatrix}=0$

or, $\begin{vmatrix} K+2 & K+2 & K+2 \\ 1 & K & 1 \\ 1 & 1 & K \end{vmatrix}=0$ $[R_1' = R_1+R_2+R_3]$

$\Rightarrow (K+2)\begin{vmatrix} 1 & 1 & 1 \\ 1 & K & 1 \\ 1 & 1 & K \end{vmatrix}=0$

$\Rightarrow (K+2)\begin{vmatrix} 1 & 0 & 0 \\ 1 & K-1 & 0 \\ 1 & 0 & K-1 \end{vmatrix}=0$ $[C_2'=C_2-C_1,\ C_3'=C_3-C_1]$

$\therefore (K+2)(K-1)^2=0$

$\therefore K=1,1,-2.$

$\therefore K=1,-2$

Th:- Prove that a set of vectors containing the null vector θ in the vector space V over a field F is linearly dependent

Proof:
Let $\{\alpha,\beta,\theta\}$ be a finite set of vectors containing null vector θ in the vector space V over the field F.

We consider the relation $c_1\alpha+c_2\beta+c_3\theta=\theta$; $c_1,c_2,c_3\in F$

If we choose $c_1=0, c_2=0, c_3\neq 0$. We have

$$L.H.S = 0.\alpha+0.\beta+c_3\theta$$
$$= \theta+\theta+\theta$$
$$= \theta.$$

Since c_1,c_2,c_3 not all zero in F such that $c_1\alpha+c_2\beta+c_3\theta=\theta$

So, the set $\{\alpha,\beta,\theta\}$ is linearly dependent.

Note:- A set containing of a single non-zero vector α in a vector space V over the field F is L.I.

Let $S=\{\alpha\}$, $\alpha\neq\theta$

Let us consider the relation

$c_1\alpha=\theta$

$\Rightarrow c_1=0$ or, $\alpha=\theta$

$\Rightarrow c_1=0$ [$\because \alpha\neq\theta$]

$\therefore$ The set $S=\{\alpha\}$ is L.I.

Theorem: A set of vectors $\{\alpha_1,\alpha_2\ldots\alpha_n\}$ in a vector space V over the field F be L.D. iff. at least one of the vectors of the set can be expressed as a linear combination of the remaining others.

Proof:- Let the set of vectors $\{\alpha_1,\alpha_2,\ldots\alpha_n\}$ are linearly dependent.

Then $\exists$ scalars $c_1,c_2,\ldots c_n$ not all zero in F such that

$$c_1\alpha_1+c_2\alpha_2+\ldots+c_n\alpha_n=\theta.$$

Let $c_j\neq 0\in F$, then c_j^{-1} exists in F and $c_j c_j^{-1}=1$. [I is identity element in F]

$$c_1\alpha_1+c_2\alpha_2+\ldots+c_{j-1}\alpha_{j-1}+c_j\alpha_j+c_{j+1}\alpha_{j+1}+\ldots+c_n\alpha_n=\theta.$$

or, $c_j\alpha_j=-c_1\alpha_1-c_2\alpha_2-\ldots-c_{j-1}\alpha_{j-1}-c_{j+1}\alpha_{j+1}-\ldots-c_n\alpha_n$

$\therefore \alpha_j=-c_j^{-1}c_1\alpha_1-c_j^{-1}c_2\alpha_2-\ldots-c_j^{-1}c_{j-1}\alpha_{j-1}-c_j^{-1}c_{j+1}\alpha_{j+1}-\ldots-c_j^{-1}c_n\alpha_n$

$=d_1\alpha_1+d_2\alpha_2+\ldots+d_{j-1}\alpha_{j-1}+d_{j+1}\alpha_{j+1}+\ldots+d_n\alpha_n$

when $d_i=c_j^{-1}c_i$, $i=1,2,\ldots j-1, j+1,$

This shows that α_j can be expressed as a linear combination of the remaining vectors of the given set.

