Study Material - Sem. 1 - C2T -

- Special Theory of Relativity -

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The theory of relativity starts by examining how measurements of physical quantities are affected by relative motion between an observer and what is being observed. The theory of relativity was first definitely formulated by Einstein. The term 'relativity' is applied to Einstein's Theory because the primary quantities in mechanics and astronomy such as space, time and mass are shown by it to be relative, i.e, any one of these factors cannot be reckoned absolutely by itself, but only in relation to others. The new Theory is in direct contradiction with the fundamental ideas of classical Newtonian mechanics, where space, time and mass are considered absolute. Einstein's theory rejects this absolute nature of fundamental quantities, space, time and mass postulated by classical mechanics by denying their independence from the position or motion of bodies or observers.

Galilean Transformation : Classical Physics

In classical physics, the motion of a body or any physical system is governed by the Newton's laws of motion. Let us consider the simplest mechanical system : a particle of mass m acted upon by a force $\vec{F}$. Then by second law of Newton,

$$\left.\begin{aligned} m\left(\frac{d^2x}{dt^2}\right) &= F_x \\ m\left(\frac{d^2y}{dt^2}\right) &= F_y \\ m\left(\frac{d^2z}{dt^2}\right) &= F_z \end{aligned}\right] \longrightarrow (1)$$

where F_x, F_y, F_z are the x, y and z components of the vector $\vec{F}$ and $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, $\frac{d^2z}{dt^2}$ are the accelerations of the particle along x, y and z axis respectively.

The above equations are valid only if the frame of reference S defined by the coordinates x, y, z, t is an inertial frame. We define an inertial frame as a frame of reference in which the law of inertia — Newton's first law — holds. In such a system, which we may also describe as an unaccelerated system, a body that is acted on by zero net external force will move with a constant velocity. Newton assumed that a frame of reference fixed with respect to the stars is an inertial system. A rocket ship drifting in outer space, without spinning and with its engines cut off, provides an ideal inertial system. Frames accelerating with respect to such a system are not inertial. In practice, we can often neglect the small (acceleration) effects due to the rotation and orbital motion of the earth and to solar motion. Thus we may regard any set of axes fixed on the earth as forming (approximately) an inertial coordinate system. Likewise, any set of axes moving at uniform velocity w.r.t. the earth, as in a train, ship or airplane, will be (nearly) inertial because motion at uniform velocity does not introduce acceleration. However, a system of axes which accelerates w.r.t the earth, such as one fixed to a spinning merry-go-round or to an accelerating car, is not an inertial system.

Let us now consider another inertial frame $S'(x', y', z', t')$ whose axes are parallel to the corresponding axes of the old frame $S(x, y, z, t)$. The new frame moves to the right with a constant velocity v along x-axis relative to old one. Here t and t' are the times measur-ed in the two frames and defined such that $t = t' = 0$ when the two frames coincide. What is the relation between the sets of numbers (x, y, z, t) and (x', y', z', t')?

Plainly,
$$\left.\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned}\right] \longrightarrow (2)$$

These equations are known as the Galilean transformation equations.

From the first equation of set (2), we have,

$$\frac{dx'}{dt} = \frac{dx}{dt} - v$$

But, because $t = t'$, the operation $\frac{d}{dt}$ is identical to the operation $\frac{d}{dt'}$, so that

$$\frac{dx'}{dt} = \frac{dx'}{dt'}$$

Therefore, $\frac{dx'}{dt'} = \frac{dx}{dt} - v$

Similarly, $\frac{dy'}{dt'} = \frac{dy}{dt}$

and $\frac{dz'}{dt'} = \frac{dz}{dt}$

Then,

$$\frac{d^2x'}{dt'^2} = \frac{d^2x}{dt^2}$$

$$\frac{d^2y'}{dt'^2} = \frac{d^2y}{dt^2}$$

$$\frac{d^2z'}{dt'^2} = \frac{d^2z}{dt^2}$$

So, in the reference frame $S'(x', y', z', t')$, Newton's second law can be written as —

$$\left.\begin{aligned} m\frac{d^2x'}{dt'^2} &= F_x \\ m\frac{d^2y'}{dt'^2} &= F_y \\ m\frac{d^2z'}{dt'^2} &= F_z \end{aligned}\right] \longrightarrow (3)$$

