1.

Vector spaces

**

Defination:- A non-empty set V is said to be form a vector space over a field F if

(i) There exists a binary composition '+' on V, called addition, satisfy the following conditions.

(a) $\alpha+\beta \in V$, $\forall \alpha, \beta \in V$

(b) $\alpha+(\beta+\gamma) = (\alpha+\beta)+\gamma$ $\forall \alpha, \beta, \gamma \in V$

(c) $\exists$ an element $\theta \in V$, such that $\alpha+\theta = \alpha$ $\forall \alpha \in V$.
Where θ is called additive identity element.

(d) for each $\alpha \in V$ $\exists$ an element $-\alpha \in V$, such that
$\alpha+(-\alpha) = \theta$.

(e) $\alpha+\beta = \beta+\alpha$; $\forall \alpha, \beta \in V$.

(ii) and there exists an external composition of F with V, called multiplication by the ~~scalar~~ elements of F, satisfy the conditions –

(f) $c\alpha \in V$, $\forall \alpha \in V, c \in F$.

(g) $c(d\alpha) = (cd)\alpha$, $\forall \alpha \in V$, $c, d \in F$.

(h) $c(\alpha+\beta) = c\alpha + c\beta$; $\forall \alpha, \beta \in V$, $c \in F$.

(i) $(c+d)\alpha = c\alpha + d\alpha$; $\forall \alpha \in V$, $c, d \in F$.

(j) $1.\alpha = \alpha$; $\forall \alpha \in V$, $1 \in F$ be the identity element.

Where the elements of V are called vectors and elements of F are called scalars.

Note:- The additive identity element θ is also known as zero vector or null vector.

Ex: Define real vector space or complex vector space.

ANS:- V is said to be real (or complex) vector space over a field F if the field F is $\mathbb{R}$ (or $\mathbb{C}$).

Theorm:- In a vector space V over a field F

(i) $0\alpha = \theta$; $\forall \alpha \in V$

(ii) $c\theta = \theta$, $\forall c \in F$

(iii) $(-1)\alpha = -\alpha$, $\forall \alpha \in V$, $-1 \in F$

** (iv) $c\alpha = \theta \Rightarrow$ either $c = 0$ or $\alpha = \theta$.

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Sub-Space:-

Th: A non-empty subset W of a vector space V over a field F is a sub-space of V iff.

(i) $\alpha \in W, \beta \in W \Rightarrow \alpha+\beta \in W$

(ii) $\alpha \in W, c \in F \Rightarrow c\alpha \in W$.

OR A non-empty subset W of a vector space V over a field F is a sub-space of V iff.

$\alpha, \beta \in W$ and $c, d \in F \Rightarrow c\alpha + d\beta \in W$.

Improper subspace:-

Let, V be a vector space over a field F. Then V itself is a subspace of V. This sub space is called improper subspace of V.

Trivial subspace:-

Let, V be a vector space over a field F. The null vector θ of V form a subspace of V. This subspace is called trivial subspace of V.

Ex:-

Examine if the set S is a subspace of $\mathbb{R}^3$

where $S = \{(x, y, z) \in \mathbb{R}^3 : x \geq 0\}$.

Soln:- Since $(0,0,0) \in S$, then S is a non empty subset of $\mathbb{R}^3$.

Let, $\alpha = (x_1, y_1, z_1) \in S$, $x_1 \geq 0$

$\beta = (x_2, y_2, z_2) \in S$, $x_2 \geq 0$.

Now, (i) $\alpha + \beta = (x_1+x_2, y_1+y_2, z_1+z_2)$

where $x_1 + x_2 \geq 0 + 0$

≥ 0.

$\therefore \alpha \in S, \beta \in S \Rightarrow \alpha+\beta \in S$. —①

(ii) Let, $c \in \mathbb{R}$

$\therefore c\alpha = c(x_1, y_1, z_1)$

$= (cx_1, cy_1, cz_1)$

where $cx_1 \geq c.0 \geq 0$.

