Semester-III

Core Course 5T

Chapter-5

Partial Differential Equations

Class Note 1 (1 hour)

(Discussion on the syllabus and Introduction)

Debasish Aich

SYLLABUS

Semester-III

Core Course (CC)

CC-5 : Mathematical Physics-II
Credits 06

C5T: Mathematical Physics-II
Credits 04

Mathematical Physics – II

Fourier Series:
Periodic functions. Orthogonality of sine and cosine functions, Dirichlet Conditions (Statement only). Expansion of periodic functions in a series of sine and cosine functions and determination of Fourier coefficients. Complex representation of Fourier series. Expansion of functions with arbitrary period. Expansion of non-periodic functions over an interval. Even and odd functions and their Fourier expansions. Application. Summing of Infinite Series. Term-by-Term differentiation and integration of Fourier Series. Parseval Identity.

Frobenius Method and Special Functions
Singular Points of Second Order Linear Differential Equations and their importance. Frobenius method and its applications to differential equations. Legendre, Bessel, Hermite and Laguerre Differential Equations. Properties of Legendre Polynomials: Rodrigues Formula, Generating Function, Orthogonality. Simple recurrence relations. Expansion of function in a series of Legendre Polynomials. Bessel Functions of the First Kind: Generating Function, simple recurrence relations. Zeros of Bessel Functions (Jo(x) and J1(x)) and Orthogonality.

Some Special Integrals
Beta and Gamma Functions and Relation between them. Expression of Integrals in terms of Gamma Functions. Error Function (Probability Integral).

Variational calculus in physics (To be taught by Debasish Aich)

Functionals. Basic ideas of functionals. Extremization of action as a basic principle in mechanics. Lagrangian formulation. Euler's equations of motion for simple systems: harmonics oscillators, simple pendulum, spherical pendulum, coupled oscillators. Cyclic coordinates. Symmetries and conservation laws. Legendre transformations and the Hamiltonian formulation of mechanics. Canonical equations of motion. Applications to simple systems.

Partial Differential Equations (To be taught by Debasish Aich)

Solutions to partial differential equations, using separation of variables: Laplace's Equation in problems of rectangular, cylindrical and spherical symmetry. Wave equation and its solution for vibrational modes of a stretched string, rectangular and circular membranes. Diffusion Equation.

Reference Books

- Mathematical Methods for Physicists: Arfken, Weber, 2005, Harris, Elsevier.
- Fourier Analysis by M.R. Spiegel, 2004, Tata McGraw-Hill.
- Mathematics for Physicists, Susan M. Lea, 2004, Thomson Brooks/Cole.
- Differential Equations, George F. Simmons, 2006, Tata McGraw-Hill.
- Partial Differential Equations for Scientists & Engineers, S.J. Farlow, 1993, Dover Pub.
- Engineering Mathematics, S.Pal and S.C. Bhunia, 2015, Oxford University Press
- Mathematical methods for Scientists & Engineers, D.A. McQuarrie, 2003, Viva Books
- Mathematical Physics, P. K. Chattopadhyay, 2014, New Academic Science.

C5P: Mathematical Physics II Lab **Credits 02**

Mathematical Physics II

Introduction to Numerical computation using numpy and scipy
Introduction to the python numpy module. Arrays in numpy, array operations, array item selection, slicing, shaping arrays. Basic linear algebra using the linalg submodule. Introduction to online graph plotting using matplotlib. Introduction to the scipy module. Uses in optimization and solution of differential equations.

