DISCRETE TIME: FIRST-ORDER DIFFERENCE EQUATIONS

In the continuous-time context, the pattern of change of a variable y is embodied in the derivatives $y'(t)$, $y''(t)$, etc. The time change involved in these is infinitesimal in magnitude. When time is, instead, taken to be a *discrete* variable, so that the variable t is allowed to take only integer values, the concept of the derivative obviously will no longer be appropriate. Then, as we shall see, the pattern of change of the variable y must be described by so-called "differences," rather than by derivatives or differentials, of $y(t)$. Accordingly, the techniques of differential equations will give way to those of difference equations.

When we are dealing with discrete time, the variable y will change its value only when the variable t changes from one integer value to the next, such as from $t = 1$ to $t = 2$. Meanwhile, nothing is supposed to happen to y. In this light, it becomes more convenient to interpret the values of t as referring to *periods*—rather than *points*—of time, with $t = 1$ denoting period 1 and $t = 2$ denoting period 2, and so forth. Then we may simply regard y as having one unique value in each time period. In view of this interpretation, the discrete-time version of economic dynamics is often referred to as *period analysis*. It should be emphasized, however, that "period" is being used here not in the calendar sense but in the analytical sense. Hence, a period may involve one extent of calendar time in a particular economic model, but an altogether different one in another. Even in the same model, moreover, each successive period should not necessarily be construed as meaning equal calendar time. In the analytical sense, a period is merely a length of time that elapses before the variable y undergoes a change.

As the reader will recall, however, discrete time and continuous time really have much in common. In particular, if the time period in the discrete-time case is very, very short, it will closely approach the continuous-time case in essence.

16.1 Discrete Time, Differences, and Difference Equations

The change from continuous time to discrete time produces no effect on the fundamental nature of dynamic analysis, although the formulation of the problem must be altered. Basically, our dynamic problem is still to find a time path from some given pattern of change of a variable y through time. But the pattern of change should now be represented by the difference quotient $\Delta y/\Delta t$, which is the discrete-time counterpart of the derivative dy/dt. Recall, however, that t can only take integer values; thus, when we are comparing the values of y in two consecutive periods, we must have $\Delta t = 1$. For this reason, the difference quotient $\Delta y/\Delta t$ can be simplified to the expression Δy; this is called the *first difference* of y. The symbol Δ, meaning difference, can accordingly be interpreted as a directive to take the first difference of (y). As such, it constitutes the discrete-time counterpart of the operator symbol d/dt.

The expression Δy can take various values, of course, depending on which two consecutive time periods are involved in the difference-taking (or "differencing"). To avoid ambiguity, let us add a time subscript to y and define the first difference more specifically, as follows:

$$\Delta y_t \equiv y_{t+1} - y_t \tag{16.1}$$

where y_t means the value of y in the tth period, and y_{t+1} is its value in the period immediately following the tth period. With this symbolism, we may describe the pattern of change of y by an equation such as

$$\Delta y_t = 2 \tag{16.2}$$

or

$$\Delta y_t = -0.1 y_t \tag{16.3}$$

Equations of this type are called *difference equations*. The reader should note the striking resemblance between the last two equations, on the one hand, and the differential equations $dy/dt = 2$ and $dy/dt = -0.1y$ on the other.

Even though difference equations derive their name from difference expressions such as Δy_t, there are alternate equivalent forms of such equations which are completely free of Δ expressions and which are more convenient to use. By virtue of (16.1), we can rewrite (16.2) as

$$(16.2') \quad y_{t+1} - y_t = 2$$

or

$$(16.2'') \quad y_{t+1} = y_t + 2$$

For (16.3), the corresponding alternate equivalent forms are

$$(16.3') \quad y_{t+1} - 0.9y_t = 0$$

or

$$(16.3'') \quad y_{t+1} = 0.9y_t$$

The double-prime-numbered versions will prove convenient when we are calculating a y value from a known y value of the preceding period. In later discussions, however, we shall employ mostly the single-prime-numbered versions, i.e., those of (16.2′) and (16.3′).

