Semester-III

Core Course 5T

Chapter-5

Partial Differential Equations

Class Note 3 (2 hours)

(Laplace equation in Spherical Polar Coordinates and Solution of its Angular Part)

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Class Note-3

6. Laplace's Equation in spherical polar coordinate systems: Obtaining the form of equation. Solution using the method of separation of variables (up to angular part).

Laplace's equation in spherical polar coordinate system:

Spherical polar coordinates are (r,θ,φ). Expression of $\vec{\nabla}$ in spherical polar coordinates is:

$$\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\varphi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}$$

Therefore:

$$\nabla^2 u(r,\theta,\varphi) = \left(\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\varphi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)\cdot\left(\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\varphi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)u(r,\theta,\varphi)$$

$$= \left(\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\varphi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)\cdot\left(\hat{r}\frac{\partial u}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial u}{\partial\theta} + \hat{\varphi}\frac{1}{r\sin\theta}\frac{\partial u}{\partial\varphi}\right)$$

$$= \hat{r}\frac{\partial}{\partial r}\cdot\left(\hat{r}\frac{\partial u}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial u}{\partial\theta} + \hat{\varphi}\frac{1}{r\sin\theta}\frac{\partial u}{\partial\varphi}\right) + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}\cdot\left(\hat{r}\frac{\partial u}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial u}{\partial\theta} + \hat{\varphi}\frac{1}{r\sin\theta}\frac{\partial u}{\partial\varphi}\right)$$
$$+\hat{\varphi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\cdot\left(\hat{r}\frac{\partial u}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial u}{\partial\theta} + \hat{\varphi}\frac{1}{r\sin\theta}\frac{\partial u}{\partial\varphi}\right)$$

Remember:

$$\hat{r} = \sin\theta\cos\varphi\,\hat{\imath} + \sin\theta\sin\varphi\,\hat{\jmath} + \cos\theta\,\hat{k};$$

$$\hat{\theta} = \cos\theta\cos\varphi\,\hat{\imath} + \cos\theta\sin\varphi\,\hat{\jmath} - \sin\theta\,\hat{k};$$

$$\hat{\varphi} = -\sin\varphi\,\hat{\imath} + \cos\varphi\,\hat{\jmath}$$

Then: $$\frac{\partial\hat{r}}{\partial r} = \frac{\partial\hat{\theta}}{\partial r} = \frac{\partial\hat{\varphi}}{\partial r} = 0;$$

$$\frac{\partial\hat{r}}{\partial\theta} = \cos\theta\cos\varphi\,\hat{\imath} + \cos\theta\sin\varphi\,\hat{\jmath} - \sin\theta\,\hat{k} = \hat{\theta},$$

$$\frac{\partial\hat{\theta}}{\partial\theta} = -\left(\sin\theta\cos\varphi\,\hat{\imath} + \sin\theta\sin\varphi\,\hat{\jmath} + \cos\theta\,\hat{k}\right) = -\hat{r}, \qquad \frac{\partial\hat{\varphi}}{\partial\theta} = 0;$$

$$\frac{\partial\hat{r}}{\partial\varphi} = (-\sin\varphi\,\hat{\imath} + \cos\varphi\,\hat{\jmath})\sin\theta = \sin\theta\,\hat{\varphi}, \qquad \frac{\partial\hat{\theta}}{\partial\varphi} = \cos\theta\,\hat{\varphi},$$

$$\frac{\partial\hat{\varphi}}{\partial\varphi} = -(\cos\varphi\,\hat{\imath} + \sin\varphi\,\hat{\jmath}) = -\left(\sin\theta\,\hat{r} + \cos\theta\,\hat{\theta}\right);$$