Conversely,

Let α_j can be expressed as a linear combination of the remaining vectors of the given set.

Then we have

$$\alpha_j = c_1\alpha_1 + c_2\alpha_2 + \dots + c_{j-1}\alpha_{j-1} + c_{j+1}\alpha_{j+1} + \dots + c_n\alpha_n$$

where $c_i \in F$, $i = 1, 2, \dots j-1, j+1 \dots n$.

$$\text{or, } c_1\alpha_1 + c_2\alpha_2 + \dots + c_{j-1}\alpha_{j-1} + (-1)\alpha_j + c_{j+1}\alpha_{j+1} + \dots + c_n\alpha_n = \theta$$

Since the above equality holds for the scalars $c_1, c_2 \dots c_{j-1}, -1, c_{j+1} \dots c_n$ in F.

Thus the relation holds for the scalars and in which all of them are not equal to zero.

Hence the set $\{\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n\}$ is L.D.

$S = \{\alpha, \beta, 2\beta\}$

$2\beta = 0.\alpha + 2.\beta \Rightarrow$ L.D.

Basis of a vector space :-

Defn:- Let V be a vector space over the field F, a set S of vectors in V is said to be basis of V if

(i) S is linearly independent in V.

(ii) S generates V, or $L(S) = V$.

~~Dimension of a vector space~~

Ex:- Prove that the set $E = \{(1,0),(0,1)\}$ is a basis of the vector space $\mathbb{R}^2$.

Soln. Let, $\epsilon_1 = (1,0)$, $\epsilon_2 = (0,1)$.

To prove linearly independent

We consider the relation

$$c_1\epsilon_1 + c_2\epsilon_2 = \theta. \quad \text{where } c_1, c_2 \in \mathbb{R}$$

$\therefore c_1(1,0) + c_2(0,1) = (0,0)$

$(c_1, c_2) = (0,0)$

$\Rightarrow c_1 = 0, \quad c_2 = 0.$

$\therefore$ The set of vectors $\{(1,0),(0,1)\}$ is L.I.

To prove $L(E) = \mathbb{R}^2$:

Let, $\xi = (a,b) \in \mathbb{R}^2$ be arbitrary vector, where a, b are real.

If possible,

Let, $\xi = c\epsilon_1 + d\epsilon_2$, where $c, d \in \mathbb{R}$.

$\Rightarrow (a,b) = c(1,0) + d(0,1)$

$\Rightarrow (a,b) = (c,d)$

$\Rightarrow c = a, \ d = b.$

This prove that —

$\therefore \xi \in L(E)$.

$\therefore \xi \in \mathbb{R}^2 \Rightarrow \xi \in L(E)$

$\therefore \mathbb{R}^2 \subseteq L(E)$ —— ①.

Again, $E \subseteq \mathbb{R}^2$ and $L(E)$ being the smallest subspace of $\mathbb{R}^2$ containing E, so, $L(E) \subseteq \mathbb{R}^2$ —— ②

From ① and ②.

$L(E) = \mathbb{R}^2$.

Hence, $E = \{(1,0),(0,1)\}$ is a basis of the vector space $\mathbb{R}^2$.

This basis is known as standard basis of $\mathbb{R}^2$ and the dimension of the vector space $\mathbb{R}^2$ is 2.

Ex: prove that the set $E = \{(1,0,0), (0,1,0), (0,0,1)\}$ is a basis of the vector space $\mathbb{R}^3$.

→ Let $\epsilon_1 = (1,0,0)$,

$\epsilon_2 = (0,1,0)$,

$\epsilon_3 = (0,0,1)$.

To prove. L.I

We consider the relation

$c_1\epsilon_1 + c_2\epsilon_2 + c_3\epsilon_3 = \theta$ where $c_1, c_2, c_3 \in \mathbb{R}$

$\Rightarrow c_1(1,0,0) + c_2(0,1,0) + c_3(0,0,1) = (0,0,0)$

$\Rightarrow (c_1, c_2, c_3) = (0,0,0)$

$\Rightarrow c_1 = c_2 = c_3 = 0$.