$\therefore c \in \mathbb{R}, \alpha \in S \Rightarrow c\alpha \in S$. —②

From ① and ② it follows that S is a subspace of $\mathbb{R}^3$.

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Examine if the set S is a subspace of $\mathbb{R}^3$ where $S = \{(x, y, z) \in \mathbb{R}^3 : x = 0, y = 0\}$.

Soln: Since $(0,0,0) \in S$, then S is a non-empty subset of $\mathbb{R}^3$.

i) Let $\alpha = (x_1, y_1, z_1) \in S$, where $x_1 = 0, y_1 = 0$.

$\beta = (x_2, y_2, z_2) \in S$ where $x_2 = 0, y_2 = 0$.

Now, $\alpha + \beta = (x_1, y_1, z_1) + (x_2, y_2, z_2)$

$= (x_1 + x_2, y_1 + y_2, z_1 + z_2)$

where, $x_1 + x_2 = 0$

$y_1 + y_2 = 0$.

$\therefore \alpha \in S, \beta \in S \Rightarrow \alpha + \beta \in S$. ——(1)

ii) Let $c \in \mathbb{R}$,

$\therefore c\alpha = c(x_1, y_1, z_1)$

$= (cx_1, cy_1, cz_1)$

where $cx_1 = 0$

$cy_1 = 0$.

$\therefore c \in \mathbb{R}, \alpha \in S \Rightarrow c\alpha \in S$. ——(2)

From (1) and (2) it follows that S is a subspace of $\mathbb{R}^3$.

Ex:- Examine if the set S is a subspace of $\mathbb{R}^3$ where $S = \{(x, y, z) \in \mathbb{R}^3 : x = 1\}$.

Soln: Since $(1,0,0) \in S$, then S is a non-empty subset of $\mathbb{R}^3$.

Let $\alpha = (x_1, y_1, z_1) \in S$, $x_1 = 1$

$\beta = (x_2, y_2, z_2) \in S$, $x_2 = 1$.

Now, (i) $\alpha + \beta = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$

where $x_1 + x_2 = 2$

$\therefore \alpha \in S, \beta \in S \Rightarrow (\alpha + \beta) \notin S$. ——(1)

(ii) Let $c \in \mathbb{R}$

$\therefore c\alpha = c(x_1, y_1, z_1)$

$= (cx_1, cy_1, cz_1)$

where $c.x_1 = c.1 = c$

$\therefore c \in \mathbb{R}, \alpha \in S \Rightarrow c\alpha \notin S$ [if $c \neq 1$]. ——(2)

From (1) and (2) it follows that S is not a subspace of $\mathbb{R}^3$

1. Examine if the set S is a subspace of $\mathbb{R}^3$.

where $S = \{(x,y,z) \in \mathbb{R}^3 : x+y+z=1\}$.

soln:- Since $(1,0,0) \in S$ then S is a non empty subset of $\mathbb{R}^3$.

Let $\alpha = (x_1, y_1, z_1) \in S$, where $x_1+y_1+z_1=1$

$\beta = (x_2, y_2, z_2) \in S$, where $x_2+y_2+z_2=1$

Now $\alpha+\beta = (x_1+x_2, y_1+y_2, z_1+z_2)$

where $(x_1+x_2)+(y_1+y_2)+(z_1+z_2) = (x_1+y_1+z_1)+(x_2+y_2+z_2)$

$= 1+1$

$= 2.$

$\therefore \alpha \in S, \beta \in S \Rightarrow \alpha+\beta \notin S.$ — (1)

(ii) Let $c \in \mathbb{R}$

$\therefore c\alpha = c(x_1, y_1, z_1)$

$= (cx_1, cy_1, cz_1)$

where $cx_1+cy_1+cz_1 = c(x_1+y_1+z_1)$

$= c \cdot 1$

$= c$

$\therefore c \in \mathbb{R}, \alpha \in S \Rightarrow c\alpha \notin S$ [if $c \neq 1$]. — (2)

From (1) and (2) it follows that S is not a subspace of $\mathbb{R}^3$.

2 (vi) Examine if the set S is a sub space of $\mathbb{R}^3$.

