Semester-III

Core Course 5T

Chapter-5

Partial Differential Equations

Class Note 2 (2 hours)

(Laplace Equation in Physics. Solution of Laplace equation in 2D & 3D Rectangular Cartesian and Cylindrical Polar Coordinate Systems, few problems of 2D XY coordinate system)

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Partial Differential Equations

Class Notes-2

2. Laplace's Equation in Physics

Gauss's Law in Electrostatics:

The net flux of the electric field through any hypothetical closed surface is equal to $\frac{1}{\epsilon_0}$ times the net electric charge within that closed surface. i.e.:

$$\oiint_S \vec{E}\cdot\hat{n}dS = \frac{1}{\epsilon_0}\int_V \rho\, dV \ldots\ldots\ldots (2.1)$$

where ϵ_0 is the permittivity of free space, $\hat{n}$ is the outward unit normal on the elementary portion dS of the closed surface S, $\vec{E}$ is the intensity of electrostatic field at the position of dS, $\oiint_S \vec{E}\cdot\hat{n}dS$ is the net flux of the electric field through the closed surface S, dV is an elementary portion of the volume V enclosed by the surface S and ρ is the electric charge density at the position of dV.

With the help of Gauss's divergence theorem, equation (A) gives:

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0} \ldots\ldots\ldots (2.2)$$

And if u is the electrostatic potential, given by $\vec{E} = -\vec{\nabla}u$, then eqn. (B) gives:

$$\nabla^2 u = -\frac{\rho}{\epsilon_0} \ldots\ldots\ldots (2.3)$$

Equation (C) is the well-known Poisson's equation. If $\rho = 0$ (in charge free space), then Poisson's equation converts to **Laplace's equation in electrostatics**:

$$\nabla^2 u = 0 \ldots\ldots\ldots\ldots (2.4)$$

The same equation applies to gravitation, in a region not containing any mass, if u represents the gravitational potential. It also applies to the case of steady state flow of heat in a region not containing any heat source, if u represents temperature. Laplace's equation applies to many other problems of physics also.

3. Laplace's equation in 2D rectangular Cartesian coordinates:

$$\frac{\partial^2 \boldsymbol{u}(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{x}^2}+\frac{\partial^2 \boldsymbol{u}(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{y}^2}=\mathbf{0}\ \ldots\ldots\ldots(\mathbf{3.1})$$

Assuming that u can be expressed as the product of three functions of respectively x only and y we can write:

$$\boldsymbol{u}(\boldsymbol{x},\boldsymbol{y})=\boldsymbol{X}(\boldsymbol{x})\boldsymbol{Y}(\boldsymbol{y})\ \ldots\ldots\ldots\ldots(\mathbf{3.2})$$

Then equation (1) can be written as:

$$Y(y)\frac{d^2X(x)}{dx^2}+X(x)\frac{d^2Y(y)}{dy^2}=0 \qquad \Rightarrow \frac{1}{X(x)}\frac{d^2X(x)}{dx^2}+\frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2}=0$$

$$\Rightarrow \frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2}=-\frac{1}{X(x)}\frac{d^2X(x)}{dx^2}\ \ldots\ldots\ldots(3.3)$$

Note that the coordinates x and y are independent of each other and the left hand side does depend on x while the right hand side does not depend on y. Therefore the two sides of the above equation can change independently. Therefore the above equation can is valid only if the two sides do not change but are equal to a constant, say k^2. i.e.:

$$\frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2}=-\frac{1}{X(x)}\frac{d^2X(x)}{dx^2}=k^2$$

$$\text{Then } -\frac{1}{X(x)}\frac{d^2X(x)}{dx^2}=k^2 \qquad \Rightarrow \frac{d^2X(x)}{dx^2}+k^2X(x)=0\ \ldots\ldots\ldots(3.4)$$

$$\textbf{with solution}: \quad \boldsymbol{X}(\boldsymbol{x})=\boldsymbol{A}_1\boldsymbol{e}^{\boldsymbol{ikx}}+\boldsymbol{B}_1\boldsymbol{e}^{-\boldsymbol{ikx}}\ \ldots\ldots\ldots.(\mathbf{3.5})$$

$$\text{And } \frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2}=k^2 \ \Rightarrow \frac{d^2Y(y)}{dy^2}-k^2Y(y)=0\ \ldots\ldots\ldots(3.6)$$

$$\textbf{with solution}: \quad \boldsymbol{Y}(\boldsymbol{y})=\boldsymbol{A}_2\boldsymbol{e}^{\boldsymbol{ky}}+\boldsymbol{B}_2\boldsymbol{e}^{-\boldsymbol{ky}}\ \ldots\ldots\ldots\ (\mathbf{3.7})$$