It is important to note that the choice of time subscripts in a difference equation is somewhat arbitrary. For instance, without any change in meaning, (16.2′) can be rewritten as $y_t - y_{t-1} = 2$, where $(t - 1)$ refers to the period immediately preceding the tth. Or, we may express it equivalently as $y_{t+2} - y_{t+1} = 2$.

Also, it may be pointed out that, although we have consistently used subscripted y symbols, it is also acceptable to use $y(t)$, $y(t + 1)$, and $y(t - 1)$ in their stead. In order to avoid using the notation $y(t)$ for both continuous-time and discrete-time cases, however, in the discussion of period analysis we shall adhere to the subscript device.

Analogous to differential equations, difference equations can be either linear or nonlinear, homogeneous or nonhomogeneous, and of the first or second (or higher) orders. Take (16.2′) for instance. It can be classified as: (1) linear, for no y term (of any period) is raised to the second (or higher) power; (2) nonhomogeneous, since the right-hand side (where there is no y term) is nonzero; and (3) of the first-order, because there exists only a *first difference* Δy_t, involving a one-period time lag only. (In contrast, a second-order difference equation, to be discussed in the ensuing chapter, involves a two-period lag and thus entails three y terms: y_{t+2}, y_{t+1}, as well as y_t.)

Actually, that equation can also be characterized as having constant coefficients and a constant term (= 2). Since the constant-coefficient case is the only one we shall consider, this characterization will henceforth be implicitly assumed. Throughout the present chapter, the constant-term feature will also be retained, though a method of dealing with the variable-term case will be discussed in the next chapter.

The reader should check that the equation (16.3′) is also linear and of the first order; but unlike (16.2′), it is homogeneous.

16.2 Solving a First-order Difference Equation

In solving a differential equation, our objective was to find a time path $y(t)$. As we know, such a time path is a function of time which is totally free from any derivative (or differential) expressions and which is perfectly consistent with the given differential equation as well as with its initial conditions. The time path we seek from a difference equation is similar in nature. Again, it should be a function of t—a formula defining the values of y in every time period—which is consistent with the given difference equation as well as with its initial conditions. Besides, it must not contain any difference expressions such as Δy_t (or expressions like $y_{t+1} - y_t$).

Solving differential equations is, in the final analysis, a matter of integration. How do we solve a difference equation?

iterative method Before developing a general method of attack, let us first explain a relatively pedestrian method, the *iterative method*—which, though crude, will prove immensely revealing of the essential nature of a so-called "solution."

In this chapter we are concerned only with the first-order case; thus the difference equation describes the pattern of change of y between *two* consecutive periods only. Once such a pattern is specified, such as by (16.2″), and once we are given an initial value y_0, it is no problem to find y_1 from the equation. Similarly, once y_1 is found, y_2 will be immediately obtainable, and so forth, by repeated application (iteration) of the pattern of change specified in the difference equation. The results of iteration will then permit us to infer a time path.

EXAMPLE 1 Find the solution of the difference equation (16.2), assuming an initial value of $y_0 = 15$. To carry out the iterative process, it is more convenient to use the alternative form of the difference equation (16.2″), namely, $y_{t+1} = y_t + 2$, with $y_0 = 15$. From this equation, we can deduce step-by-step that

$$y_1 = y_0 + 2$$
$$y_2 = y_1 + 2 = (y_0 + 2) + 2 = y_0 + 2(2)$$
$$y_3 = y_2 + 2 = [y_0 + 2(2)] + 2 = y_0 + 3(2)$$
$$\vdots$$

and, in general, for any period t,

$$(16.4) \quad y_t = y_0 + t(2) = 15 + 2t$$

This last equation specifies the y value of any time period (including the initial period $t = 0$); it therefore constitutes the solution of (16.2).