$S = \{(x,y,z) \in \mathbb{R}^3 : x+2y-z=0, 2x-y+z=0\}$.

soln since $(0,0,0) \in S$, then S is a non-empty subset of $\mathbb{R}^3$.

Let $\alpha = (x_1, y_1, z_1) \in S$ when $x_1+2y_1-z_1=0$

$2x_1-y_1+z_1=0$.

$\beta = (x_2, y_2, z_2) \in S$ when $x_2+2y_2-z_2=0$; $2x_2-y_2+z_2=0$.

(i) $\alpha+\beta = (x_1+x_2, y_1+y_2, z_1+z_2)$

Where $x_1+x_2+2(y_1+y_2)-(z_1+z_2)$

$= x_1+2y_1-z_1+x_2+2y_2-z_2$

$= 0+0$

$= 0$

and $2(x_1+x_2)-(y_1+y_2)+(z_1+z_2)$

$= 2x_1-y_1+z_1+2x_2-y_2+z_2$

$= 0+0 = 0.$

$\therefore \alpha \in S, \beta \in S \Rightarrow \alpha+\beta \in S$ —— (1)

(ii) Let $c \in \mathbb{R}$

Now, $c\alpha = ~~c(x_1+y_1+z_1)~~$ $c(x_1, y_1, z_1)$

$= (cx_1, cy_1, cz_1)$

When

$cx_1 + 2cy_1 - cz_1$

$= c(x_1+2y_1-z_1)$

$= c.0$

$= 0$

and $2cx_1 - cy_1 + cz_1$

$= c(2x_1 - y_1 + z_1)$

$= c.0$

$= 0$

$\therefore c \in \mathbb{R}, \alpha \in S \Rightarrow c\alpha \in S$, —— (2).

From (1) and (2), it follows that S is a subspace of $\mathbb{R}^3$.

<u>Note:</u>

Every subspace W of a vector space V must contain the null vector θ of V.

** <u>Theorem:</u> Prove that the intersection of two subspace of a vector space V over a field F is a subspace of V.

<u>Proof:</u>

Let, W_1 and W_2 be two subspace of V

Then, $\theta \in W_1$ and $\theta \in W_2$

$\therefore \theta \in W_1 \cap W_2$

Hence $W_1 \cap W_2$ is non-empty subset of V.

<u>Case-I:</u> When $W_1 \cap W_2 = \{\theta\}$, then $W_1 \cap W_2$ is a subspace of V.

<u>Case-II:</u> When $W_1 \cap W_2 \neq \{\theta\}$.

Let. $\alpha_1, \alpha_2 \in W_1 \cap W_2$

$\therefore \alpha_1, \alpha_2 \in W_1$ and $\alpha_1, \alpha_2 \in W_2$

Since, W_1 is a subspace of V (i) $\alpha_1 \in W_1, \alpha_2 \in W_1 \Rightarrow \alpha_1 + \alpha_2 \in W_1$

(ii) $c \in F, \alpha_1 \in W_1 \Rightarrow c\alpha_1 \in W_1$

again, since W_2 is a subspace of V

(iii) $\alpha_1 \in W_2, \alpha_2 \in W_2 \Rightarrow \alpha_1+\alpha_2 \in W_2$

(iv) $c \in F, \alpha_1 \in W_2 \Rightarrow c\alpha_1 \in W_2$,

From (i) and (iii) we have $\alpha_1 \in W_1 \cap W_2, \alpha_2 \in W_1 \cap W_2 \Rightarrow (\alpha_1+\alpha_2) \in W_1 \cap W_2$,

and from (iii) and (iv), we have
$c \in F, \alpha_1 \in W_1 \cap W_2 \Rightarrow c\alpha_1 \in W_1 \cap W_2$.
Hence $W_1 \cap W_2$ is a subspace of V.

Note:-
But union of two subspace may or may not be a subspace of V over the field F.

Ex:-
Let, $S_1 = \{(x,y,z) \in \mathbb{R}^3 : x=0, y=0\}$
$S_2 = \{(x,y,z) \in \mathbb{R}^3 : x=0, z=0\}$ be two subspace of V.