Then the general solution of eqn. (14) becomes:

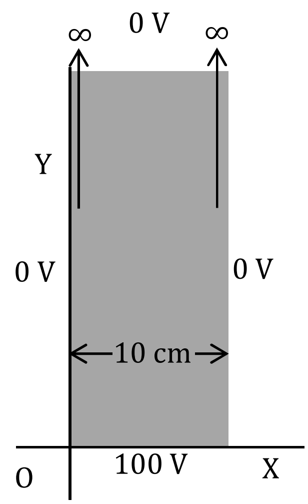
$$\boldsymbol{u}(\boldsymbol{x},\boldsymbol{y})=\boldsymbol{X}(\boldsymbol{x})\boldsymbol{Y}(\boldsymbol{y})=\left(\boldsymbol{A}_1\boldsymbol{e}^{\boldsymbol{ikx}}+\boldsymbol{B}_1\boldsymbol{e}^{-\boldsymbol{ikx}}\right)\left(\boldsymbol{A}_2\boldsymbol{e}^{\boldsymbol{ky}}+\boldsymbol{B}_2\boldsymbol{e}^{-\boldsymbol{ky}}\right)\ldots\ldots\ldots(\mathbf{3.8})$$

Note that nothing has been said about whether the constant k is real or imaginary. If it is real then $X(x)=A_1e^{ikx}+B_1e^{-ik}$ is oscillatory or sinusoidal and $Y(y)=A_2e^{ky}+B_2e^{-ky}$ is non-oscillatory. On the other hand if k is imaginary then ik is real and $X(x)=A_1e^{ikx}+B_1e^{-ikx}$ is non oscillatory but $Y(y)=A_2e^{ky}+B_2e^{-ky}$ is oscillatory.

Problem-1

An infinitely long rectangular metal plate has its two long sides and the far end at potential $0\ V$ and the base at $100\ V$ (Fig. 1). The width of the plate is 10 cm. Find the steady-state potential distribution on the plate.

[**Similar Problem:** An infinitely long rectangular metal plate has its two long sides and the far end at 0°C and the base at 100°C (Fig. 1). The width of the plate is 10 cm. Find the steady-state temperature distribution inside the plate. (Mathematical Methods in the Physical Sciences, 3rd Ed.. M. L. BOAS, Page-621).]



Solution:

There is no charge distribution on the plate. Therefore Laplace's equation applies to this problem. Let u represents the potential on the plate. Symmetry of the problem suggests that the potential is not oscillatory along Y axis. Therefore the solution of Laplace's equation in this problem will be given by:

$$u(x,y) = X(x)Y(y) = \left(A_1e^{ikx} + B_1e^{-ikx}\right)\left(A_2e^{ky} + B_2e^{-k}\ \right), \quad \text{where } k \text{ is real.}$$

Since the potential becomes zero at $y = \infty$, we must have $A_2 = 0$, otherwise the term A_2e^{ky} will make the potential ∞ at $y = \infty$.

Therefore the solution reduces to:

$$u(x,y) = X(x)Y(y) = \left(A_1e^{ikx} + B_1e^{-ikx}\right)B_2e^{-ky}$$

$$= \left(Ae^{ikx} + Be^{-i}\ \ \right)e^{-ky}, \quad \text{with} \quad A_1B_2 = A, \qquad \text{and } B_1B_2 = B.$$

$$= (a\cos kx + b\sin kx)e^{-ky} \quad \text{with } A + B = a, \qquad \text{and } i(A - B) = b.$$

$$= ae^{-ky}\cos kx + be^{-ky}\sin kx.$$

Now since $u = 0$ at $x = 0$, therefore we must have $a = 0$, since otherwise the term $ae^{-k}\ \cos kx$ does not vanish at $x = 0$.

$$\text{Therefore: } u(x,y) = be^{-ky}\sin kx$$

$$Also\ u = 0\ at\ x = 10$$

$$\text{Therefore: } 0 = be^{-ky} \sin 10k$$

Now $b = 0$ makes $u = 0$ everywhere and therefore is not acceptable. And e^{-ky} cannot be zero for all y. Therefore:

$$\sin 10k = 0 \qquad \Rightarrow 10k = n\pi \Rightarrow k = \frac{n\pi}{10}, \qquad \text{with } n = 1,2 \ldots \ldots \ldots \ldots$$

We do not take $n = 0$, which makes $u(x, y) = 0$ everywhere.

Then the solution becomes:

$$u(x, y) = \sum_{n=1}^{\infty} b_n e^{-\frac{n\pi y}{10}} \sin\left(\frac{n\pi x}{10}\right).$$

Finally we have $u = 100$ at $y = 0$ for $0 < x < 10$. i.e.