The process of iteration is crude—it corresponds roughly to solving simple differential equations by straight integration—but it serves to point out clearly the manner in which a time path is generated. In general, the value of y_t will depend in a specified way on the value of y in the immediately preceding period (y_{t-1}); thus a given initial value y_0 will successively lead to $y_1, y_2, \ldots$, via the prescribed pattern of change.

EXAMPLE 2 Solve the difference equation (16.3); this time, let the initial value be unspecified and denoted simply by y_0. Again it is more convenient to work with the alternative version in (16.3″), namely, $y_{t+1} = 0.9y_t$. By iteration, we have

$$y_1 = 0.9y_0$$
$$y_2 = 0.9y_1 = 0.9(0.9y_0) = (0.9)^2 y_0$$
$$y_3 = 0.9y_2 = 0.9(0.9)^2 y_0 = (0.9)^3 y_0$$
$$\vdots$$

In general, we can summarize these into the solution

$$(16.5) \quad y_t = (0.9)^t y_0$$

To heighten interest, we can lend some economic content to this example. In the simple multiplier analysis, a single investment expenditure in period 0 will call forth successive rounds of spending, which in turn will bring about varying amounts of income increment in succeeding time periods. Using y to denote *income increment*, we have y_0 = the amount of investment in period 0; but the subsequent income increments will depend on the marginal propensity to consume (MPC). If MPC = 0.9 and if the income of each period is consumed only in the next period, then 90 percent of y_0 will be consumed in period 1, resulting in an income increment in period 1 of $y_1 = 0.9y_0$. By similar reasoning, we can find $y_2 = 0.9y_1$, etc. These, we see, are precisely the results of the iterative process cited above. In other words, the multiplier process of income generation can be described by a difference equation such as (16.3″), and a solution like (16.5) will tell us what the magnitude of income increment is to be in any time period t.

EXAMPLE 3 Solve the difference equation

$$my_{t+1} - ny_t = 0$$

Upon normalizing and transposing, this may be written as

$$y_{t+1} = \left(\frac{n}{m}\right) y_t$$

which is the same as (16.3″) in Example 2 except for the replacement of 0.9 by n/m. Hence, by analogy, the solution should be

$$y_t = \left(\frac{n}{m}\right)^t y_0$$

The reader is requested to watch the term $\left(\frac{n}{m}\right)^t$. It is through this term that various values of t will lead to their corresponding values of y. It therefore corresponds to the expression e^{rt} in the solutions to differential equations. If we write it more generally as b^t (b for base) and attach the more general multiplicative constant A (instead of y_0), then we see that the solution of the general homogeneous difference equation of Example 3 will be in the form

$$y_t = Ab^t$$

We shall find that this expression Ab^t will play the same important role in difference equations as the expression Ae^{rt} did in differential equations. However, even though both are exponential expressions, the former is to the base b, whereas the latter is to the base e. It stands to reason that, just as the type of the continuous-time path $y(t)$ depends heavily on the value of r, the discrete-time path y_t will hinge principally upon the value of b.

general method By this time, the reader must have become quite impressed with the various similarities between differential and difference equations. As might be conjectured, the general method of solution presently to be explained will again parallel that for differential equations.

Suppose that we are seeking the solution to the first-order difference equation

$$(16.6) \quad y_{t+1} + ay_t = c$$

where a and c are two constants. The general solution will consist of the sum of two components: a *particular integral* y_p,† which is *any* solution of the complete nonhomogeneous equation (16.6), and a *complementary function* y_c, which is the general solution of the reduced equation of (16.6):

$$(16.7) \quad y_{t+1} + ay_t = 0$$

The y_p component will again represent the equilibrium level of y, and the y_c component signifies the deviations of the time path from the equilibrium level. The sum of y_c and y_p will constitute a *general* solution, because of the presence of an arbitrary constant. As before, in order to definitize the solution, an initial condition will be needed.