Therefore:

$$\nabla^2 u(r,\theta,\varphi) = \hat{r}\frac{\partial}{\partial r}\cdot\left(\hat{r}\frac{\partial u}{\partial r}+\hat{\theta}\frac{1}{r}\frac{\partial u}{\partial \theta}+\hat{\varphi}\frac{1}{r\sin\theta}\frac{\partial u}{\partial \varphi}\right)$$

$$+\hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta}\cdot\left(\hat{r}\frac{\partial u}{\partial r}+\hat{\theta}\frac{1}{r}\frac{\partial u}{\partial \theta}+\hat{\varphi}\frac{1}{r\sin\theta}\frac{\partial u}{\partial \varphi}\right)$$

$$+\hat{\varphi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}\cdot\left(\hat{r}\frac{\partial u}{\partial r}+\hat{\theta}\frac{1}{r}\frac{\partial u}{\partial \theta}+\hat{\varphi}\frac{1}{r\sin\theta}\frac{\partial u}{\partial \varphi}\right)$$

$$= \hat{r}\cdot\hat{r}\frac{\partial^2 u}{\partial r^2}+\hat{\theta}\cdot\frac{1}{r}\frac{\partial \hat{r}}{\partial \theta}\frac{\partial u}{\partial r}+\hat{\theta}\cdot\hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial u}{\partial \theta}\right)+\hat{\varphi}\cdot\frac{1}{r\sin\theta}\frac{\partial \hat{r}}{\partial \varphi}\frac{\partial u}{\partial r}$$

$$+\hat{\varphi}\cdot\frac{1}{r\sin\theta}\frac{\partial \hat{\theta}}{\partial \varphi}\frac{1}{r}\frac{\partial u}{\partial \theta}+\hat{\varphi}\cdot\hat{\varphi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}\left(\frac{1}{r\sin\theta}\frac{\partial u}{\partial \varphi}\right)$$

$$= \frac{\partial^2 u}{\partial r^2}+\hat{\theta}\cdot\frac{1}{r}\hat{\theta}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

$$+\hat{\varphi}\cdot\frac{1}{r\sin\theta}\sin\theta\,\hat{\varphi}\frac{\partial u}{\partial r}+\hat{\varphi}\cdot\frac{1}{r\sin\theta}\cos\theta\,\hat{\varphi}\frac{1}{r}\frac{\partial u}{\partial \theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \varphi^2}$$

$$= \frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\cot\theta\frac{1}{r^2}\frac{\partial u}{\partial \theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \varphi^2}$$

$$= \frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\cot\theta\frac{1}{r^2}\frac{\partial u}{\partial \theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \varphi^2}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial u}{\partial \theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \varphi^2}$$

Or, $$\nabla^2 u(r,\theta,\varphi) = \left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \varphi^2}\right]u(r,\theta,\varphi)$$

Or, $$\boldsymbol{\nabla^2 \equiv \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \varphi^2}}$$

Therefore **Laplace's equation in spherical polar coordinates** can be written as:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial u}{\partial \theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \varphi^2}=0$$

$$\Rightarrow \boldsymbol{\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial u}{\partial \theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial \varphi^2}=0} \quad \ldots\ldots\ldots (6.1)$$

Separation of variables:

To solve the differential eqn. we use method of separation of variables by assuming:

$$\psi(r,\theta,\varphi) = R(r)Y(\theta,\varphi) \;\dots\dots\dots (6.2)$$

Then Laplace's eqn. becomes, after some rearrangements:

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{\partial R(r)}{\partial r}\right) = -\frac{1}{Y}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y(\theta,\varphi)}{\partial\theta}\right) - \frac{1}{Y}\frac{1}{\sin^2\theta}\frac{\partial^2 Y(\theta,\varphi)}{\partial\varphi^2}$$

Since two sides of the equation are functions of different variables, they are independent of each other. Therefore both of them are equal to a constant, say, λ. Thus the above equation gives two equations, one angular and other radial:

$$\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y(\theta,\varphi)}{\partial\theta}\right) + \frac{\partial^2 Y(\theta,\varphi)}{\partial\varphi^2} + \lambda\sin^2\theta\, Y(\theta,\varphi) = 0 \;\dots\dots (6.3)\; [\textbf{Angular eqn.}]$$