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{10}\right) = 100$$

b_n can be evaluated with the help of Fourier series as:

$$b_n = \frac{2}{10}\int_0^{10} 100\, sin\left(\frac{n\pi x}{10}\right) dx = 20\int_0^{10} sin\left(\frac{n\pi x}{10}\right) dx = -\frac{200}{n\pi}\left[cos\left(\frac{n\pi x}{10}\right)\right]_{x=0}^{10}$$

$$= -\frac{200}{n\pi}[\cos n\pi - \cos 0] = \begin{cases} \dfrac{400}{n\pi} \text{ for } n = \text{odd} \\ 0 \text{ for } n = \text{even} \end{cases}$$

Now finally the solution becomes:

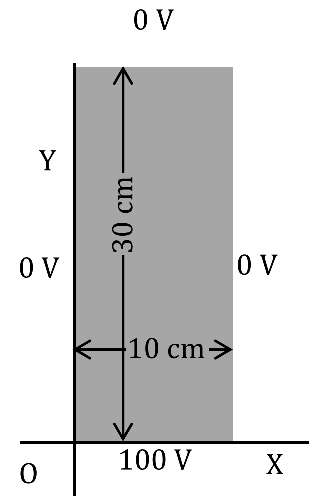
$$u(x, y) = \sum_{n=1,3,5\ldots\infty} \frac{400}{n\pi} e^{-\frac{n\pi y}{10}} \sin\left(\frac{n\pi x}{10}\right)$$

$$= \frac{400}{\pi}\left(e^{-\frac{\pi y}{10}} \sin\left(\frac{\pi x}{10}\right) + \frac{1}{3}e^{-\frac{3\pi y}{10}} \sin\left(\frac{3\pi x}{10}\right) + \frac{1}{5}e^{-\frac{5\pi}{10}} \sin\left(\frac{5\pi x}{10}\right) \ldots \ldots .\right).$$

Problem-2

A 30 cm long and 10 cm wide rectangular metal plate has its two long sides and the far end at potential $0\ V$ and the base at $100\ V$ (Fig. 2). Find the steady-state potential distribution on the plate.

Solution:



This is like Problem-1 with finite length. Therefore the solution of Laplace's equation will be:

$$u(x,y) = X(x)Y(y)$$

$$= \left(A_1 e^{ikx} + B_1 e^{-ikx}\right)\left(A_2 e^{ky} + B_2 e^{-ky}\right), \text{ with } k \text{ real.}$$

Since the potential becomes zero at $y = 30$, we must have:

$$A_2 e^{30k} + B_2 e^{-30k} = 0$$

This condition can be satisfied if we take:

$$A_2 = -A_3 e^{-30k} \ \ and\ B_2 = A_3 e^{30k},$$

where A_3 is a constant to be determined from other conditions. Therefore the solution reduces to:

$$u(x,y) = \left(A_1 e^{ikx} + B_1 e^{-ikx}\right) A_3 \left(e^{k(30-y)} - e^{-k(30-y)}\right)$$

[Verify whether $u(x,30) = 0$]

$$= \left(A_1 e^{ikx} + B_1 e^{-ikx}\right) 2A_3 \sinh\left(k(30-y)\right)$$

$$= \left(A e^{ikx} + B e^{-ikx}\right) \sinh\left(k(30-y)\right), \text{ with } \ 2A_1A_3 = A, \quad \text{ and } 2B_1A_3 = B.$$

$$= (a\cos kx + b\sin kx)\sinh\left(k(30-y)\right) \ \text{ with } A + B = a, \quad \text{ and } i(A-B) = b.$$

$$= a\sinh\left(k(30-y)\right)\cos kx + b\sinh\left(k(30-y)\right)\sin kx$$

Now, since $u = 0$ at $x = 0$, therefore we must have $a = 0$, since otherwise the term $a\sinh\left(k(30-y)\right)\cos kx$ does not vanish at $x = 0$.

$$\text{Therefore: } u(x,y) = b\sinh\left(k(30-y)\right)\sin kx$$

Also $u = 0$ at $x = 10$.

$$\text{Therefore } \ 0 = b\sinh\left(k(30-y)\right)\sin 10k$$

Now $b = 0$ makes $u = 0$ everywhere and therefore is not acceptable. And $\sinh k(y-30)$ cannot be zero for all y. Therefore:

$$\sin 10k = 0$$

$$\Rightarrow 10k = n\pi \Rightarrow k = n\pi/10\,, \qquad \text{with } n = 1,2 \ldots\ldots\ldots\ldots$$

We do not take $n = 0$, which makes $u(x,y) = 0$ everywhere.