$\therefore$ The set of vector $\{(1,0,0), (0,1,0), (0,0,1)\}$ is L.I.

To prove $L(E) = \mathbb{R}^3$.

Let $\xi = (a,b,c) \in \mathbb{R}^3$ be arbitrary vector, where a, b, c are real.

If possible,

Let $\xi = r_1\epsilon_1 + r_2\epsilon_2 + r_3\epsilon_3$, where $r_1, r_2, r_3 \in \mathbb{R}$

$\Rightarrow (a,b,c) = r_1(1,0,0) + r_2(0,1,0) + r_3(0,0,1)$

$\Rightarrow (a,b,c) = (r_1, r_2, r_3)$

$\therefore r_1 = a, \; r_2 = b, \; r_3 = c$

This prove that

$\therefore \xi \in L(E)$.

$\therefore \xi \in \mathbb{R}^3 \Rightarrow \xi \in L(E)$.

$\therefore \mathbb{R}^3 \subseteq L(E)$ ——— (1)

again, $E \subseteq \mathbb{R}^3$ and $L(E)$ being the smallest subspace of $\mathbb{R}^3$ containing E, so $L(E) \subseteq \mathbb{R}^3$. ——— (2)

From (1) and (2).

$$\mathbb{R}^3 = L(E).$$

Hence the set of vectors $E = \{(1,0,0), (0,1,0), (0,0,1)\}$ is a basis of $\mathbb{R}^3$.

Which is known as standard basis of $\mathbb{R}^3$ and dimension of the vector space $\mathbb{R}^3$ is 3.

Ex. prove that the set $S = \{(0,1,1), (1,0,1), (1,1,0)\}$ is a basis of $\mathbb{R}^3$.

Soln: Let $\alpha_1 = (0,1,1)$,

$\alpha_2 = (1,0,1)$

$\alpha_3 = (1,1,0)$

To prove L.I :-

We consider the relation,

$$c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 = \theta \qquad c_1, c_2, c_3 \in F$$

$$\Rightarrow c_1(0,1,1) + c_2(1,0,1) + c_3(1,1,0) = (0,0,0)$$

$$\Rightarrow (c_2+c_3, \; c_1+c_3, \; c_1+c_2) = (0,0,0)$$

$\therefore c_2 + c_3 = 0$ ——— (1), $c_1 + c_3 = 0$ ——— (2), $c_1 + c_2 = 0$ ——— (3)

Adding (1), (2) and (3) we get

$$c_1 + c_2 + c_3 = 0 \quad \text{——— (4)}$$

Subtracting (1), (2) and (3) from (4), we obtain

$$c_1 = 0, \; c_2 = 0, \; c_3 = 0.$$

$\therefore$ The set of vectors is L.I.

To prove $L(S) = \mathbb{R}^3$.

Let $\xi = (a,b,c) \in \mathbb{R}^3$ be arbitrary vectors where a, b, c are real.

If possible let

$$\xi = r_1\alpha_1 + r_2\alpha_2 + r_3\alpha_3 \; ; r_1, r_2, r_3 \in \mathbb{R}$$

$$\Rightarrow (a,b,c) = r_1(0,1,1) + r_2(1,0,1) + r_3(1,1,0).$$

$$= (r_2+r_3, \; r_1+r_3, \; r_1+r_2)$$

$\therefore r_2+r_3 = a$ —(5), $r_1+r_3 = b$ —(6), $r_1+r_2 = c$ —(7).

Adding (5), (6), (7), we have.

$$r_1+r_2+r_3 = \frac{a+b+c}{2} \text{ —— (8)}$$

subtracting (5), (6), (7) from (8), we get

$$r_1 = \frac{1}{2}(b+c-a), \; r_2 = \frac{1}{2}(a+c-b), \; r_3 = \frac{1}{2}(a+b-c).$$

This prove that –

$\therefore \xi \in L(S)$.