Introduction to OCTAVE (if time permits)

Curve fitting, Least square fit, Goodness of fit, standard deviation

Ohms law to calculate R, Hooke's law to calculate spring constant

Solution of Linear system of equations by Gauss elimination method and Gauss Seidal method. Diagonalization of matrices, Inverse of a matrix, Eigen vectors, eigen values problems

Solution of mesh equations of electric circuits (3 meshes)
Solution of coupled spring mass systems (3 masses)

Generation of Special functions using User defined functions

Generating and plotting Legendre Polynomials Generating and plotting Bessel function

Solution of ODE First order Differential equation Euler, modified Euler and Runge-Kutta second order methods Second order differential equation Fixed difference method

First order differential equation

1. Radioactive decay
2. Current in RC, LC circuits with DC source
3. Newton's law of cooling
4. Classical equations of motion Second order Differential Equation
5. Harmonic oscillator (no friction)
6. Damped Harmonic oscillator
7. Over damped
8. Critical damped
9. Oscillatory

10. Forced Harmonic oscillator
11. Transient and
12. Steady state solution
13. Apply above to LCR circuits also
14. Solve $x^2 \frac{d^2y}{dx^2} - 4x(1+x)\frac{dy}{dx} + 2(1+x)y = x^3$ with the boundary condition at x=1, y=$\frac{1}{2}e^2, \frac{dy}{dx} = \frac{-3}{2}e^2 - 0.5$, in the range 1≤x≤3. Plot y and $\frac{dy}{dx}$ against x in the given range in the same graph.

Partial differential equations

1. **Wave equation**
2. **Heat equation**
3. **Poisson equation**
4. **Laplace equation**

Reference Books

- Mathematical Methods for Physics and Engineers, K.F Riley, M.P. Hobson and S. J. Bence, 3rd ed., 2006, Cambridge University Press
- Complex Variables, A.S. Fokas & M.J. Ablowitz, 8th Ed., 2011, Cambridge Univ. Press
- Numpy beginners guide, Idris Alba, 2015, Packt Publishing
- Computational Physics, D.Walker, 1st Edn., 2015, Scientific International Pvt. Ltd.
- Simulation of ODE/PDE Models with MATLAB®, OCTAVE and SCILAB: Scientific and Engineering Applications: A.V. Wouwer, P. Saucez, C.V. Fernández. 2014 Springer

Partial Differential Equations

Syllabus

Partial Differential Equations

Solutions to partial differential equations, using separation of variables: Laplace's Equation in problems of rectangular, cylindrical and spherical symmetry. Wave equation and its solution for vibrational modes of a stretched string, rectangular and circular membranes. Diffusion Equation.

Teaching Plan

Week -1

Partial differential equations (PDEs). Method of Separation of variables. PDEs frequently encountered in physics: Laplace's Equation, Wave equation, Diffusion Equation.

Week -2

Solution of Laplace's Equation in rectangular Cartesian coordinate system by the method of separation of variables.

Laplace's Equation in cylindrical polar coordinate systems: Obtaining the form of equation. Solution using the method of separation of variables.

Week -3

Laplace's Equation in spherical polar coordinate systems: Obtaining the form of equation. Solution using the method of separation of variables.

Week -4

Laplace’s equation for solving problems of rectangular, cylindrical and spherical symmetry.

Week -5

Wave equation and its solution for vibrational modes of a stretched string.

Week -6

Application of wave equation in the problem of vibrational modes of rectangular and circular membranes.

Week -7

Solution of Diffusion Equation.

Partial Differential Equations

1. Introduction

(Based on Arfken & Weber and Charlie Harper)

Partial Differential Equations:

In physics, to deal with the force we require differential equation. Thus, the equation of motion is a differential equation. Since the conception of force and equation of motion is very fundamental in physics, almost all the elementary and advanced parts of theoretical physics are formulated in terms of differential equations. Differential equations in one variable are ordinary differential equations (**ODEs**). More frequently we encounter differential equations involving more than one variable. These are partial differential equations (**PDEs**). In quantum mechanics, which appears in vast parts of physics, we have to use PDEs almost in each problem. We also have to use partial differential equations in other parts of physics. Since the dynamics of many physical systems involve just two derivatives, for example, acceleration in classical mechanics and the kinetic energy operator, $-\frac{\hbar^2}{2m}\nabla^2$, in quantum mechanics, differential equations of second order occur most frequently in physics.