Then the solution becomes:

$$u(x,y) = \sum_{n=1}^{\infty} b_n \sinh\left(\frac{n\pi(30-y)}{10}\right) \sin\left(\frac{n\pi x}{10}\right).$$

Finally we have $u = 100$ at $y = 0$ for $0 < x < 10$. i.e.

$$\sum_{n=1}^{\infty} b_n \sinh(3n\pi) \sin\left(\frac{n\pi x}{10}\right) = 100$$

b_n can be evaluated with the help of Fourier series as:

$$b_n = \frac{2}{10\sinh(3n\pi)} \int_0^{10} 100\, sin\left(\frac{n\pi x}{10}\right)\, dx = \frac{20}{\sinh(3n\pi)} \int_0^{10} sin\left(\frac{n\pi x}{10}\right)\, dx$$

$$= -\frac{200}{n\pi \sinh(3n\pi)} \left[cos\left(\frac{n\pi x}{10}\right)\right]_{x=0}^{10}$$

$$= -\frac{200}{n\pi\sinh(3n\pi)}[\cos n\pi - \cos 0] = \begin{cases} \dfrac{400}{n\pi\sinh(3n\pi)} & \text{for } n = \text{odd} \\ 0 \text{ for } n = \text{even} \end{cases}$$

Now finally the solution becomes:

$$u(x,y) = \sum_{n=1,3,5\ldots\infty} \frac{400}{n\pi\sinh(3n\pi)} \sinh\left(\frac{n\pi(30-y)}{10}\right) \sin\frac{n\pi x}{10}$$

$$= \frac{400}{\pi}\left(e^{-\frac{\pi y}{10}} \sin\frac{\pi x}{10} + \frac{1}{3} e^{-\frac{3\pi y}{10}} \sin\frac{3\pi x}{10} + \frac{1}{5} e^{-\frac{5\pi y}{10}} \sin\frac{5\pi x}{10} \ldots\ldots.\right)$$

Problem-3 (Homework)

Solve the Problem-1 if the plate is 30 cm wide and the potential at the bottom edge is:

$$\begin{array}{ll} x\,Volt, & \text{for } \quad 0\,cm < x < 15\,cm \\ (30 - x)\,Volt, & \text{for } \quad 15\,cm < x < 30\,cm \end{array}$$

and the potentials at the other sides are 0 $Volt$.

4. Laplace's equation in 3D rectangular Cartesian coordinates:

$$\nabla^2 u(x, y, z) = 0$$

$$\boldsymbol{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \; \ldots\ldots\ldots (4.1)}$$

Assuming that u can be expressed as the product of three functions of respectively x only, y only and z only, we can write:

$$\boldsymbol{u(x, y, z) = X(x)Y(y)Z(z) \; \ldots\ldots\ldots\ldots (4.2)}$$

Then equation (22) can be written as:

$$Y(y)Z(z)\frac{d^2X(x)}{dx^2} + Z(z)X(x)\frac{d^2Y(y)}{dy^2} + X(x)Y(y)\frac{d^2Z(z)}{dz^2} = 0$$

$$\Rightarrow \frac{1}{X(x)}\frac{d^2X(x)}{dx^2} + \frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} + \frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2} = 0 \; \ldots\ldots\ldots (4.3)$$

$$\Rightarrow \frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} + \frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2} = -\frac{1}{X(x)}\frac{d^2X(x)}{dx^2}$$

Note that the coordinates x, y and z are independent of each other and the left hand side does depend on x while the right hand side does not depend on y, z. Therefore the two sides of the above equation can change independently. Therefore the above equation can is valid only if the two sides do not change but are equal to a constant, say ${k_1}^2$. i.e.:

$$\frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} + \frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2} = -\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = {k_1}^2$$

Then $-\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = {k_1}^2 \quad \Rightarrow \frac{d^2X(x)}{dx^2} + {k_1}^2 X(x) = 0 \; \ldots\ldots\ldots (4.4)$

with solution: $\boldsymbol{X(x) = A_1 e^{ik_1x} + B_1 e^{-ik_1x} \; \ldots\ldots\ldots . (4.5)}$

And $\frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} + \frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2} = {k_1}^2$

$$\Rightarrow \frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} = {k_1}^2 - \frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2}$$

Again, like before, the two sides of the above equation are independent of each other and the equation can be valid only if the two sides are equal to a constant, say $-{k_2}^2$. i.e.:

$$\frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} = {k_1}^2 - \frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2} = -{k_2}^2$$