$\therefore \xi \in \mathbb{R}^3 \Rightarrow \xi \in L(S)$.

$\therefore \mathbb{R}^3 \subseteq L(S)$. —(9)

again $S \subseteq \mathbb{R}^3$ and $L(S)$ being the smallest subspace of $\mathbb{R}^3$ containing S, so $L(S) \subseteq \mathbb{R}^3$ —— (10)

From (9) and (10), we have $L(S) = \mathbb{R}^3$.

Hence, the set of vectors $S = \{(0,1,1), (1,0,1), (1,1,0)\}$ is a basis of $\mathbb{R}^3$.

Ex: Show that the set of vectors $S = \{(1,2,1), (3,6,2), (0,0,1)\}$ is ~~a~~ not a basis of the vector space $\mathbb{R}^3$.

~~Soln Let α_1~~ Let $\alpha_1 = (1,2,1)$, $\alpha_2 = (3,6,2)$, $\alpha_3 = (0,0,1)$

~~To prove L.I.~~

Let us consider ~~the relation~~ $c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 = \theta$

$$c_1(1,2,1) + c_2(3,6,2) + c_3(0,0,1) = \theta$$

$c_1 + 3c_2 = 0$, $2c_1 + 6c_2 = 0$, $c_1 + 2c_2 + c_3 = 0$

$c_1 + 3c_2 = 0$, $c_3 = 0$

$$\begin{vmatrix} 1 & 3 & 0 \\ 2 & 6 & 0 \\ 1 & 2 & 1 \end{vmatrix} = 0$$

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9.(i)

$S = \{(k,1,1),(1,k,1),(1,1,k)\}$.

$\therefore$ Let. $\alpha_1 = (k,1,1)$,

$\alpha_2 = (1,k,1)$

$\alpha_3 = (1,1,k)$.

~~To prove L.I~~ :

The set S is a basis of $\mathbb{R}^3$ if S is L.I and $L(S) = \mathbb{R}^3$.

Since S is L.I, we have

$$\begin{vmatrix} k & 1 & 1 \\ 1 & k & 1 \\ 1 & 1 & k \end{vmatrix} \neq 0$$

$$\text{or, } \begin{vmatrix} k+2 & k+2 & k+2 \\ 1 & k & 1 \\ 1 & 1 & k \end{vmatrix} \neq 0$$

$$\text{or } (k+2) \begin{vmatrix} 1 & 1 & 1 \\ 1 & k & 1 \\ 1 & 1 & k \end{vmatrix} \neq 0$$

$\Rightarrow k \neq 1, -2$.

Dimension of a vector space :

The number of vectors in a basis of a vector space V is said to be the dimension or rank of V. it is denoted by dim V.

(i) The dimension of the null space $\{\theta\}$ is 0.

(ii) " " " " vector space $\mathbb{R}^2$ is 2.

(iii) " " " " " " $\mathbb{R}^3$ is 3.

(iv) " " " " " " $\mathbb{R}^n$ is n.

(v) " " " " " " $\mathbb{R}_{m\times n}$ is mn.

(vi) " " " " " " P_n of all real poly. in x of degree < n together with zero poly. is n basis V.

Theorm:- Let, V be a vectorspace of dimension n over a field F then any linearly independent set of n vectors is a basis of V.

Ex: Show that the set $S = \{(1,1,0),(1,0,1),(0,1,1)\}$ is a basis of $\mathbb{R}^3$.

Soln: Let $\alpha_1 = (1,1,0)$

$\alpha_2 = (1,0,1)$.

$\alpha_3 = (0,1,1)$.

To prove L.I:

$\therefore$ The set of vectors S is L.I.

Since, $\mathbb{R}^3$ is a vectorspace of dimension 3 and S is a L.I set containing 3 vectors of $\mathbb{R}^3$ then S is a basis of $\mathbb{R}^3$.

1,0.