And $$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR(r)}{dr}\right) - \frac{\lambda}{r^2}R(r) = 0 \;\dots\dots\dots (6.4)\; [\textbf{Radial or } \boldsymbol{r} \textbf{ eqn.}]$$

Orbital Angular Momentum Operator
(You can avoid this)

Multiplying throughout by $-\hbar^2$ and rearranging eqn. (6.3) can be written as:

$$Or, \qquad -\hbar^2\left(\frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)Y(\theta,\varphi) = \lambda\hbar^2 Y(\theta,\varphi) \;\dots\dots (6.3A)$$

In quantum mechanics,

$$-\hbar^2\left(\frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right) = \hat{L}^2$$

is an operator, which is the square of the orbital angular momentum operator $\hat{\vec{L}}$.

Therefore Eqn. (6.3A) can be written as:

$$\hat{L}^2 Y(\theta,\varphi) = \lambda\hbar^2 Y(\theta,\varphi) \;\dots\dots\dots (6.3.B)$$

Eqn. (6.3B) is the eigen value equation of the operator $\hat{L}^2$ and $\lambda\hbar^2$ is the eigen value of $\hat{L}^2$.

To solve eqn. (6.3) again we apply the method of separation of variables by assuming $Y(\theta,\varphi)=\Theta(\theta)\Phi(\varphi)$. Then this eqn. becomes, after rearrangements:

$$\frac{sin\,\theta}{\Theta(\theta)}\frac{d}{d\theta}\left(sin\,\theta\frac{d\Theta(\theta)}{d\theta}\right)+\lambda\,sin^2\theta=-\frac{1}{\Phi(\varphi)}\frac{d^2\Phi(\varphi)}{d\varphi^2}.$$

As before, the two sides of the equation are functions of different variables. So they are independent of each other and so are equal to a constant, say, m^2. Thus the above equation gives two equations:

$$\frac{\sin\theta}{\Theta(\theta)}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta(\theta)}{d\theta}\right)+\lambda\sin^2\theta=m^2$$

Or, $$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta(\theta)}{d\theta}\right)+\left(\lambda-\frac{m^2}{\sin^2\theta}\right)\Theta(\theta)=0\ \ldots\ldots\ldots(6.5)\ [\boldsymbol{\theta}\ \mathbf{eqn.}]$$

And $$\frac{d^2\Phi(\varphi)}{d\varphi^2}+m^2\Phi(\varphi)=0\ \ldots\ldots\ldots(6.6)\ [\boldsymbol{\varphi}\ \mathbf{eqn.}]$$

To solve the θ eqn. and the φ eqn. we don't need the expression or functional form of the potential $V(r)$. It will be required to solve the radial equation.

Solution of the $\boldsymbol{\varphi}$ eqn.:

Equation (6.6) has solutions:

$$\Phi(\varphi)=Be^{\pm im\varphi}\ \text{for}\ m\neq0\quad\text{and,}\quad\Phi(\varphi)=C+D\varphi\quad\text{for}\ m=0.$$

Φ and it's derivative must be continuous within $0\leq\varphi\leq2\pi$. Also for Φ to be single valued, one must have $\Phi(\varphi+2\pi)=\Phi(\varphi)$.

Therefore

$(i)\qquad Be^{\pm im(\varphi+2\pi)}=Be^{\pm im\varphi}\ \Rightarrow e^{\pm2\pi im}=1\quad\Rightarrow\ m=0,\pm1,\pm2,\pm3.$

$(ii)\qquad D=0.$

Then, for all possible values of m, the solutions of eqn. (6.6) can be written as:

$$\boldsymbol{\Phi(\varphi)=N_\varphi e^{im\varphi},\quad \text{with}\ m=0,\pm1,\pm2,\pm3\ \ldots\ldots\ldots\quad\ldots\ldots\ldots(6.7).}$$

Where N_φ is a constant to be determined from boundary conditions.