Then:

$$\frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} = -{k_2}^2 \qquad \Rightarrow \frac{d^2Y(y)}{dy^2} + {k_2}^2Y(y) = 0 \ \ldots\ldots\ldots (4.6)$$

with solution: $\boldsymbol{Y(y) = A_2e^{ik_2y} + B_2e^{-ik_2y}} \ldots\ldots\ldots.\ \mathbf{(4.7)}$

and $\frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2} = {k_1}^2 + {k_2}^2 = -{k_3}^2\ (say) \qquad \Rightarrow \frac{d^2Z(z)}{dz^2} + {k_3}^2Z(z) = 0 \ldots\ldots (4.8)$

with solution: $\boldsymbol{Z(z) = A_3e^{ik_3z} + B_3e^{-ik_3z}} \ldots\ldots\ldots.\ \mathbf{(4.9)}$

Then the general solution of eqn. (1) becomes is given by:

$$\boldsymbol{u(x,y,z) = X(x)Y(y)Z(z)}$$

$$= \boldsymbol{\left(A_1e^{ik_1x} + B_1e^{-ik_1x}\right)\left(A_2e^{ik_2y} + B_2e^{-ik_2y}\right)\left(A_3e^{ik_3z} + B_3e^{-ik_3z}\right)} \ldots\ldots\ldots \mathbf{(4.10)}$$

Note that from eqns. (25), (27) and (29) we have:

$${k_1}^2 + {k_2}^2 + {k_3}^2 = 0 \ldots\ldots\ldots (4.11)$$

Which implies that at least one of k_1, k_2 and k_3 must be imaginary and at least one of them must be real for eqn. (32) to be valid. And when k_1 is real, $e^{\pm ik_1x}$ are sinusoidal or oscillatory but if k_1 is imaginary, ik_1 is real and $e^{\pm ik_1x}$ are not sinusoidal. Same thing can be said about k_2 and k_3. Thus all of $X(x)$, $Y(y)$ and $Z(z)$ are not oscillatory or all of them are not non oscillatory.

5. Laplace's Equation in cylindrical polar coordinate systems: Obtaining the form of equation. Solution using the method of separation of variables.

Obtaining the form of equation

Cylindrical polar coordinates are (ρ, φ, z). Expression of $\vec{\nabla}$ in cylindrical polar coordinates is:

$$\vec{\nabla} = \hat{\rho}\frac{\partial}{\partial \rho} + \hat{\varphi}\frac{1}{\rho}\frac{\partial}{\partial \varphi} + \hat{z}\frac{\partial}{\partial z}$$

Therefore: $$\nabla^2 u(\rho, \varphi, z) = \left(\hat{\rho}\frac{\partial}{\partial \rho} + \hat{\varphi}\frac{1}{\rho}\frac{\partial}{\partial \varphi} + \hat{z}\frac{\partial}{\partial z}\right) \cdot \left(\hat{\rho}\frac{\partial}{\partial \rho} + \hat{\varphi}\frac{1}{\rho}\frac{\partial}{\partial \varphi} + \hat{z}\frac{\partial}{\partial z}\right) u(\rho, \varphi, z)$$

$$= \left(\hat{\rho}\frac{\partial}{\partial \rho} + \hat{\varphi}\frac{1}{\rho}\frac{\partial}{\partial \varphi} + \hat{z}\frac{\partial}{\partial z}\right) \cdot \left(\hat{\rho}\frac{\partial u}{\partial \rho} + \hat{\varphi}\frac{1}{\rho}\frac{\partial u}{\partial \varphi} + \hat{z}\frac{\partial u}{\partial z}\right)$$

$$= \hat{\rho}\frac{\partial}{\partial \rho} \cdot \left(\hat{\rho}\frac{\partial u}{\partial \rho} + \hat{\varphi}\frac{1}{\rho}\frac{\partial u}{\partial \varphi} + \hat{z}\frac{\partial u}{\partial z}\right) + \hat{\varphi}\frac{1}{\rho}\frac{\partial}{\partial \varphi} \cdot \left(\hat{\rho}\frac{\partial u}{\partial \rho} + \hat{\varphi}\frac{1}{\rho}\frac{\partial u}{\partial \varphi} + \hat{z}\frac{\partial u}{\partial z}\right)$$
$$+\hat{z}\frac{\partial}{\partial z} \cdot \left(\hat{\rho}\frac{\partial u}{\partial \rho} + \hat{\varphi}\frac{1}{\rho}\frac{\partial u}{\partial \varphi} + \hat{z}\frac{\partial u}{\partial z}\right)$$