Let us first deal with the complementary function. Our experience with Example 3, suggests that we may try a solution of the form $y_t = Ab^t$ (with $A,b \neq 0$); in that case, we also have $y_{t+1} = Ab^{t+1}$. If these values of y_t and y_{t+1} hold, the homogeneous equation (16.7) will become

$$Ab^{t+1} + aAb^t = 0$$

which, upon canceling the nonzero common factor Ab^t, yields

$$b + a = 0 \qquad \text{or} \qquad b = -a$$

This means that, for the trial solution to work, we must set $b = -a$; thus the complementary function should be written as

$$y_c(= Ab^t) = A(-a)^t$$

Now let us search for the particular integral which has to do with the complete equation (16.6). In this regard, Example 3 is of no help at all, because that example relates only to a homogeneous equation. However, we note that for y_p we can choose *any* solution of (16.6), so that, if a trial solution of the simplest form $y_t = k$ (a constant) can work out, no real difficulty will be encountered. Now, if $y_t = k$, then y will maintain the same constant value through time, and we must have $y_{t+1} = k$ also. Substitution of these values into (16.6) yields

$$k + ak = c \qquad \text{and} \qquad k = \frac{c}{1+a}$$

† We are borrowing this term from differential equations, even though no "integral" of any sense is involved here. Some writers call it a *particular solution.*

Since this particular k value satisfies the equation, the particular integral can be written as

$$y_p \; (= k) = \frac{c}{1 + a} \qquad (a \neq -1)$$

This being a constant, a stationary equilibrium is indicated in this case.

If it happens that $a = -1$, however, the particular integral $c/(1 + a)$ is not defined, and some other solution of the nonhomogeneous equation (16.6) must be sought. In this event, we employ the now-familiar trick of trying a solution of the form $y_t = kt$. This implies, of course, that $y_{t+1} = k(t + 1)$. Substituting these into (16.6), we find

$$k(t + 1) + akt = c \qquad \text{and} \qquad k = \frac{c}{t + 1 + at} = c$$

[because $a = -1$]

thus $\quad y_p \; (= kt) = ct$

This form of the particular integral is a nonconstant function of t; it therefore represents a moving equilibrium.

Adding y_c and y_p together, we may now write the general solution in one of the two following forms:

$$(16.8) \quad y_t = A(-a)^t + \frac{c}{1 + a} \qquad (a \neq -1)$$

$$(16.9) \quad y_t = A(-a)^t + ct = A + ct \qquad (a = -1)$$

Neither of these is completely determinate, in view of the arbitrary constant A. To eliminate this arbitrary constant, we resort to the initial condition that $y_t = y_0$ when $t = 0$. Letting $t = 0$ in (16.8), we have

$$y_0 = A + \frac{c}{1 + a} \qquad \text{and} \qquad A = y_0 - \frac{c}{1 + a}$$

Consequently, the definite version of (16.8) is

$$(16.8') \quad y_t = \left(y_0 - \frac{c}{1 + a}\right)(-a)^t + \frac{c}{1 + a} \qquad (a \neq -1)$$

Letting $t = 0$ in (16.9), on the other hand, we find $y_0 = A$, so that the definite version of (16.9) is

$$(16.9') \quad y_t = y_0 + ct \qquad (a = -1)$$

The reader should check the validity of each of these solutions by the following two steps: First, by letting $t = 0$ in (16.8′), see that the latter equation reduces to the identity $y_0 = y_0$, signifying the satisfaction of the initial condition. Second, by substituting the y_t formula (16.8′) and a similar y_{t+1} formula—obtained by replacing t with $(t+1)$ in (16.8′)—into (16.6), see that the latter reduces to the identity $c = c$, signifying that the time path is consistent with the given difference equation. The check on the validity of solution (16.9′) is analogous.