(i)

Given that $S = \{\alpha, \beta, \gamma\}$ is a basis of V, then set of vectors S is L.I and dimension of V is 3.

Let $\alpha_1 = c\alpha$ $\quad S' = \{\alpha_1, \alpha_2, \alpha_3\}$.

$\alpha_2 = c\beta$

$\alpha_3 = c\gamma$

$c_1(\alpha + c\beta) + c_2\beta + c_3\gamma = 0$

To prove L.I:-

we consider the relation,

$$c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 = 0 \quad ; \text{ where } c_1, c_2, c_3 \in F.$$

$$\text{or, } c_1c\alpha + c_2c\beta + c_3c\gamma = 0.$$

Since α, β, γ are L.I, we have $cc_1 = 0$, $cc_2 = 0$, $cc_3 = 0$.

$$\Rightarrow c_1 = 0,\ c_2 = 0,\ c_3 = 0 \quad [\because c \neq 0]$$

Hence $\alpha_1, \alpha_2, \alpha_3$ are linearly independent.

Since V is a vector space of dimension 3 and S' is a L.I set containing 3 vectors of V, then S' is basis of V.

ii.

Given that $\{\alpha_1, \alpha_2, \alpha_3\}$ is a basis of V.

Then $\{\alpha_1, \alpha_2, \alpha_3\}$ are linearly independent and $\dim V = 3$

Let, $S = \{\beta_1, \beta_2, \beta_3\}$. where $\beta_1 = \alpha_1 + \alpha_3$

$$\beta_2 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3$$

$$\beta_3 = \alpha_1 + 2\alpha_2 + 3\alpha_3.$$

To prove L.I.

We consider the relation,

$$c_1\beta_1 + c_2\beta_2 + c_3\beta_3 = 0 \quad , \text{ where } c_1, c_2, c_3 \in R$$

$$\text{or, } c_1(\alpha_1 + \alpha_3) + c_2(2\alpha_1 + 3\alpha_2 + 4\alpha_3) + c_3(\alpha_1 + 2\alpha_2 + 3\alpha_3) = 0$$

$$\text{or, } (c_1 + 2c_2 + c_3)\alpha_1 + \alpha_2(3c_2 + 2c_3) + \alpha_3(c_1 + 4c_2 + 3c_3) = 0.$$

Since, $\alpha_1, \alpha_2, \alpha_3$ are L.I we have

$$\therefore \quad c_1 + 2c_2 + c_3 = 0 \ ; \ 3c_2 + 2c_3 = 0 ; \quad c_1 + 4c_2 + 3c_3 = 0.$$

These are the homogeneous system of equations in c_1, c_2, c_3

The coeff. determinant of this system is, $\begin{vmatrix} 1 & 2 & 1 \\ 0 & 3 & 2 \\ 1 & 4 & 3 \end{vmatrix}$

$$= 1(9-8) - 2(0-2) + 1(0-3)$$

$$= 1 + 4 - 3$$

$$= 2 (\neq 0).$$

By cramer's rule, the system have unique sol^n.

and the sol^n is $c_1 = c_2 = c_3 = 0$.

Hence $\beta_1, \beta_2, \beta_3$ are L.I.

Since V is a vector space of dimension 3. and S is L.I. set containing 3 vectors of V, then S is a basis of V.

[8 ① Find the basis and dimension of the subspace S of $\mathbb{R}^3$ defined by
$S = \{(x,y,z) \in \mathbb{R}^3 ; 2x+y-z=0\}$.

Sol^n:- Let $\xi = (a,b,c) \in S$ be any vector, where $a,b,c \in \mathbb{R}$.

$\therefore 2a+b-c=0$, $a,b,c \in \mathbb{R}$.

$\Rightarrow c = 2a+b$

$\therefore \xi = (a,b,c)$

$= (a,b,2a+b)$

$= a(1,0,2)+b(0,1,1)$.