Orbital Magnetic Quantum Number
(You can avoid this)

The φ eqn i.e. equation (6.6) can be modified as:

$$\frac{d^2\Phi(\varphi)}{d\varphi^2} + m^2\Phi(\varphi) = 0 \quad \Rightarrow \quad -\hbar^2\frac{\partial^2\Phi(\varphi)}{\partial\varphi^2} = m^2\hbar^2\Phi(\varphi)$$

$$\left[\text{Since } \Phi(\varphi) \text{ is single variable function, therefore } \frac{d}{d\varphi} \text{ can be replaced by } \frac{\partial}{\partial\varphi}\right]$$

$$\Rightarrow \left(-i\hbar\frac{\partial}{\partial\varphi}\right)^2 \Phi(\varphi) = (m\hbar)^2\Phi(\varphi) \quad \ldots\ldots\ldots (6.6A)$$

But

$$-i\hbar\frac{\partial}{\partial\varphi} = \hat{L}_z \ldots\ldots\ldots (6.6B)$$

is called the Z component of angular momentum operator in quantum mechanics.

Therefore Eqn. (6.6A) can be written as:

$$\hat{L}_z{}^2\Phi(\varphi) = (m\hbar)^2\Phi(\varphi) \quad \ldots\ldots (6.6C)$$

Eqn. (6.6C) is the eigen value equation of the operator $\hat{L}_z{}^2$ for eigen function $\Phi(\varphi)$ and $(m\hbar)^2$ is the eigen value of $\hat{L}_z{}^2$.

From eqn. (6.7) and eqn. (6.6B) we get:

$$\hat{L}_z\Phi_m(\varphi) = -i\hbar\frac{\partial}{\partial\varphi}\left(N_\varphi e^{im\varphi}\right) = m\hbar\left(N_\varphi e^{im\varphi}\right) = m\hbar\Phi_m(\varphi)$$

Thus we see that the eigen values of the Z component of orbital angular momentum operator are $m\hbar$, where m is zero or integer (positive or negative) but cannot have any half integer value. **In quantum mechanics m is called orbital magnetic quantum number.**

Solution of the θ eqn.:

In eqn. (6.5), let $\cos\theta = w$ and $\Theta(\theta) = P(w)$.

Then

$$\frac{d}{d\theta} = \frac{d}{dw}\frac{dw}{d\theta} = -\sin\theta\frac{d}{dw} = -\sqrt{1-w^2}\frac{d}{dw}$$

and eqn. (6.5) becomes:

$$-(1-w^2)\frac{d}{dw}\left(-(1-w^2)\frac{dP(w)}{dw}\right) + \lambda(1-w^2)P(w) - m^2P(w) = 0$$

$$\text{Or,}\quad \frac{d}{dw}\left((1-w^2)\frac{dP(w)}{dw}\right) + \left(\lambda - \frac{m^2}{1-w^2}\right)P(w) = 0 \;\ldots\ldots\ldots\ldots (6.8)$$

$$\text{Or,}\quad (1-w^2)\frac{d^2P(w)}{dw^2} - 2w\frac{dP(w)}{dw} + \left(\lambda - \frac{m^2}{1-w^2}\right)P(w) = 0 \;\ldots\ldots\ldots\ldots (6.9)$$

This may be called the general form (with no restriction on λ) of associated Legendre differential equation. To solve this eqn. we shall first obtain the solutions of the equation for $m = 0$. Then those solutions will be used to find the solutions of associated Legendre differential equation. Now with $m = 0$ we have:

$$(1-w^2)\frac{d^2P(w)}{dw^2} - 2w\frac{dP(w)}{dw} + \lambda P(w) = 0 \ldots\ldots\ldots (6.10)$$