EXAMPLE 4 Solve the first-order difference equation

$$y_{t+1} - 5y_t = 1 \qquad (y_0 = \tfrac{7}{4})$$

Following the procedure used in deriving (16.8′), we can find y_c by trying a solution $y_t = Ab^t$ (which implies $y_{t+1} = Ab^{t+1}$). Substituting these values into the homogeneous version $y_{t+1} - 5y_t = 0$ and canceling the common factor Ab^t, we get $b = 5$. Thus,

$$y_c = A(5)^t$$

To find y_p, try the solution $y_t = k$, which implies $y_{t+1} = k$. Substituting these into the complete difference equation, we find $k = -\frac{1}{4}$. Hence

$$y_p = \frac{-1}{4}$$

It follows that the general solution is

$$y_t = y_c + y_p = A(5)^t - \tfrac{1}{4}$$

Letting $t = 0$ here and utilizing the initial condition $y_0 = \frac{7}{4}$, we obtain $A = 2$. Thus the definite solution may finally be written as

$$y_t = 2(5)^t - \tfrac{1}{4}$$

Since the given difference equation of this example is a special case of (16.6), with $a = -5$, $c = 1$, and $y_0 = \frac{7}{4}$, and since (16.8′) is the solution "formula" for this type of difference equation, we could have found our solution by inserting the specific parameter values into (16.8′), with the result that

$$y_t = \left(\frac{7}{4} - \frac{1}{1-5}\right)(5)^t + \frac{1}{1-5} = 2(5)^t - \frac{1}{4}$$

which checks perfectly with the earlier answer.

Note that the y_{t+1} term in (16.6) has a coefficient equal to unity. If a given difference equation has a nonunity coefficient for this term, it must be normalized before using the solution formula (16.8′).

EXERCISE 16.2

1 Convert the following difference equations into the form of (16.2″):

(a) $\Delta y_t = 7$ (b) $\Delta y_t = 0.2y_t$ (c) $\Delta y_t = 2y_t - 9$

(d) $y_{t+1} - y_t = 3 \qquad (y_0 = 2)$

16.4 The Cobweb Model

To illustrate the use of first-order difference equations in economic analysis, we shall cite two variants of the market model for a single commodity. The first variant, known as the *cobweb model*, differs from our earlier market models in that it treats Q_s as a function not of current price but of the price of the preceding time period.

the model Consider a situation in which the producer's output decision must be made one period in advance of the actual sale—such as in agricultural production, where planting must precede by an appreciable length of time the harvesting and sale of the output. Let us assume that the output decision in period t is based on the then-prevailing price P_t. Since this output will not be available for sale until period $(t + 1)$, however, P_t will determine not Q_{st} but $Q_{s,t+1}$. Thus we now have a "lagged" supply function[1]

$$Q_{s,t+1} = S(P_t)$$

or, equivalently,

$$Q_{st} = S(P_{t-1})$$

When such a supply function interacts with a demand function of the form

$$Q_{dt} = D(P_t)$$

interesting dynamic price patterns will result.

Taking the linear versions of these (lagged) supply and (unlagged) demand functions, and assuming that in each time period the market price is always set at a level which clears the market, we shall have a market

[1] We are making the implicit assumption here that the entire output of a period will be placed on the market, with no part of it held in storage. Such an assumption is appropriate when the commodity in question is perishable or when no inventory is ever kept. A model with inventory will be considered in the next section.

model with the following three equations:

$$Q_{dt} = Q_{st}$$

(16.10) $$Q_{dt} = \alpha - \beta P_t \qquad (\alpha,\beta > 0)$$

$$Q_{st} = \gamma + \delta P_{t-1} \qquad (\gamma < 0;\ \delta > 0)$$

By substituting the last two equations into the first, however, the model can be reduced to a single first-order difference equation as follows:

$$\beta P_t + \delta P_{t-1} = \alpha - \gamma$$

In order to solve this equation, it is desirable first to normalize it and shift the time subscripts ahead by one period [alter t to $(t + 1)$, etc.]. The result,

(16.11) $$P_{t+1} + \frac{\delta}{\beta} P_t = \frac{\alpha - \gamma}{\beta}$$

will then be a replica of (16.6), with the substitutions

$$y = P \qquad a = \frac{\delta}{\beta} \qquad \text{and} \qquad c = \frac{\alpha - \gamma}{\beta}$$

Inasmuch as δ and β are both positive, it follows that $a \neq -1$. Consequently, we can apply formula (16.8′), to get the time path