$= a\alpha + b\beta$, where $\alpha = (1,0,2) \in S$
$\beta = (0,1,1) \in S$

This proves that $\xi \in L\{\alpha, \beta\}$.

$\therefore \xi \in S \Rightarrow \xi \in L\{\alpha, \beta\}$.

$\therefore S \subseteq L\{\alpha, \beta\}$ —— ①.

again, $\alpha \in S$, $\beta \in S$.

Since S is a subspace of $\mathbb{R}^3$, then $a\alpha + b\beta \in S$.

$\therefore L\{\alpha, \beta\} \subseteq S$. —— ②

From ① and ②, we have $L\{\alpha,\beta\} = S$.

To prove L.I:

We consider the relation $c_1\alpha + c_2\beta = \theta$.

i.e. $c_1(1,0,2) + c_2(0,1,1) = \theta$

i.e. $(c_1, c_2, 2c_1+c_2) = (0,0,0)$.

$\therefore c_1 = 0$, $c_2 = 0$, $2c_1 + c_2 = 0$

$\therefore \{\alpha, \beta\}$ is L.I.

Hence $\{\alpha,\beta\} = \{(1,0,2), (0,1,1)\}$ is a basis of S.

and dimension of S is 2.

1.8. (11)

Find the basis and dimension of the subspace S of $\mathbb{R}^3$ defined by

$S = \{(x,y,z) \in \mathbb{R}^3 : x+2y = z,\ 2x+3z = y\}$.

+ Let $\xi = \{a,b,c\} \in S$, where $a,b,c \in \mathbb{R}$

then $a+2b-c=0$, $2a-b+3c=0$ —(2)

$2a-b+3c$ —(1)

Solving (1) and (2),

$$\frac{a}{6-1} = \frac{b}{-2-3} = \frac{c}{-1-4}$$

$$\therefore \frac{a}{5} = \frac{b}{-5} = \frac{c}{-5}$$

$$\therefore \frac{a}{1} = \frac{b}{-1} = \frac{c}{-1} = k \text{ (say)}$$ [k is arbitrary real number]

$\therefore a = k,\ b = -k,\ c = -k.$

$\xi = (a,b,c)$

$= (k,-k,-k)$

$= k(1,-1,-1) = k\alpha$, where $\alpha = (1,-1,-1) \in S$

This prove that $\xi \in L\{\alpha\}$

$\xi \in S \Rightarrow \xi \in L\{\alpha\}$

$\therefore S \subseteq L(\alpha)$. —(1)

again, $\alpha \in S$ and since S is a subspace of $\mathbb{R}^3$, we have

$c\alpha \in S$, where $c \in \mathbb{R}$

$\therefore L\{\alpha\} \subseteq S$. —(2)

From (1) and (2), $L(\alpha) = S$.

<u>To prove L.I :-</u>

Since singleton $\{\alpha\}$ is a set of non-zero vector then it is L.I.

Hence $\{\alpha\}$ i.e $\{(1,-1,-1)\}$ is a basis of S and $\dim S = 1$.

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2(i). $S = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}_{2\times2} : a+b=0 \right\}$.

Soln:- Let, $\xi = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in S$, where $a+b=0$
or, $b=-a$.

$= \begin{pmatrix} a & -a \\ c & d \end{pmatrix}$

$= a\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$= a\alpha + c\beta + d\gamma$, where $\alpha = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$, $\beta = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $\gamma = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in S$

This prove that $\xi \in L\{\alpha, \beta, \gamma\}$.

$\xi \in S \Rightarrow \xi \in L\{\alpha \beta \gamma\}$

$\therefore S \subseteq L\{\alpha \beta \gamma\}$ ——— (1)

again, $\alpha, \beta, \gamma \in S$ and since S is a subspace of $\mathbb{R}_{2\times2}$.

we have

$a\alpha + b\beta + c\gamma \in S$, $a, b, c \in \mathbb{R}$

$\therefore L\{\alpha, \beta, \gamma\} \subseteq S$. ——— (2).

from (1) and (2) we have

$L\{\alpha, \beta, \gamma\} = S$.