Which can be called the general form (with no restriction on λ) of Legendre differential equation. Using Frobenius method, we assume the trial solution:

$$P(w) = \sum_{\nu=0}^{\infty} a_\nu w^{\nu+s}, \text{with } a_0 \neq 0 \ldots\ldots\ldots (6.11)$$

$$\text{Then:}\quad \frac{dP(w)}{dw} = \sum_{\nu=0}^{\infty} a_\nu(\nu+s)w^{\nu+s-1} \quad , \quad \frac{d^2P(w)}{dw^2} = \sum_{\nu=0}^{\infty} a_\nu(\nu+s)(\nu+s-1)w^{\nu+s-2}$$

$$\Rightarrow (1-w^2)\sum_{\nu=0}^{\infty} a_\nu(\nu+s)(\nu+s-1)w^{\nu+s-2} - 2w\sum_{\nu=0}^{\infty} a_\nu(\nu+s)w^{\nu+s-1} + \lambda\sum_{\nu=0}^{\infty} a_\nu w^{\nu+s} = 0$$

$$\Rightarrow \sum_{\nu=0}^{\infty} a_\nu(\nu+s)(\nu+s-1)w^{\nu+s-2} - \sum_{\nu=0}^{\infty} a_\nu(\nu+s)(\nu+s-1)w^{\nu+s} - \sum_{\nu=0}^{\infty} 2a_\nu(\nu+s)w^{\nu+s} + \sum_{\nu=0}^{\infty} \lambda a_\nu w^{\nu+s} = 0$$

$$\Rightarrow \sum_{\nu=0}^{\infty} a_\nu(\nu+s)(\nu+s-1)w^{\nu+s-2} - \sum_{\nu=0}^{\infty} a_\nu[(\nu+s)(\nu+s-1)+2(\nu+s)-\lambda]w^{\nu+s} = 0$$

$$\Rightarrow \sum_{\nu=0}^{\infty} a_\nu(\nu+s)(\nu+s-1)w^{\nu+s-2} - \sum_{\nu=0}^{\infty} a_\nu[(\nu+s)(\nu+s+1)-\lambda]w^{\nu+s} = 0 \ldots\ldots(6.12)$$

Eqn. (15) should be valid for all values of w. Therefore the coefficients of each power of w must vanish separately.

Equating the coefficient of w^{s-2} to zero we get (remember: ν can not be negative):

$$a_0 s(s-1) = 0 \ldots\ldots\ldots(6.13)$$

Equating the coefficient of w^{s-1} to zero we get:

$$a_1(s+1)s = 0 \ldots\ldots\ldots(6.14)$$

Equating the coefficients of $w^{\nu+s}$ equal to zero we have:

$$a_{\nu+2}(\nu+s+2)(\nu+s+1) = a_\nu(\nu+s)(\nu+s-1) + 2a_\nu(\nu+s) - \lambda a_\nu$$

$$\Rightarrow a_{\nu+2}(\nu+s+2)(\nu+s+1) = [(\nu+s)(\nu+s+1)-\lambda]a_\nu$$

$$\Rightarrow a_{\nu+2} = \frac{(\nu+s)(\nu+s+1)-\lambda}{(\nu+s+2)(\nu+s+1)}a_\nu \quad \ldots\ldots\ldots(6.15)$$

Since $a_0 \neq 0$, from eqn. (6.13), we must have:

$$s = 0 \text{ or } s = 1.$$

Up to this step, a_0 is arbitrary.

From (6.14) we have:

$$a_1 = \text{ arbitrary for } s = 0$$

$$\text{and } a_1 = 0 \text{ for } s = 1$$

Therefore $s = 1$ leads to $a_1 = 0$ and hence according to eqn. (6.15) all $a_{odd} = 0$. Then only one series, which contains only odd powers of w, will be obtained as the solution:

$$P(w) = \sum_{\nu=0,2,4\dots} a_\nu w^{\nu+1} = a_0 w + a_2 w^3 + a_4 w^5 + \cdots \quad (6.16)$$

But for $s = 0$, a_1 is not necessarily zero and we obtain the general solution as the sum of one odd and one even series as:

$$P(w) = a_0 + a_2 w^2 + a_4 w^4 + \cdots + a_1 w^1 + a_3 w^3 + a_5 w^5 + \cdots$$

$$P(w) = \sum_{\nu=0,2,4\dots} a_\nu w^\nu + \sum_{\nu=1,3,5\dots} a_\nu w^\nu \quad \dots\dots\dots (6.17)$$

Thus the more general solution (6.17), which is a sum of two series, is obtained for $s = 0$. However it should be noted that each of the two series of eqn. (6.17) or the odd series of eqn. (6.16) alone can satisfy equation (6.10) and can be taken as a solution of this equation.

