3.4: Jacobian

Def: If u and v are two differentiable functions of two independent variables x and y. Then the Jacobian of u and v with respect to x and y is denoted by $J\left(\frac{u,v}{x,y}\right)$ or, $\frac{\partial(u,v)}{\partial(x,y)}$ and defined as

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

Similarly, we cand define

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

Note: $\frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(u,v)} = 1$

Problems:

1. If $u = x^2 + y^2$ and $v = xy$, find $\frac{\partial(u,v)}{\partial(x,y)}$

Sol: Suppose $u = x^2 + y^2$ and $v = xy$.

Then, $u_x = \frac{\partial u}{\partial x} = 2x$

$u_y = \frac{\partial u}{\partial y} = 2y$

$v_x = \frac{\partial v}{\partial x} = y$

$v_y = \frac{\partial v}{\partial y} = x.$

Therefore, $\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$

$= \begin{vmatrix} 2x & 2y \\ y & x \end{vmatrix}$

$= 2x^2 - 2y^2$

$= 2(x^2 - y^2)$

2. If $x = r\cos\theta$, $y = r\sin\theta$, find $\frac{\partial(r,\theta)}{\partial(x,y)}$.

Sol: Suppose $x = r\cos\theta$ and $y = r\sin\theta$. $\longrightarrow$ (1)

we have

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} r_x & r_y \\ \theta_x & \theta_y \end{vmatrix}$$

From (1), we have

$$x^2+y^2 = r^2(\cos^2\theta + \sin^2\theta) \Rightarrow x^2+y^2 = r^2$$

$$\Rightarrow \boxed{r = \sqrt{x^2+y^2}}$$

and

$$\frac{y}{x} = \frac{\sin\theta}{\cos\theta} \Rightarrow \tan\theta = \frac{y}{x}$$

$$\Rightarrow \boxed{\theta = \tan^{-1}\left(\frac{y}{x}\right)}$$

Now,

$$r_x = \frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{r}$$

$$r_y = \frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} = \frac{y}{r}$$

$$\theta_x = \frac{\partial \theta}{\partial x}$$

$$= \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{\partial}{\partial x}\left(\frac{y}{x}\right)$$

$$= \frac{x^2}{x^2+y^2} \cdot \left(\frac{-y}{x^2}\right)$$

$$= \frac{-y}{x^2+y^2}$$

$$= \frac{-y}{r^2}$$

$$\theta_y = \frac{\partial \theta}{\partial y}$$

$$= \frac{1}{1+\left(\frac{y}{x}\right)^2} \frac{\partial}{\partial y}\left(\frac{y}{x}\right)$$

$$= \frac{x^2}{x^2+y^2}\left(\frac{1}{x}\right)$$

$$= \frac{x}{x^2+y^2}$$

$$= \frac{x}{r^2}$$

Therefore, $\dfrac{\partial(r, \theta)}{\partial(x, y)} = \begin{vmatrix} \frac{x}{r} & \frac{y}{r} \\ \frac{-y}{r^2} & \frac{x}{r^2} \end{vmatrix}$

$$= \frac{x^2}{r^3} + \frac{y^2}{r^3}$$

$$= \frac{1}{r^3}\left(x^2+y^2\right)$$

$$= \frac{1}{r^3} \cdot r^2 = \frac{1}{r}$$

$$\boxed{\text{So, } \frac{\partial(x, y)}{\partial(r, \theta)} = r}$$

3. If $x = r\cos\theta$, $y = r\sin\theta$ and $z = z$,

find $\dfrac{\partial(x,y,z)}{\partial(r,\theta,z)}$.

Hint: $\dfrac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} x_r & x_\theta & x_z \\ y_r & y_\theta & y_z \\ z_r & z_\theta & z_z \end{vmatrix}$

Ans: $\boxed{r}$

4. In spherical polar coordinates,

$x = r.\sin\theta.\cos\phi$, $y = r\sin\theta.\sin\phi$

and $z = r\cos\theta$, find $\dfrac{\partial(x,y,z)}{\partial(r,\theta,\phi)}$

Ans: $\boxed{r^2\sin\theta}$

3.5: Functional Dependence

If $J\left(\frac{u,v}{x,y}\right)=0$, then we say that u and v are functionally dependent. Otherwise (i.e., $J\left(\frac{u,v}{x,y}\right)\neq 0$), u and v are functionally independent.

Problems: 1. Verify $u=x(1-y)$ and $v=xy$ are functionally dependent or not. If u and v are functionally dependent, find the relation between u and v.

Sol: Given $u=x(1-y)$ and $v=xy$

So, $u_x = 1-y$ | $v_x = y$

$u_y = -x$ | $v_y = x$

Therefore, $$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

$$= \begin{vmatrix} 1-y & -x \\ y & x \end{vmatrix} = x(1-y)+xy = x \neq 0$$

Hence u and V are not functionally dependent. They are functionally independent.

2. If $u = \tan^{-1}x + \tan^{-1}y$ and $V = \frac{x+y}{1-xy}$, prove that u and V are functionally dependent and hence find the relation between u and V.

Sol: Suppose $u = \tan^{-1}x + \tan^{-1}y$

and $V = \frac{x+y}{1-xy}$.

Then,
$$u_x = \frac{\partial u}{\partial x} = \frac{1}{1+x^2}, \quad u_y = \frac{1}{1+y^2}$$
$$V_x = \frac{\partial V}{\partial x} = \frac{1+y^2}{(1-xy)^2}, \quad V_y = \frac{1+x^2}{(1-xy)^2}$$

Therefore,

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1}{1+x^2} & \frac{1}{1+y^2} \\ \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \end{vmatrix}$$

$$= \frac{1}{(1-xy)^2} \begin{vmatrix} \frac{1}{1+x^2} & \frac{1}{1+y^2} \\ 1+y^2 & 1+x^2 \end{vmatrix}$$

$$= 0$$

Hence u and v are functionally dependent.

Now,

$$u = \tan^{-1}x + \tan^{-1}y$$

$$= \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

$$= \tan^{-1}v$$

$$\boxed{\text{i.e., } u = \tan^{-1}v}$$