(16.12) $$P_t = \left(P_0 - \frac{\alpha - \gamma}{\beta + \delta}\right)\left(\frac{-\delta}{\beta}\right)^t + \frac{\alpha - \gamma}{\beta + \delta}$$

where P_0 represents the initial price.

the cobwebs Three points may be observed in regard to this time path. In the first place, the expression $(\alpha - \gamma)/(\beta + \delta)$, which constitutes the particular integral of the difference equation, can be taken as the equilibrium price of the model:[1]

$$\bar{P} = \frac{\alpha - \gamma}{\beta + \delta}$$

Being a constant, this is a stationary equilibrium. Substituting $\bar{P}$ into our solution, we can express the time path P_t alternatively in the form

(16.12′) $$P_t = (P_0 - \bar{P})\left(\frac{-\delta}{\beta}\right)^t + \bar{P}$$

This leads us to the second point, namely, the significance of the expression $(P_0 - \bar{P})$. Since this corresponds to the constant A in the Ab^t term, its

[1] The reader may verify this as follows: When price is in equilibrium, we must have $P_t = P_{t-1} = \bar{P}$. By setting $P_t = P_{t-1} = \bar{P}$ in (16.11) and solving for $\bar{P}$, we obtain $\bar{P} = (\alpha - \gamma)/(\beta + \delta)$. This procedure is, of course, the familiar one of finding a particular integral by the use of the simplest (constant) trial solution.