For $s = 0$, the recurrence relation (6.15) becomes:

$$a_{\nu+2} = \frac{\nu(\nu+1) - \lambda}{(\nu+2)(\nu+1)} a_\nu \quad \dots\dots\dots (6.18)$$

Since $\lim\limits_{\nu \to \infty} \frac{a_{\nu+2}}{a_\nu} w^2 = \lim\limits_{\nu \to \infty} \frac{\nu(\nu+1) - \lambda}{(\nu+1)(\nu+2)} w^2 = w^2$,

the series of equation (6.17) [and also of eqn. (6.16)] converge for $w^2 < 1$. **But they become indeterminate at $w^2 = 1$ or $w = \pm 1$.**

Legendre Differential equation and Legendre polynomials:

In physics, we often encounter problems, where the boundary conditions require solutions to be finite or in other words, if series solution method is followed, then either the infinite series has to converge or the series has to terminate after finite number of terms. Therefore to have physically meaningful solutions within the range $-1 \le w \le 1$ or $0 \le \theta \le \pi$, we must have the series terminated after finite number of terms. We will now see that the series solutions terminate if λ is restricted to be a product of two consecutive integers, like:

$$\lambda = l(l+1) \quad \dots\dots\dots (6.19)$$

Where l is 0 or a positive integer. Then:

$$a_{\nu+2} = \frac{\nu(\nu+1) - l(l+1)}{(\nu+2)(\nu+1)} a_\nu = \frac{\nu^2 + \nu - l^2 - l}{(\nu+2)(\nu+1)} a_\nu = \frac{(\nu - l)(\nu + l) + \nu - l}{(\nu+2)(\nu+1)} a_\nu$$

$$= \frac{(\nu - l)(\nu + l + 1)}{(\nu+2)(\nu+1)} a_\nu$$

Or

$$a_2 = \frac{(0 - l)(0 + l + 1)}{(0+2)(0+1)} a_0 = -\frac{l(l+1)}{2} a_0$$

$$a_3 = \frac{(1 - l)(1 + l + 1)}{(1+2)(1+1)} a_1 = \frac{(1 - l)(l + 2)}{3!} a_1$$

$$\dots.etc.$$

A closer observation reveals that **any single value of *l* cannot simultaneously terminate both the series**. When l is even, the even series terminates, but the odd series does not. And when l is odd, the odd series terminates. For physically meaningful solution, the terminated series is taken and the other is rejected. $P(w)$ is suffixed by l, which is the highest power of w in the accepted expansion.

For example, for $l = 0, 1, 2, 3$ ...., as physically acceptable solution, we get:

$$P_0(w) = a_0$$

$$P_1(w) = a_1 w$$

$$P_2(w) = a_0 + a_2 w^2 = a_0 - \frac{2(2+1)}{2} a_0 w^2 = a_0 - 3a_0 w^2$$

$$P_3(w) = a_1 w + a_3 w^3 = a_1 w + \frac{(1-3)(3+2)}{3!} a_1 w^3 = a_1 w - \frac{5}{3} a_1 w^3$$

$$\dots\dots$$

Note that a_0 of $P_0(w)$ and a_0 of $P_2(w)$ are not necessarily same. Similarly a_1 of $P_1(w)$ and a_1 of $P_3(w)$ are not necessarily same etc [see Table-1].