The equation have exactly the same mathematical form as equation (1). Newton's laws, which govern the behaviour of the system, do not change when we make Galilean transformation. <u>Newton's laws</u>

are therefore said to remain invariant in Galilean transformation. The behaviour of all mechanical system will thus be identical in all inertial frames in uniform translations w.r.t each other. A series of mechanical experiments, whether performed in a car at rest w.r.t the ground or moving smoothly over a straight track at constant speed, gives therefore the same result.

Newtonian mechanics thus predicts that all inertial frames in uniform translation relative to each other are equivalent so far as the mechanical phenomena are concerned. A corollary to this is that no mechanical phenomena can distinguish between two inertial frames, so much so that it is impossible by mechanical experiments to prove that one of the frames is in a state of absolute rest while the other moves relative to it. This equivalence of all inertial frames is called Newtonian relativity.

Galilean Transformation & Electromagnetic Theory—

Electromagnetic phenomena are discussed in classical physics in term of the Maxwell's equations which govern their behaviour. Maxwell's relation comprise a set of differential equations and have the similar role in electromagnetic phenomena as the Newton's laws in mechanical phenomena. Without actually carrying out the Galilean transformation of Maxwell's equations, the calculation being too complicated owing to the nature of the equation, we shall just state the result. It is found that, on Galilean transformation, Maxwell's equations do change their mathematical form, in sharp contrast to Newton's equations. This is a very important point in the development of the theory of relativity.

Maxwell's equations predict the existence of electromagnetic waves that propagate with a velocity not dependent on the frequency of wave motion. With the mechanistic outlook of nineteenth century, physicists felt quite sure that the electromagnetic waves require the existence of a propagation medium. The medium was given the name ether. It appeared from the very start that ether must be attributed some strange properties in order to be in agreement with certain known facts. For instance, as these waves can travel through a vacuum, ether must be massless. But they must at the same time be highly elastic to sustain the vibrations. Despite the difficulties, the ether concept was more attractive than the alternative that electromagnetic waves require no medium for propagation. Now, once the ether is assumed, electromagnetic waves would expectedly propagate with a fixed velocity with respect to it, as sound waves do with respect to air.

It was assumed the Maxwell's electromagnetic equations were valid for the frame of reference which was at rest with respect to ether – the so-called ether frame. A solution of these equations gives the propagation velocity of the electromagnetic waves : c ($= 3 \times 10^8$

m/sec.) and this is in agreement with experimental observation. But in a new frame in uniform translation with respect to ether, Maxwell's equations, we have seen, do change their form and when the propagation velocity of electromagnetic waves as seen from the new frame was evaluated using the changed form of Maxwell's equations, it was found to have a value different from c.

If $\bar{c}$ be the fixed velocity of light relative to ether, $\bar{c}_0$ the velocity of light relative to the new frame and $\bar{v}$ the velocity of the new frame relative to ether, then $\bar{c}_0 = \bar{c} - \bar{v}$. The velocity $-\bar{v}$ sometimes called the ether wind velocity. The above vector addition of velocities accepts tacitly the validity of the Galilean transformations.

To summarise: Towards the end of the nineteenth century, physics was based upon the following three fundamental assumptions :

(i) The validity of Newton's laws,

(ii) the validity of Maxwell's equations, and

(iii) The validity of Galilean transformation

Everything that was derivable from the above assumptions, was in good experimental agreement. These assumptions predicted that —