Examples of PDEs

Most frequently encountered PDEs in physics are the following:

1. (A) Laplace's equation:

$$\boldsymbol{\nabla^2 u(x,y,z) = 0 \ldots\ldots\ldots (1.1)}$$

This is a homogeneous PDE and most common and important equation, appearing in the studies of:

(i) Electromagnetism, including electrostatics, dielectrics, steady currents and magnetostatics,

(ii) Hydrodynamics (irrotational flow of perfect fluid and surface waves),

(iii) Heat flow,

(iv) Gravitation.

1. (B) Poisson's equation:

$$\boldsymbol{\nabla^2 u(x,y,z) = -\frac{\rho}{\epsilon_0} \ldots\ldots\ldots (1.2)}$$

2. **Wave equation:**

1D: $$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 u(x,t)}{\partial t^2}, \quad \text{3D: } \nabla^2 u(x,y,z,t) = \frac{1}{v^2}\frac{\partial^2 u(x,y,z,t)}{\partial t^2} \ldots (1.3A \& 1.3B)$$

3. **Heat Conduction Equation or Diffusion Equation:**

1D: $$\frac{\partial^2 T(x,t)}{\partial x^2} = \frac{1}{\sigma}\frac{\partial T(x,t)}{\partial t}, \quad \text{3D: } \nabla^2 u(x,y,z,t) = \frac{1}{\sigma}\frac{\partial T(x,y,z,t)}{\partial t} \ldots (1.4A \& 1.4B)$$

4. **The Schrödinger wave equation:**

$$-\frac{\hbar^2}{2m}\nabla^2\psi(x,y,z,t) + V\psi(x,y,z,t) = i\hbar\frac{\partial\psi(x,y,z,t)}{\partial t} \ldots\ldots\ldots (1.5)$$

Etc.

Separation of Variables:

Method of separation of variables is the oldest and still most useful systematic technique for solving PDEs. In this technique the function of a number ($say\ n$) of variables is assumed to be expressible as the product of n functions, each of which is a function of a single variable. For example, we can consider the case of two dimensional Laplace equation in rectangular Cartesian coordinates:

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0 \ldots\ldots\ldots (1.6)$$

Assuming that, u can be expressed as the product of two functions of respectively x only and y only, we can write:

$$u(x,y) = X(x)Y(y) \ldots\ldots\ldots\ldots (1.7)$$

Then equation (6) can be written as:

$$Y(y)\frac{d^2X(x)}{dx^2} + X(x)\frac{d^2Y(y)}{dy^2} = 0 \qquad \Rightarrow \frac{1}{X(x)}\frac{d^2X(x)}{dx^2} + \frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} = 0$$

$$\Rightarrow \frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} = -\frac{1}{X(x)}\frac{d^2X(x)}{dx^2}$$

Note that the coordinates x and y are independent of each other and the left hand side does depend on x while the right hand side does not depend on y. Therefore the two sides of the above equation can change independently. Therefore the above equation can be valid only if the two sides do not change but are equal to a constant, say k^2. i.e.:

$$\frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} = -\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = k^2$$

Rearranging we get:

$$\frac{d^2}{dx^2}X(x) = -k^2X(x) \dots\dots\dots (\mathbf{1.8})$$

And

$$\frac{d^2}{dy^2}Y(y) = k^2Y(y) \dots\dots\dots (\mathbf{1.9})$$

Thus we obtain two ordinary differential equations. After solving these ODEs the product of their solutions can be taken as the solution of the PDE.

Note that both the equations (8) and (9) have a common form. On the left hand side the second order differential operator operates on the function $X(x)$ (or $Y(y)$ in eqn. (9)) and this operation produces the same function multiplied by a constant $-k^2$ or (or k^2 in eqn. (9)). Such equations are called eigen value equations, in which an operator operates on a function, called eigen function of this operator, and produces the same eigen function multiplied by a constant, called the eigen value of that operator belonging to that eigen function. Therefore in equation $(\mathbf{8})$ $\frac{d^2}{dx^2}$ is the operator, $X(x)$ is the eigen function of that operator $\frac{d^2}{dx^2}$ and $-k^2$ is the eigen value of $\frac{d^2}{dx^2}$ belonging to the eigen function $X(x)$. In solving PDEs by the method of separation of variables we always obtain the ODEs in the form of eigen value equations.