With $\lambda = l(l+1)$, the equation (6.10) takes the form:

$$(1 - w^2)\frac{d^2P(w)}{dw^2} - 2w\frac{dP(w)}{dw} + l(l+1)P(w) = 0 \dots\dots\dots (6.20)$$

where $l = 0, 1, 2 \dots..$

This is the standard form of well-known Legendre differential equation. The solutions $P_l(w)$ are called Legendre polynomials. They can be obtained from the Rodrigues formula:

$$P_l(w) = \frac{1}{2^l\, l!}\left(\frac{d}{dw}\right)^l (w^2-1)^l \dots\dots\dots\dots\dots (6.21)$$

It has been already mentioned that the suffix l of $P_l(w)$ is the highest power of w in the expression of $P_l(w)$. It is also evident from the Rodrigues formula. Highest power w of in the expression of $(w^2-1)^l$ is $2l$. And $P_l(w)$ is obtained by l times differentiating $(w^2-1)^l$ with respect to w. Thus in $P_l(w)$ the highest power of w is l.

Table-1

First few Legendre polynomials

$\boldsymbol{l}$	$\boldsymbol{P_l(w)} = \frac{1}{2^l\, l!}\left(\frac{d}{dw}\right)^l (w^2-1)^l$	$\boldsymbol{\Theta_l(\theta)} = P_l(w) = P_l(\cos\theta)$
0	$\boldsymbol{P_0} = \frac{1}{2^0\, 0!}\left(\frac{d}{dw}\right)^0 (w^2-1)^0 = \mathbf{1}$	**1**
1	$\boldsymbol{P_1} = \frac{1}{2^1\, 1!}\left(\frac{d}{dw}\right)^1 (w^2-1)^1 = \frac{1}{2}\cdot 2w = \boldsymbol{w}$	$\boldsymbol{\cos\theta}$
2	$\boldsymbol{P_2} = \frac{1}{2^2\, 2!}\left(\frac{d}{dw}\right)^2 (w^4-2w^2+1) = \boldsymbol{\frac{1}{2}(3w^2-1)}$	$\boldsymbol{\frac{1}{2}(3cos^2\theta-1)}$
3	$\boldsymbol{P_3} = \frac{1}{2^3\, 3!}\left(\frac{d}{dw}\right)^3 (w^6-3w^4+3w^2-1) = \frac{1}{48}(120w^3-72w) = \boldsymbol{\frac{1}{2}(5w^3-3w)}$	$\boldsymbol{\frac{1}{2}(5cos^3\theta-3\cos\theta)}$

Associated Legendre Differential equation and Associated Legendre polynomials:

With $\lambda = l(l+1)$, the equation (6.9) takes the form:

$$(1-w^2)\frac{d^2P(w)}{dw^2} - 2w\frac{dP(w)}{dw} + \left(l(l+1) - \frac{m^2}{1-w^2}\right)P(w) = 0 \;\; \ldots\ldots\ldots\ldots (6.22)$$

$$\text{where } l = 0, 1, 2 \ldots..$$

This is the standard form of well-known associated Legendre differential equation. It can be shown that (the details of the solution are not required here) the associated Legendre differential equation is satisfied by the associated Legendre polynomials given by the Rodrigues formula:

$$P_l^m(w) = (1-w^2)^{\frac{|m|}{2}}\left(\frac{d}{dw}\right)^{|m|}[P_l(w)] \ldots\ldots\ldots\ldots\ldots\ldots (6.23)$$

To obtain $P_l^m(w)$, one has to differentiate $P_l(w)$ by $|m|$ times. It has been shown that the highest power of w in $P_l(w)$ is w^l, therefore one cannot differentiate $P_l(w)$ more than l times for nonzero result. Thus for nonzero solution of the associated Legendre differential equation one must have:

$$|m| \leq l, \qquad \text{Or } \; m = -l \; \text{ to } + l \text{ in integer steps.}$$

First few associated Legendre polynomials are given in **Table-2**.