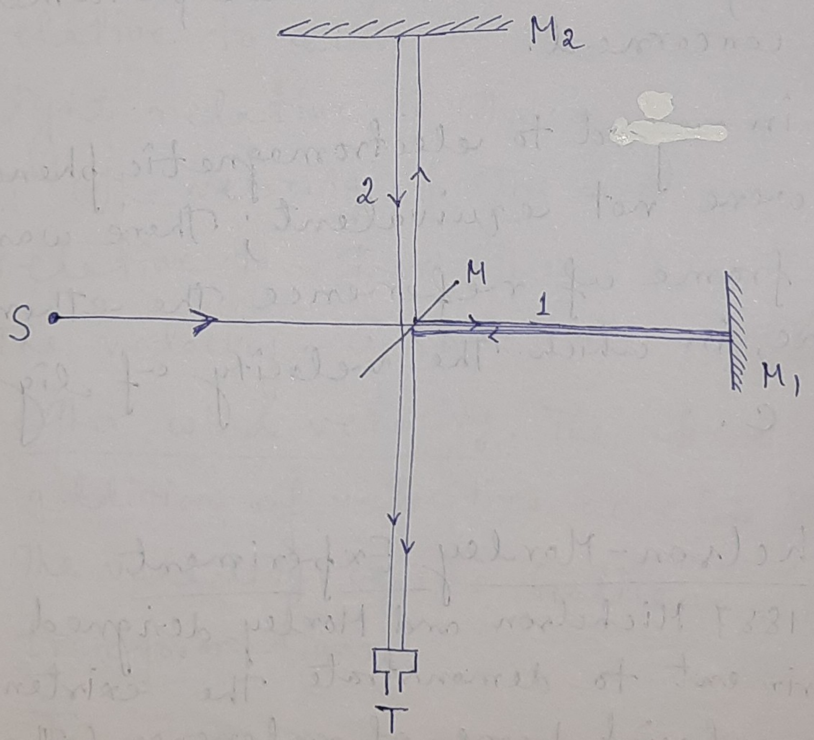
(a) all inertial frames in uniform translation with respect to each other were equivalent in so far as the mechanical phenomena were concerned.

(b) but in regard to electromagnetic phenomena, they were not equivalent; there was only one frame of reference, the ether frame, in which the velocity of light was c.

Michelson - Morley Experiment

In 1887 Michelson and Morley designed an experiment to demonstrate the existence of the special frame of reference (The ether frame) and to determine the motion of the earth with respect to the ether frame. The basic idea of the experiment was to measure the velocity of light in a frame fixed relative to earth in two mutually ⊥r direction.

Let us now describe the Michelson-Morley experiment. The Michelson interferometer is fixed on the earth. If we imagine the ether to be fixed w.r.t the sun, then the earth (and interferometer) moves through the ether at a speed of 30 km/sec, in different direction in different seasons. [The earth moves at an orbital speed of 30 km/sec along its nearly circular orbit about the sun, reversing the direction of its velocity every six months.] For the moment, neglect the earth's spinning motion.



[A simplified version of the Michelson interferometer is shown in the above figure. The beam from the source S is split into two beams by the partially silvered mirror M. The beams are reflected by mirrors 1 and 2, returning to the partially silvered mirror. The beams are then transmitted to the telescope T where they interfere, giving rise to a fringe pattern].

The beam of light (plane waves or parallel rays) from the laboratory source S (fixed w.r.t. the instrument) is split by the partially silvered mirror M into two coherent beams, beam 1 being transmitted through M and beam 2 being reflected off of M. Beam 1 is reflected back to M by mirror M_1 and beam 2 by mirror M_2. Then the returning beam 1 is partially reflected and the returning beam 2 is partially transmitted by M back to a telescope T where they interfere. The interference is constructive or destructive depending on the phase difference of the beams. The partially silvered mirror surface M is inclined at 45° to the beam directions. If M_1 and M_2 are very nearly (but not quite) at right angles, we shall observe a fringe system in the telescope consisting of nearly parallel lines.

Let us now compute the phase difference between the beams 1 and 2. These difference can arise from two causes, the different path lengths travelled, l_1 and l_2, and the different speeds of travel.