Then $\alpha = (0,0,1) \in S_1$ and $\beta = (0,1,0) \in S_2$. $\alpha \in S_1 \cup S_2$, $\beta \in S_1 \cup S_2$

$\alpha + \beta = (0,0,1) + (0,1,0)$
$= (0,1,1) \notin S_1$ and $\notin S_2$.
$\therefore \alpha \in S_1 \cup S_2, \beta \in S_1 \cup S_2 \Rightarrow \alpha + \beta \notin S_1 \cup S_2$
$\therefore \alpha + \beta \notin S_1 \cup S_2$
Hence $S_1 \cup S_2$ is not a subspace of V over the field F.

Ex:- Let V be a vector space over the field F and let $\alpha, \beta \in V$ then the set $W = \{c\alpha + d\beta : c, d \in F\}$ forms a sub-space of V.

Soln:- Since $\theta = (0\alpha + 0\cdot\beta) \in W$ then W is a non-empty subset of V.

Let, $\alpha_1 = c_1\alpha + d_1\beta \in W$ where $c_1, d_1 \in F$
$\alpha_2 = c_2\alpha + d_2\beta \in W$ where $c_2, d_2 \in F$

Now, $\alpha_1 + \alpha_2 = c_1\alpha + d_1\beta + c_2\alpha + d_2\beta$
$= (c_1 + c_2)\alpha + (d_1 + d_2)\beta \in W$.
where $c_1 + c_2 \in F$
$d_1 + d_2 \in F$

$\therefore \alpha_1 \in W, \alpha_2 \in W \Rightarrow \alpha_1 + \alpha_2 \in W$. ⟵ (i)

(ii) Let, $c \in F$
$\therefore c\alpha_1 = c(c_1\alpha + d_1\beta)$
$= ((cc_1)\alpha + (cd_1)\beta) \in W$
and $cc_1 \in F$
$cd_1 \in F$

$\therefore c \in F, \alpha_1 \in W \Rightarrow c\alpha_1 \in W$. ⟵ (ii)

From (i) and (ii) it follows that W is a subspace of V over a field of R

Note:-

This subspace is said to be generated by the vectors α and β. then the set $\{\alpha, \beta\}$ is called generating set or spanning set of the subspace W.

Linear combination:-

let, V be a vector space over the field F

let $\alpha_1, \alpha_2 \dots \alpha_r \in V$. A vector $\beta \in V$ is said to be a linear combination of the vectors $\alpha_1, \alpha_2 \dots \alpha_r$ if β can be expressed as $\beta = c_1\alpha_1 + c_2\alpha_2 + \dots + c_r\alpha_r$ for some scalers $c_1, c_2 \dots c_r \in F$

Ex:- If $\alpha = (1,1,2)$, $\beta = (0,2,1)$; $\gamma = (2,2,4)$

Determine whether

(i) α is a linear combination of β and γ.

(ii) β " " " " of α and γ.

(iii) γ " " " " " α and β.

Sol:- (i)

let, $\alpha = c\beta + d\gamma$, where $c, d \in F$.

or $(1,1,2) = c(0,2,1) + d(2,2,4)$

or $(1,1,2) = (2d, 2c+2d, c+4d)$

$\therefore 2d = 1$, $2c+2d = 1$, $c+4d = 2$

or $d = \frac{1}{2}$ or $c = 0$,

$\therefore$ we write $\alpha = 0.\beta + \frac{1}{2}.\gamma$

Hence, α can be expressed as a linear combination of β and γ.

(ii) let $\beta = c\alpha + d\gamma$, where $c, d \in F$.

or $(0,2,1) = c(1,1,2) + d(2,2,4)$

or $(0,2,1) = (c+2d, c+2d, 2c+4d)$

$\therefore c+2d = 0$, $c+2d = 2$, $2c+4d = 1$

or $c+2d = 1/2$

The above equations are inconsistent.

Hence, β can not be expressed as a linear combination of α and γ.