Remember:

$$\hat{\rho} = \cos\varphi\,\hat{\imath} + \sin\varphi\,\hat{\jmath}, \quad \hat{\varphi} = -\sin\varphi\,\hat{\imath} + \cos\varphi\,\hat{\jmath}, \quad \hat{z} = \hat{k}$$

$$\text{Then: } \frac{\partial \hat{\rho}}{\partial \rho} = \frac{\partial \hat{\varphi}}{\partial \rho} = \frac{\partial \hat{z}}{\partial \rho} = 0$$

$$\frac{\partial \hat{\rho}}{\partial \varphi} = -\sin\varphi\,\hat{\imath} + \cos\varphi\,\hat{\jmath} = \hat{\varphi}, \qquad \frac{\partial \hat{\varphi}}{\partial \varphi} = -\cos\varphi\,\hat{\imath} - \sin\varphi\,\hat{\jmath} = -\hat{\rho}, \qquad \frac{\partial \hat{z}}{\partial \varphi} = 0;$$

$$\text{And } \frac{\partial \hat{\rho}}{\partial z} = \frac{\partial \hat{\varphi}}{\partial z} = \frac{\partial \hat{z}}{\partial z} = 0.$$

$$\text{Then } \nabla^2 u(\rho, \varphi, z) = \hat{\rho} \cdot \hat{\rho}\frac{\partial^2 u}{\partial \rho^2} + \hat{\rho} \cdot \hat{\varphi}\frac{\partial}{\partial \rho}\left(\frac{1}{\rho}\frac{\partial u}{\partial \varphi}\right) + \hat{\rho} \cdot \hat{z}\frac{\partial}{\partial \rho}\left(\frac{\partial u}{\partial z}\right)$$

$$+\hat{\varphi}\frac{1}{\rho} \cdot \frac{\partial \hat{\rho}}{\partial \varphi}\frac{\partial u}{\partial \rho} + \hat{\varphi}\frac{1}{\rho} \cdot \hat{\rho}\frac{\partial}{\partial \varphi}\left(\frac{\partial u}{\partial \rho}\right) + \hat{\varphi}\frac{1}{\rho} \cdot \frac{\partial \hat{\varphi}}{\partial \varphi}\frac{1}{\rho}\frac{\partial u}{\partial \varphi} + \hat{\varphi}\frac{1}{\rho} \cdot \hat{\varphi}\frac{\partial}{\partial \varphi}\left(\frac{1}{\rho}\frac{\partial u}{\partial \varphi}\right) + \hat{\varphi}\frac{1}{\rho} \cdot \hat{z}\frac{\partial}{\partial \varphi}\left(\frac{\partial u}{\partial z}\right)$$

$$+\hat{z} \cdot \hat{\rho}\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial \rho}\right) + \hat{z} \cdot \hat{\varphi}\frac{\partial}{\partial z}\left(\frac{1}{\rho}\frac{\partial u}{\partial \varphi}\right) + \hat{z} \cdot \hat{z}\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\right)$$

$$= \frac{\partial^2 u}{\partial \rho^2} + \hat{\varphi}\frac{1}{\rho} \cdot \hat{\varphi}\frac{\partial u}{\partial \rho} + \hat{\varphi}\frac{1}{\rho} \cdot (-\hat{\rho})\frac{1}{\rho}\frac{\partial u}{\partial \varphi} + \hat{\varphi} \cdot \hat{\varphi}\frac{1}{\rho}\frac{\partial}{\partial \varphi}\left(\frac{1}{\rho}\frac{\partial u}{\partial \varphi}\right) + \hat{z} \cdot \hat{z}\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\right)$$

$$\nabla^2 u(\rho,\varphi,z) = \frac{\partial^2 u}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial\varphi^2} + \frac{\partial^2 u}{\partial z^2} \ldots\ldots\ldots(5.1)$$

$$\boldsymbol{\nabla^2} \equiv \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial z^2} \equiv \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial z^2} \ldots\ldots\ldots(5.1A)$$

Therefore Laplace's equation in cylindrical polar coordinate system can be written as:

$$\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial z^2}\right)\boldsymbol{u(\rho,\varphi,z) = 0} \ldots\ldots\ldots(5.2)$$

Assume that $u(\rho,\varphi,z)$ can be expressed as the product of $R(\rho)$, $\Phi(\varphi)$ and $Z(z)$ which are functions of ρ only, φ only and z only. i.e.:

$$\boldsymbol{u(\rho,\varphi,z) = R(\rho)\Phi(\varphi)Z(z)} \ldots\ldots\ldots(5.3)$$