The second cause, for the moment, is the crucial one. The time for beam 1 to travel from M to M_1 and back is ____

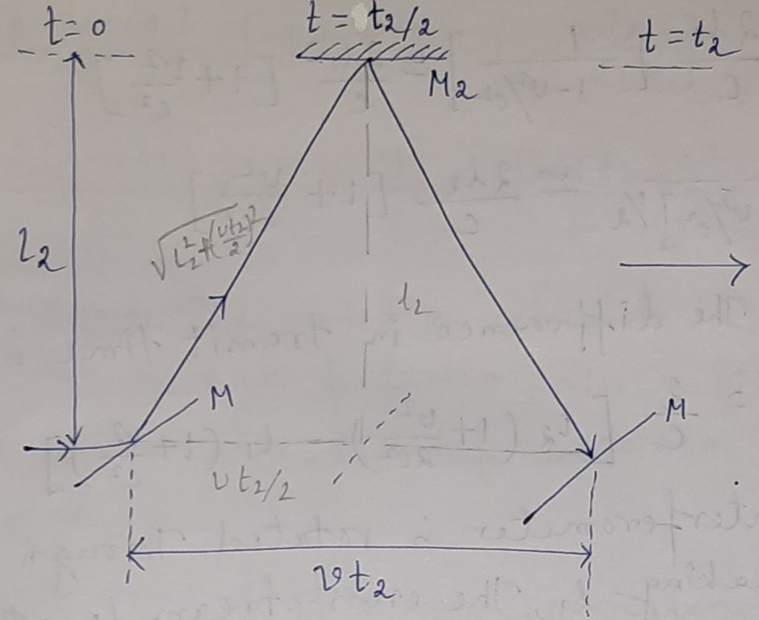
$$t_1 = \frac{l_1}{c-v} + \frac{l_1}{c+v}$$

$$= l_1 \left[\frac{c+v+c-v}{(c^2-v^2)} \right] = \frac{2cl_1}{c^2-v^2}$$

$$= \frac{2cl_1}{c^2(1-v^2/c^2)} = \frac{2l_1}{c} \left[\frac{1}{1-v^2/c^2} \right] \rightarrow (1)$$

[$v \rightarrow$ velocity of earth relative to ether]

for light, whose speed is c in the ether, has an "upstream" speed of $(c-v)$ w.r.t the apparatus and a "downstream" speed of $(c+v)$. The path of beam 2, travelling from M to M_2 and back is a "crossstream" path through ether, enabling the beam to return to the (advancing) mirror M.



[The cross stream path of beam 2. The mirrors move through the ether at a speed v, the light moving through the ether at speed c. Reflection from the moving mirror automatically gives the cross stream path.
$v \rightarrow$ velocity of interferometer w.r.t ether.]

The transit time is given by

$$2\left[l_2^2 + \left(\frac{vt_2}{2}\right)^2\right]^{1/2} = ct_2$$

or, $t_2 = \frac{2}{c}\left[l_2^2 + \left(\frac{vt_2}{2}\right)^2\right]^{1/2}$

or, $t_2^2 = \frac{4}{c^2}\left[l_2^2 + \frac{v^2 t_2^2}{4}\right] = \frac{4l_2^2}{c^2} + t_2^2 \frac{v^2}{c^2}$

or, $t_2^2 - \frac{v^2}{c^2} t_2^2 = \frac{4l_2^2}{c^2}$ or, $t_2^2\left[1 - \frac{v^2}{c^2}\right] = \frac{4l_2^2}{c^2}$

or, $t_2^2 = \frac{4l_2^2}{c^2\left[1 - \frac{v^2}{c^2}\right]}$ or, $t_2 = \frac{2l_2}{c\sqrt{1 - v^2/c^2}} \rightarrow (2)$

Now, $t_1 = \frac{2l_1}{c}\left[\frac{1}{1-v^2/c^2}\right] \simeq \frac{2l_1}{c}\left[1+\frac{v^2}{c^2}\right]$

$$t_2 = \frac{2l_2}{c\left[1-v^2/c^2\right]^{1/2}} \simeq \frac{2l_2}{c}\left[1+\frac{v^2}{2c^2}\right]$$