Thus the associated Legendre polynomials $P_l^m(w) = P_l^m(\cos\theta)$ are the solution of the θ equation:

$$\boldsymbol{\Theta(\theta) = \Theta_l^m(\theta) = N_\theta P_l^m(\cos\theta) = P_l^m(w) = N_\theta(1-w^2)^{\frac{|m|}{2}}\left(\frac{d}{dw}\right)^{|m|}[P_l(w)]}$$

$$\textbf{with } \boldsymbol{w = \cos\theta;} \quad \boldsymbol{l = 0, 1, 2 \ldots.. \; and \; m = 0, \pm 1, \pm 2 \ldots. \pm l}$$

$$\textbf{and } \boldsymbol{N_\theta} \textbf{ is a constant to be determined from boundary conditions.}$$

Thus the solution of the angular part of Laplace equation is given by:

$$\boldsymbol{Y(\theta,\varphi) = Y_{lm}(\theta,\varphi) = N_\varphi N_\theta P_l^m(\cos\theta)e^{im\varphi} = N_\varphi N_\theta(1-w^2)^{\frac{|m|}{2}}\left(\frac{d}{dw}\right)^{|m|}[P_l(w)]e^{im\varphi}}$$

$$\textbf{with } \boldsymbol{w = \cos\theta;} \quad \boldsymbol{l = 0, 1, 2 \ldots.. \; and \; m = 0, \pm 1, \pm 2 \ldots. \pm l.}$$

Table-2

First few associated Legendre polynomials: $P_l^m(w)$ for few small values of l and m

l	$P_l(w)$	m	$P_l^m(w) = (1-w^2)^{\frac{\lvert m\rvert}{2}}\left(\frac{d}{dw}\right)^{\lvert m\rvert}[P_l(w)] = \Theta_l^m(\theta)$
0	$P_0 = 1$	**0**	$P_0^0(\cos\theta) = (1-w^2)^0\left(\frac{d}{dw}\right)^0[P_0(w)] = 1$
1	$P_1 = w = \cos\theta$	**0**	$P_1^0 = (1-w^2)^0\left(\frac{d}{dw}\right)^0[P_1(w)] = P_1(w)$ $= w = \cos\theta$
		± 1	$P_1^{\pm 1} = (1-w^2)^{\frac{\lvert\pm 1\rvert}{2}}\left(\frac{d}{dw}\right)^{\lvert\pm 1\rvert}[P_1(w)]$ $= \sqrt{(1-w^2)}\frac{d}{dw}w = \sqrt{(1-w^2)} = \sin\theta$
2	$P_2 = \frac{1}{2}(3w^2-1)$ $= \frac{1}{2}(3\cos^2\theta - 1)$	**0**	$P_2^0 = (1-w^2)^0\left(\frac{d}{dw}\right)^0[P_2(w)] = [P_2(w)]$ $= \frac{1}{2}(3w^2-1) = \frac{1}{2}(3\cos^2\theta - 1)$
		± 1	$P_2^{\pm 1} = (1-w^2)^{\frac{\lvert\pm 1\rvert}{2}}\left(\frac{d}{dw}\right)^{\lvert\pm 1\rvert}\left[\frac{1}{2}(3w^2-1)\right]$ $= \sqrt{(1-w^2)}\frac{d}{dw}\left[\frac{1}{2}(3w^2-1)\right]$ $= 3w\sqrt{(1-w^2)} = 3\cos\theta\sin\theta$
		± 2	$P_2^{\pm 2} = (1-w^2)^{\frac{\lvert\pm 2\rvert}{2}}\left(\frac{d}{dw}\right)^{\lvert\pm 2\rvert}\left[\frac{1}{2}(3w^2-1)\right]$ $= 3(1-w^2) = 3\sin^2\theta$

Orbital Angular Momentum Quantum Number and Eigen Values

You can avoid this

With $\lambda = l(l+1)$ Eqn. (6.3) becomes:

$$\hat{L}^2 Y(\theta,\varphi) = -\hbar^2\left(\frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)Y(\theta,\varphi) = l(l+1)\hbar^2 Y(\theta,\varphi)$$

Therefore eigen value of $\hat{L}^2$ is $l(l+1)\hbar^2$. Thus the orbital angular momentum L has the values: $L = \sqrt{l(l+1)}\hbar$.

The number l, which can be zero or positive integer, is called orbital angular momentum quantum number in quantum mechanics.

End of notes on angular part