To prove L.I

Let us consider the relation,

$c_1\alpha + c_2\beta + c_3\gamma = O_{2\times2}$

or, $c_1\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} + c_2\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + c_3\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} c_1 & -c_1 \\ c_2 & c_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow c_1 = c_2 = c_3 = 0$.

$\therefore \alpha, \beta, \gamma$ are L.I.

Hence the basis of S is $\{\alpha, \beta, \gamma\}$

$= \left\{ \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$.

The $\dim S = 3$.

(III) $S =$ the set of all 2×2 real diagonal matrices.

$= \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in \mathbb{R}_{2\times2} : a, d \in \mathbb{R} \right\}$.

Let $\xi = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$

$= a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

$= a\alpha + d\beta$, where $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\beta = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

(IV) $S =$ the set of all 2×2 real symmetric matrices

$= \left\{ \begin{pmatrix} a & b \\ b & d \end{pmatrix} \in \mathbb{R}_{2\times2} : a, b, d \in \mathbb{R} \right\}$

Let $\xi = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$

$= a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$= a\alpha + b\beta + d\gamma$.

(V) $S =$ the set of all 2×2 real skew symmetric matrices.

$= \left\{ \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} \in \mathbb{R}_{2\times2} : b \in \mathbb{R} \right\}$.

Let $\xi = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}$

$= b\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

$= b\beta$

23.

(i) $S = \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \right\}$.

Let $\alpha = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $\beta = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\gamma = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $\delta = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$.

To prove L.I.

We consider the relation

$c_1\alpha + c_2\beta + c_3\gamma + c_4\delta = O_{2\times2}$

or, $c_1\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + c_2\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + c_3\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} + c_4\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

or, $\begin{pmatrix} c_1 + c_2 + c_3 & c_1 + c_2 + c_4 \\ c_1 + c_3 + c_4 & c_2 + c_3 + c_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\therefore$ $c_1 + c_2 + c_3 = 0$. — (1) $c_1 + c_2 + c_4 = 0$ — (2) $c_1 + c_3 + c_4 = 0$, — (3) $c_2 + c_3 + c_4 = 0$ — (4).

Adding, $c_1 + c_2 + c_3 + c_4 = 0$ — (5)

Subtracting ①, ②, ③, ④ from ⑤, we have

$$c_1 = c_2 = c_3 = c_4 = 0.$$

$\therefore$ The set of vectors are L.I.

Since $\mathbb{R}_{2\times 2}$ is a vector space of all 2×2 real matrices and then $\dim \mathbb{R}_{2\times 2} = 4$ and S is a L.I set, containing 4 matrices then S is a basis of $\mathbb{R}_{2\times 2}$

Ex:- V is a vector space of all 3×1 real matrices. Show that the set $S = \left\{ \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix} \right\}$ is a basis of V.

Soln:- Let $\alpha = \begin{pmatrix}1\\1\\1\end{pmatrix}$, $\beta = \begin{pmatrix}1\\1\\0\end{pmatrix}$, $\gamma = \begin{pmatrix}1\\0\\0\end{pmatrix}$.

To prove L.I.

We consider the relation

$$c_1\alpha + c_2\beta + c_3\gamma = O_{3\times 1}$$

$$\text{i.e. } c_1\begin{pmatrix}1\\1\\1\end{pmatrix} + c_2\begin{pmatrix}1\\1\\0\end{pmatrix} + c_3\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}c_1+c_2+c_3\\c_1+c_2\\c_1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}.$$

$$\therefore c_1+c_2+c_3 = 0, \quad c_1+c_2 = 0, \quad c_1 = 0.$$

$$\Rightarrow c_1 = c_2 = c_3 = 0.$$

$\therefore$ α, β, γ are L.I.

Since V is a vector space of all 3×1 real matrices and $\dim V = 3$. and S is a L.I set, containing 3 matrices, then S is a basis of V.