Therefore, the difference in transit time is

$$\Delta t = t_2 - t_1 = \frac{2}{c}\left[l_2\left(1+\frac{v^2}{2c^2}\right) - l_1\left(1+\frac{v^2}{c^2}\right)\right].$$

Now the interferometer is rotated through 90°, thereby making l_1, the cross-stream length and l_2, the downstream length. If the corresponding times are now designated by primes, the same analysis as above gives the transit time difference as —

$$\Delta t' = t_2' - t_1' = \frac{2}{c}\left[l_2\left(1+\frac{v^2}{c^2}\right) - l_1\left(1+\frac{v^2}{2c^2}\right)\right]$$

Hence, the rotation changes the difference by —

$$\Delta T = \Delta t' - \Delta t$$

$$= \frac{2}{c}\left[l_2\left(1+\frac{v^2}{c^2}\right) - l_1\left(1+\frac{v^2}{2c^2}\right) - l_2\left(1+\frac{v^2}{2c^2}\right) + l_1\left(1+\frac{v^2}{c^2}\right)\right]$$

$$= \frac{2}{c}\left[(l_2+l_1)\left(1+\frac{v^2}{c^2}\right) - (l_1+l_2)\left(1+\frac{v^2}{2c^2}\right)\right]$$

$$= \frac{2(l_1+l_2)}{c}\left[1+\frac{v^2}{c^2}-1-\frac{v^2}{2c^2}\right]$$

$$= \frac{2(l_1+l_2)}{c} \times \frac{1}{2} \times \frac{v^2}{c^2} = \left(\frac{l_1+l_2}{c}\right)\frac{v^2}{c^2}$$

Therefore, the rotation should cause a shift in the fringe pattern, since it changes the phase relationship between beams 1 and 2.

This time difference corresponds to a path difference $c\Delta T$. If this corresponds to a shift of n fringes, then $n\lambda = c\Delta T$ where, λ is the wavelength of light used.

$$\therefore n = \frac{c\Delta T}{\lambda} = \frac{(l_1+l_2)c}{c\lambda}\left(\frac{v^2}{c^2}\right) = \frac{l_1+l_2}{\lambda}\left(\frac{v^2}{c^2}\right)$$

If $l_1 = l_2 = d$, then,

$$n = \frac{2d}{\lambda}\left(\frac{v^2}{c^2}\right)$$

In Michelson's experiment, $d = 10\,m$, $\lambda = 5000\,Å$, earth's orbital speed, $v = 3\times10^4\,m/sec$. and $c = 3\times10^8\,m/sec$.

$$\therefore n = \frac{2\times10\times(3\times10^4)^2}{5000\times10^{-10}\times(3\times10^8)^2} = 0.4$$

Thus the time difference would amount to a shift of 0.4 fringe and the interferometer was capable of detecting a displacement less than even a tenth of the expected amount. To the extreme surprise, Michelson and Morley and the entire community of physicists, no fringe shift whatever was observed. Observations during various times of the day and various seasons of the year also ended in negative results. There is no ether wind! Despite the prediction of the classical theory, the experiment showed that the velocity of light is the same when measured along two perpendicular axes in a frame which, presumably, is moving relative to ether frame in different directions at different times of the year.

Explanation of Null Result

A possible interpretation of the null result is to conclude that the speed of light has the same value c in all directions in all inertial frames. That is, in any frame of reference the speed is c and not $|\bar{c}+\bar{v}|$. This assertion automatically leads to $n=0$. But such an assertion contradicts the Galilean transformation and was hardly acceptable at that time. If the measured speed of light is independent of the motion of the observer, the existence of a preferred reference frame, i.e, the ether has to be ruled out. However, in the beginning scientists were hesitant to give up the concept of the ether frame. Many attemps were made by different people to explain, within the frame-work of classical physics, the null result of Michelson-Morley's experiment. Of these the following three deserve special mention:

a. 'ether drag' hypothesis

b. Lorentz contraction hypothesis

c. 'emission Theory'

But all these patch works could not save the situation. What is needed is a radical change and fortunately this was brought about by Einstein in 1905 in the form of the special Theory of relativity.