Rules for determining I.F.s:-

Rule-I:- If $\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)$ is a function of x alone, $f(x)$ (say), then $e^{\int f(x)\,dx}$ is an I.F. of $M\,dx+N\,dy=0$.

Proof: Let $\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=f(x)$

$$\therefore \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}+N f(x).$$

Multiplying both sides by IF $e^{\int f(x)dx}$, we have

$$\frac{\partial M}{\partial y}e^{\int f\,dx}=\frac{\partial N}{\partial x}e^{\int f\,dx}+N f(x)\,e^{\int f\,dx}$$

$$\therefore \frac{\partial}{\partial y}\left\{M e^{\int f dx}\right\}=\frac{\partial}{\partial x}\left\{N e^{\int f dx}\right\}$$

$$\frac{\partial M'}{\partial y}=\frac{\partial N'}{\partial x}, \quad \text{where } M'=Me^{\int f dx}$$
$$N'=Ne^{\int f dx}$$

This shows that $M'dx+N'dy=0$ is an exact diff. eqn,

i.e., $(M\,dx+N\,dy)e^{\int f(x)dx}=0$ satisfy the condition of exactness.

Hence $e^{\int f(x)dx}$ is an IF of $M\,dx+N\,dy=0$.

Rule-II: If $\frac{1}{M}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)$ is a function of y alone, $f(y)$ say, then $e^{-\int f(y)dy}$ is an IF of $M\,dx+N\,dy=0$

Proof: Similar proof of Rule-I.

Rule-III:- If the diff. eqn $M\,dx+N\,dy=0$ is of the form $y\,f(xy)\,dx+x\,g(xy)\,dy=0$ and if $Mx-Ny\neq 0$, then $\frac{1}{(Mx-Ny)}$ is an IF of $M\,dx+N\,dy=0$.

Another Approach:-

~~Proof~~ We have

$$M\,dx+N\,dy=0$$

$$\therefore \frac{1}{2}\left\{(Mx+Ny)\left(\frac{dx}{x}+\frac{dy}{y}\right)+(Mx-Ny)\left(\frac{dx}{x}-\frac{dy}{y}\right)\right\}=0$$

$\therefore \frac{1}{2}\left\{(Mx+Ny)\, d(\log xy) + (Mx-Ny)\, d(\log x/y)\right\} = 0.$

$\therefore \frac{1}{2}\left\{\frac{Mx+Ny}{Mx-Ny}\, d(\log xy) + d(\log x/y)\right\} = 0$

$\therefore \quad \frac{Mx+Ny}{Mx-Ny}\, d(\log xy) + d(\log x/y) = 0.$

Ex: Solve: $(x^3y^3 + x^2y^2 + xy + 1)\,y\,dx + x(x^3y^3 - x^2y^2 - xy + 1)\,dy = 0$

Sol^n: Let the given diff. eq^n is of the form $M\,dx + N\,dy = 0$

where $M = (x^3y^3 + x^2y^2 + xy + 1)\,y$

$N = (x^3y^3 - x^2y^2 - xy + 1)\,x$

Now, $Mx + Ny = 2(x^4y^4 + xy)$

$Mx - Ny = 2(x^3y^3 + x^2y^2)$

We have from -

$$\frac{Mx+Ny}{Mx-Ny}\, d(\log xy) + d(\log x/y) = 0$$

$$\frac{\cancel{2}\cancel{xy}(x^3y^3+1)}{\cancel{2}x^2y^2(x^2y^2+1)}\, \frac{d(xy)}{\cancel{xy}} + d(\log x/y) = 0$$

$$\therefore \frac{x^3y^3+1}{x^2y^2(xy+1)}\, d(xy) + d(\log x/y) = 0$$

$$\therefore \frac{\cancel{(xy+1)}(x^2y^2 - xy + 1)}{x^2y^2\cancel{(xy+1)}}\, d(xy) + d(\log x/y) = 0$$

$$\therefore \left(1 - \frac{1}{xy} + \frac{1}{x^2y^2}\right) d(xy) + d(\log x/y) = 0$$

Int.

$$\int \left(1 - \frac{1}{xy} + \frac{1}{(xy)^2}\right) d(xy) + \int d(\log x/y) = C$$

$$\therefore xy - \log(xy) - \frac{1}{xy} + \log(x/y) = C$$

$$\therefore xy - \frac{1}{xy} - 2\log y = C \quad \text{Ans}$$

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Ex: Solve: $(3x^2y^4 + 2xy)\,dx + (2x^3y^3 - x^2)\,dy = 0$ ——— (1)

Sol^n: Let the given eq^n is $M\,dx + N\,dy = 0$,

where $M = 3x^2y^4 + 2xy$ and $N = 2x^3y^3 - x^2$

$$\therefore \frac{\partial M}{\partial y} = 12x^2y^3 + 2x$$

$$\frac{\partial N}{\partial x} = 6x^2y^3 - 2x$$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 6x^2y^3 + 4x$$

$$\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{6x^2y^3 + 4x}{3x^2y^4 + 2xy} = \frac{2}{y}$$

$\therefore$ The IF is $e^{-\int \frac{2}{y}dy} = e^{-2\log y} = \frac{1}{y^2}$.

Multiplying both sides of (1) by IF we get

$$\frac{1}{y^2}(3x^2y^4 + 2xy)\,dx + \frac{1}{y^2}(2x^3y^3 - x^2)\,dy = 0$$

$$\left(3x^2y^2 + \frac{2x}{y}\right)dx + \left(2x^3y - \frac{x^2}{y^2}\right)dy = 0.$$

which is an exact equation.