Then:

$$\Phi(\varphi)Z(z)\frac{d^2R(\rho)}{d\rho^2} + \Phi(\varphi)Z(z)\frac{1}{\rho}\frac{dR(\rho)}{d\rho} + R(\rho)Z(z)\frac{1}{\rho^2}\frac{d^2\Phi(\varphi)}{d\varphi^2} + R(\rho)\Phi(\varphi)\frac{d^2Z(z)}{dz^2} = 0$$

$$\Rightarrow \frac{1}{R(\rho)}\frac{d^2R(\rho)}{d\rho^2} + \frac{1}{R(\rho)}\frac{1}{\rho}\frac{dR(\rho)}{d\rho} + \frac{1}{\Phi(\varphi)}\frac{1}{\rho^2}\frac{d^2\Phi(\varphi)}{d\varphi^2} + \frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2} = 0$$

$$\Rightarrow \frac{1}{R(\rho)}\frac{d^2R(\rho)}{d\rho^2} + \frac{1}{R(\rho)}\frac{1}{\rho}\frac{dR(\rho)}{d\rho} + \frac{1}{\Phi(\varphi)}\frac{1}{\rho^2}\frac{d^2\Phi(\varphi)}{d\varphi^2} = -\frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2}$$

Since the two sides of the above eqn. are independent of each other therefore both sides must be equal to a constant, say, $-k^2$. i.e.:

$$\frac{1}{R(\rho)}\frac{d^2R(\rho)}{d\rho^2} + \frac{1}{R(\rho)}\frac{1}{\rho}\frac{dR(\rho)}{d\rho} + \frac{1}{\Phi(\varphi)}\frac{1}{\rho^2}\frac{d^2\Phi(\varphi)}{d\varphi^2} = -\frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2} = -k^2$$

Then:

$$-\frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2} = -k^2 \Rightarrow \frac{d^2Z(z)}{dz^2} - k^2Z(z) = 0 \ldots\ldots\ldots(5.4)$$

With solution

$$\boldsymbol{Z(z) = A_1e^{kz} + B_1e^{-kz}} \ldots\ldots\ldots(5.5)$$

where A_1 and B_1 are constants to be determined from boundary conditions.

And $$\frac{1}{R(\rho)}\frac{d^2R(\rho)}{d\rho^2}+\frac{1}{R(\rho)}\frac{1}{\rho}\frac{dR(\rho)}{d\rho}+\frac{1}{\Phi(\varphi)}\frac{1}{\rho^2}\frac{d^2\Phi(\varphi)}{d\varphi^2}=-k^2$$

$$\Rightarrow\frac{1}{R(\rho)}\rho^2\frac{d^2R(\rho)}{d\rho^2}+\frac{1}{R(\rho)}\rho\frac{dR(\rho)}{d\rho}+\frac{1}{\Phi(\varphi)}\frac{d^2\Phi(\varphi)}{d\varphi^2}=-k^2\rho^2$$

$$\Rightarrow\frac{1}{R(\rho)}\rho^2\frac{d^2R(\rho)}{d\rho^2}+\frac{1}{R(\rho)}\rho\frac{dR(\rho)}{d\rho}+k^2\rho^2=-\frac{1}{\Phi(\varphi)}\frac{d^2\Phi(\varphi)}{d\varphi^2}$$

Again, since the two sides of the above eqn. are independent of each other therefore both sides must be equal to a constant, say, m^2. i.e.:

$$\frac{1}{R(\rho)}\rho^2\frac{d^2R(\rho)}{d\rho^2}+\frac{1}{R(\rho)}\rho\frac{dR(\rho)}{d\rho}+k^2\rho^2=-\frac{1}{\Phi(\varphi)}\frac{d^2\Phi(\varphi)}{d\varphi^2}=m^2\ \ldots\ldots\ldots(5.6)$$

Then:

$$-\frac{1}{\Phi(\varphi)}\frac{d^2\Phi(\varphi)}{d\varphi^2}=m^2$$

$$\Rightarrow\boldsymbol{\frac{d^2\Phi(\varphi)}{d\varphi^2}+m^2\Phi(\varphi)=0\ \ldots\ldots\ldots(5.7)}$$

And $$\frac{1}{R(\rho)}\rho^2\frac{d^2R(\rho)}{d\rho^2}+\frac{1}{R(\rho)}\rho\frac{dR(\rho)}{d\rho}+k^2\rho^2=m^2$$

$$\Rightarrow\rho^2\frac{d^2R(\rho)}{d\rho^2}+\rho\frac{dR(\rho)}{d\rho}+(k^2\rho^2-m^2)R(\rho)=0\ \ldots\ldots\ldots(5.8)$$