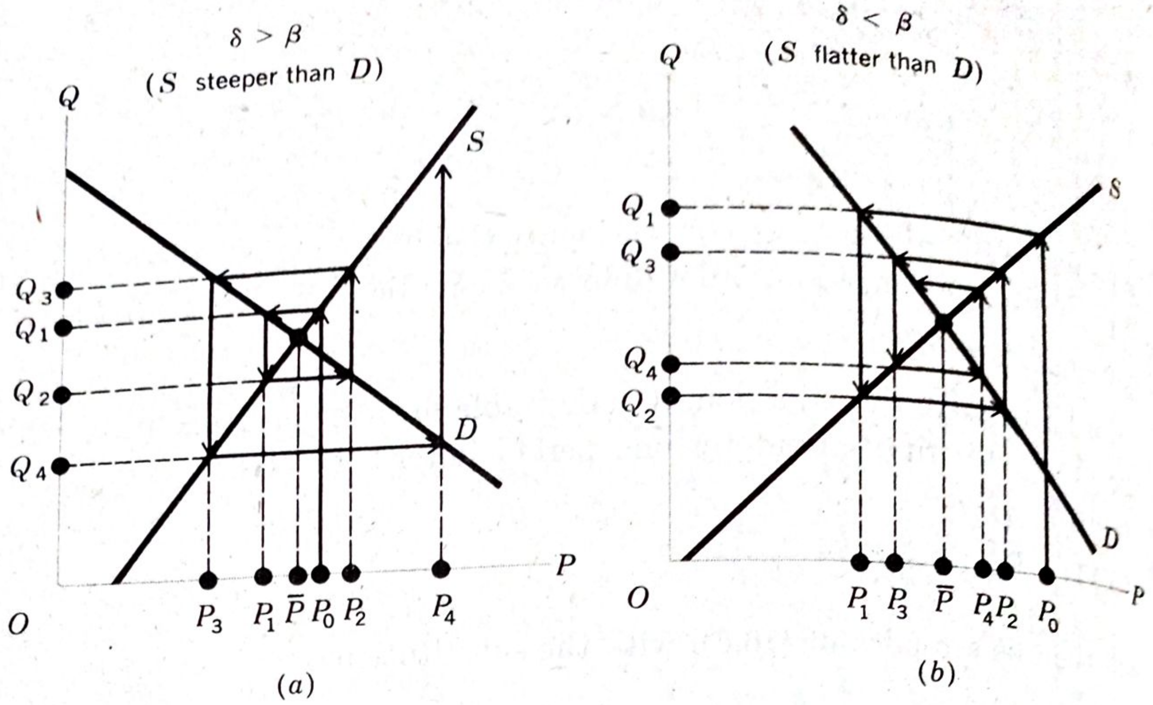

FIGURE 16.2

sign will bear on the question of whether the time path will commence above or below the equilibrium (mirror effect), whereas its magnitude will decide how far above or below (scale effect). Lastly, there is the expression $(-\delta/\beta)$, which corresponds to the b component of Ab^t. Since our model specification has it that $\beta, \delta > 0$, we must have an oscillatory time path. It is this fact which gives rise to the cobweb phenomenon, as we shall presently see. There can, of course, arise *three* possible varieties of oscillation patterns in the model. According to Table 16.1 or Fig. 16.1, the oscillation will be

$$\left.\begin{matrix}\text{explosive}\\ \text{regular}\\ \text{damped}\end{matrix}\right\} \quad \text{if } \delta \gtreqless \beta$$

In order to visualize the cobwebs, let us depict the model (16.10) in Fig. 16.2. The second equation of (16.10) plots as a downward-sloping linear demand curve, with its slope numerically equal to β. Similarly, a linear supply curve with a slope equal to δ can be drawn from the third equation, if we let the Q axis represent in this instance a *lagged* quantity supplied. The case of $\delta > \beta$ (S steeper than D) and the case of $\delta < \beta$ (S flatter than D) are illustrated in diagrams a and b, respectively. In either case, however, the intersection of D and S will yield the equilibrium price $\bar{P}$.

When $\delta > \beta$, as in diagram a, the interaction of demand and supply will produce an explosive price path as follows. Given an initial price P_0

ıere assumed above $\bar{P}$), we can follow the arrowhead and read off on the S urve that the quantity supplied in the next period (period 1) will be Q_1. In rder to clear the market, the quantity demanded in period 1 also must be $_1$, which is possible if and only if price is set at the level of P_1 (see downward rrow). Now, via the S curve, the price P_1 will lead to Q_2 as the quantity upplied in period 2, and to clear the market in the latter period, price must e set at the level of P_2 according to the demand curve. Repeating this easoning, we can trace out the prices and quantities in subsequent periods y simply following the arrowheads in the diagram, thereby weaving a "cobweb" around the demand and supply curves. By comparing the price levels, $P_0, P_1, P_2, \ldots$, we observe in this case not only an oscillatory pattern of change but also a tendency for price to widen its deviation from $\bar{P}$ as time goes by. With the cobweb being woven from inside out, the time path is divergent and explosive.

By way of contrast, in the case of diagram b, where $\delta < \beta$, a similar weaving process will create a cobweb that is center-oriented. From P_0, if we follow the arrowheads, we shall be led ever closer to the intersection of the demand and supply curves, where $\bar{P}$ is. While still oscillatory, this price path is convergent.

In Fig. 16.2 we have not shown a third possibility, namely, that of $\delta = \beta$. The procedure of graphical analysis involved, however, is perfectly analogous to the other two cases. It is therefore left to the reader as an exercise.

The above discussion has dealt only with the time path of P (that is, P_t); after P_t is found, however, it takes but a short step to get to the time path of Q. The second equation of (16.10) relates Q_{dt} to P_t, so that if (16.12) or (16.12′) is substituted into the demand equation, the time path of Q_{dt} can be obtained immediately. Moreover, since Q_{dt} must be equal to Q_{st} in each time period (clearance of market), we can simply refer to the time path as Q_t rather than Q_{dt}. On the basis of Fig. 16.2, the rationale of this substitution is easily seen. Each point on the D curve relates a P_i to a Q_i pertaining to the same time period; therefore the demand function can serve to map the time path of price into the time path of quantity.

The reader will note that the graphical technique of Fig. 16.2 is applicable even when the D and S curves are nonlinear.

EXERCISE 16.4

1 On the basis of (16.10), find the time path of Q, and analyze the condition for its convergence.