Now,

$$\int\left(3x^2y^2 + \frac{2x}{y}\right)dx = x^3y^2 + \frac{x^2}{y} \quad \text{[Assuming y to be const.]}$$

and $$\int\left(2x^3y - \frac{x^2}{y^2}\right)dy = x^3y^2 + \frac{x^2}{y} \quad \text{[Assuming x to be const.]}$$

$\therefore$ The general soln is $x^3y^2 + \frac{x^2}{y} = c$.

Ex: $2xy\,dx + (y^2 - x^2)\,dy = 0$

where $M = 2xy$ and $N = y^2 - x^2$

$\therefore \frac{\partial M}{\partial y} = 2x$, $\quad \frac{\partial N}{\partial x} = -2x$,

$$\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{4x}{2xy} = \frac{2}{y}$$

$\therefore$ IF is $e^{-\int \frac{2}{y}dy} = e^{-2\log y} = \frac{1}{y^2}$.

Multiplying both sides by I.F we have

$$\frac{2x}{y}\,dx + \left(\frac{y^2 - x^2}{y^2}\right)dy = 0.$$

which is an exact eqn.

Now, $\int \frac{2x}{y}\,dx = \frac{x^2}{y}$ [y is const.]

and $\int \frac{y^2-x^2}{y^2}\,dy = \int\left(1-\frac{x^2}{y^2}\right)dy = y + \frac{x^2}{y}$ [x is const.]

$\therefore$ The solⁿ is $\frac{x^2}{y} + y = c$.

$x^2 + y^2 = cy$.

Ex) Solve: $(xy^2 - e^{1/x^3})\,dx - x^2 y\,dy = 0$.

Solⁿ: Let given eqn is $M\,dx + N\,dy = 0$,

where $M = xy^2 - e^{1/x^3}$ and $N = -x^2 y$

$\frac{\partial M}{\partial y} = 2xy$, $\frac{\partial N}{\partial x} = -2xy$

$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 4xy$

$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{4xy}{-x^2 y} = -\frac{4}{x}$.

$\therefore$ The IF is $e^{\int(-\frac{4}{x})dx} = e^{-4\log x} = \frac{1}{x^4}$.

Multiplying both sides by IF we get

$$\frac{1}{x^4}(xy^2 - e^{1/x^3})\,dx - \frac{x^2 y\,dy}{x^4} = 0$$

$$\left(\frac{y^2}{x^3} - \frac{1}{x^4}e^{1/x^3}\right)dx - \frac{y}{x^2}\,dy = 0$$

Which is an exact equation

Now, $\int\left(\frac{y^2}{x^3} - \frac{1}{x^4}e^{1/x^3}\right)dx = -\frac{y^2}{2x^2} + \frac{1}{3}\int e^{1/x^3}\,d\left(\frac{1}{x^3}\right)$ [y is const.]

$= -\frac{y^2}{2x^2} + \frac{1}{3}e^{1/x^3}$

and $\int -\frac{y}{x^2}\,dy = -\frac{y^2}{2x^2}$ [x is const.]

$\therefore$ The general solⁿ is $-\frac{y^2}{2x^2} + \frac{1}{3}e^{1/x^3} = c$

$2x^2 e^{1/x^3} - 3y^2 = 6cx^2$.

Ans

Ex: Solve: $(y^2+2x^2y)\,dx + (2x^3 - xy)\,dy = 0$

Sol^n: Let $x^h y^k$ be its IF, we multiply both sides of given eqn by it. Then

$$x^h y^k (y^2+2x^2y)\,dx + x^h y^k (2x^3 - xy)\,dy = 0$$

$$\Rightarrow (x^h y^{k+2} + 2x^{h+2} y^{k+1})\,dx + (2x^{h+3} y^k - x^{h+1} y^{k+1})\,dy = 0 \quad \text{——— (1)}$$

Which $M\,dx + N\,dy = 0$, (Say) is an exact eqn.

Thus $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

$$\Rightarrow \frac{\partial}{\partial y}\left\{x^h y^{k+2} + 2x^{h+2} y^{k+1}\right\} = \frac{\partial}{\partial x}\left\{2x^{h+3} y^k - x^{h+1} y^{k+1}\right\}$$

$$\Rightarrow (k+2)\,x^h y^{k+1} + 2(k+1)\,x^{h+2} y^k = 2(h+3)\,x^{h+2} y^k - (h+1)\,x^h y^{k+1}$$

Comparing both side, we have—

$k+2 = -(h+1)$ | $\cancel{2}(k+1) = \cancel{2}(h+3)$

$\Rightarrow h+k = -3$ | $h-k = -2$

Solving, we have, $h = -5/2$, $k = -1/2$

$\therefore$ The IF is $x^{-5/2} y^{-1/2}$.

put this values in (1) we get:–

$$(x^{-5/2} y^{3/2} + 2x^{-1/2} y^{1/2})\,dx + (2x^{1/2} y^{-1/2} - x^{-3/2} y^{1/2})\,dy = 0$$

Which is an exact eqn.

Now $$\int (x^{-5/2} y^{3/2} + 2x^{-1/2} y^{1/2})\,dx = -\frac{2}{3} x^{-3/2} y^{3/2} + 4x^{1/2} y^{1/2}$$

and $$\int (2x^{1/2} y^{-1/2} - x^{-3/2} y^{1/2})\,dy = 4x^{1/2} y^{1/2} - \frac{2}{3} x^{-3/2} y^{3/2}$$

$\therefore$ The general sol^n is $4x^{1/2} y^{1/2} - \dfrac{2}{3} x^{-3/2} y^{3/2} = c'$

$$\Rightarrow 6\sqrt{xy} - x^{-3/2} y^{3/2} = c.$$