Eqn. (39) has solution:

$$\boldsymbol{\Phi(\varphi)=A_2e^{im\varphi}+B_2e^{-im\varphi}}\ \ldots\ldots\ldots(5.9)$$

By changing φ to $\varphi+2\pi$ we get the same point. If $\Phi(\varphi)$ is a single valued function, i.e. at every point of space it has a single value, therefore:

$$\Phi(\varphi)=\Phi(\varphi+2\pi)$$

$$A_2e^{im\varphi}+B_2e^{-im\varphi}=A_2e^{im(\varphi+2\pi)}+B_2e^{-im(\varphi+2\pi)}=A_2e^{im\varphi}e^{im2\pi}+B_2e^{-im\varphi}e^{-im2\pi}$$

This above equation will be valid if:

$$e^{\pm im2\pi}=1$$

which requires that m must be a positive or negative integer or zero. i.e.

$$\boldsymbol{m=0,\pm1,\pm2\ \ldots\ \ldots\ldots\ldots(5.9A)}$$

To make eqn. (5.8) look like some known form, let us introduce a variable ξ, given by:

$$\xi = k\rho \text{ ... ... } (5.10)$$

Then:

$$\frac{dR}{d\rho} = \frac{dR}{d\xi}\frac{d\xi}{d\rho} = \frac{dR}{d\xi}k; \qquad \rho\frac{dR}{d\rho} = k\rho\frac{dR}{d\xi} = \xi\frac{dR}{d\xi}$$

$$\frac{d^2R}{d\rho^2} = \frac{d}{d\rho}\left(k\frac{dR}{d\xi}\right) = k\frac{d}{d\xi}\left(k\frac{dR}{d\xi}\right) = (k)^2\frac{d^2R}{d\xi^2}$$

$$\rho^2\frac{d^2R}{d\rho^2} = (k\rho)^2\frac{d^2R}{d\xi^2} = \xi^2\frac{d^2R}{d\xi^2}$$

Then eqn. (40) becomes:

$$\xi^2\frac{d^2R(\xi/k)}{d\xi^2} + \xi\frac{dR(\xi/k)}{d\xi} + (\xi^2 - m^2)R(\xi/k) = 0$$

$$\xi^2\frac{d^2P(\xi)}{d\xi^2} + \xi\frac{dP(\xi)}{d\xi} + (\xi^2 - m^2)P(\xi) = 0 \text{ ... } (\mathbf{5.11}), \quad \text{where } P(\xi) = R(\xi/k) \text{ ... } (\mathbf{5.12})$$

Eqn. **(43)** is the well-known **Bessel differential equation** with solution:

$$P(\xi) = A_3 J_m(\xi) + B_3 J_{-m}(\xi) \text{ when m is not an integer ... ... } (5.13)$$

And

$$P(\xi) = A_3 J_m(\xi) + B_3 N_m(\xi) \text{ when m is an integer ... ... } (5.14)$$

Where $J_m(\xi)$ is the Bessel function of first kind of order m. $J_m(\xi)$ can be given by:

$$J_m(\xi) = \sum_{s=0}^{\infty}\frac{(-1)^s}{s!\,\Gamma(s+m+1)}\left(\frac{\xi}{2}\right)^{2s+m} \text{ ... ... } (5.15)$$

$J_{-m}(\xi)$ is given by:

$$J_{-m}(\xi) = \sum_{s=0}^{\infty}\frac{(-1)^s}{s!\,\Gamma(s-m+1)}\left(\frac{\xi}{2}\right)^{2s-m} \text{ ... ... } (5.16)$$

And $N_m(\xi)$ is called the Bessel function of second kind or Neumann function and is given by:

$$N_m(\xi) = \frac{\cos m\pi\, J_m(\xi) - J_{-m}(\xi)}{\sin m\pi} \text{ ... ... } (5.17)$$

You may remember:

For $\xi \ll 1, \qquad J_m(\xi) \approx \frac{1}{2^m m!}\xi^m,$

and for $\xi \gg 1, \qquad J_m(\xi) \approx \sqrt{\frac{2}{\pi x}}\cos\left[x - \frac{(2m+1)\pi}{4}\right]$

Note: For $\xi \ll 1, \qquad N_m(\xi) \approx \begin{cases} \frac{2^m(m-1)!}{\pi}\xi^{-m} & \text{for } m \neq 0 \\ \frac{2}{\pi}\ln\left(\frac{\gamma\xi}{2}\right) & \text{for } m = 0 \end{cases}$

With $\gamma \approx 1.78$

and for $\xi \gg 1, \qquad N_m(\xi) \approx \sqrt{\frac{2}{\pi x}}\cos\left[x - \frac{(2m+1)\pi}{4}\right]$