(iii) $\gamma = c\alpha + d\beta$.

or $(2,2,4) = c(1,1,2) + d(0,2,1)$

$= (c, c+2d, 2c+d)$ $\quad \therefore \gamma = 2\alpha + 0.\beta$

$\therefore c = 2$, $c+2d = 2$, $2c+d = 4$ $\quad \therefore$ Hence, γ can be expressed as a linear combination of α and β.

or $d = 0$.

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Linear span of a set :-

Let, S be a non-empty subset of a vector space V over a field F. The set of all finite linear combinations of the elements of S is said to be the linear span of S. And it is denoted by L(S).

Where S is known as spanning set or generating set of L(S)

$S = \{\alpha, \beta, \gamma\}$.

$L(S) = \{\alpha + a\beta + 0.\gamma ; 0.\alpha + a\beta + 0.\gamma, 0.\alpha + 0.\beta + 0.\gamma \ldots\}$

$= \{c\alpha + d\beta + e\gamma ; c, d, e \in \mathbb{R}\}$

L(S) also a subspace of V over the field F and it is the smallest subspace of V containing the set S.

~~$L(S) = \{c\alpha + d\beta + e\gamma : c, d, e \in F\}$~~.

Theorem: If S and T be two non-empty finite subset of a vector space V, over the field F and $S \subset T$

Then $L(S) \subset L(T)$.

Th: If S and T be two non-empty finite subsets of a vector space V over the field F and each element of T is a linear combination of the vectors of S, then $L(T) \subset L(S)$.

Ex: Let $S = \{\alpha, \beta, \gamma\}$ and $T = \{\alpha, \beta, \alpha+\beta, \beta+\gamma\}$ be two subspace of a real vector space V, Show that $L(S) = L(T)$.

Soln Here, S and T both are non-empty subset of V.

We have, $\alpha = \alpha + 0.\beta + 0.(\alpha+\beta) + 0.(\beta+\gamma)$.

$\beta = 0.\alpha + 1.\beta + 0.(\alpha+\beta) + 0.(\beta+\gamma)$.

$\gamma = 0.\alpha + (-1).\beta + 0.(\alpha+\beta) + 1.(\beta+\gamma)$.

$\therefore$ Each element of S is a linear combination of the vectors of T

Then $L(S) \subset L(T)$. ——— (1)

again we have $\alpha = 1.\alpha + 0.\beta + 0.\gamma$.

$\beta = 0.\alpha + 1.\beta + 0.\gamma$,

$\alpha + \beta = 1.\alpha + 1.\beta + 0.\gamma$.

$\beta + \gamma = 0.\alpha + 1.\beta + 1.\gamma$.

Each element of T is a linear combination of the vector of S.

$\therefore L(T) \subset L(S)$ ——— (2)

From ① and ②, $L(S)=L(T)$.

Q.5.

Let $S=\{\alpha_1,\alpha_2\}$, $T=\{\alpha_2,\alpha_3\}$, $W=\{\alpha_3,\alpha_1\}$.

Given that, $\alpha_1+\alpha_2+\alpha_3=\theta$. —— ①

We have to prove $L(S)=L(T)=L(W)$.

To prove $L(S)=L(T)$

$\alpha_1=\theta-\alpha_2-\alpha_3$

$=-\alpha_2-\alpha_3$.

$=(-1).\alpha_2+(-1).\alpha_3$,

$\alpha_2=$ 1.$\alpha_2+0.\alpha_3$

$\therefore$ Each element of S is a linear combination of the elements of T.

$\therefore L(S)\subset L(T)$, —— ②.

again,

$\alpha_2=0.\alpha_1+1.\alpha_2$

$\alpha_3=\theta-\alpha_1-\alpha_2$

$=-\alpha_1-\alpha_2$

$=(-1).\alpha_1+(-1).\alpha_2$.

$\therefore$ Each element of T is a linear combination of the elements of S.

$\therefore L(T)\subset L(S)$ —— ③

From ② and ③ $L(T)=L(S)$. —— ④

To prove $L(T)=L(W)$.

$\alpha_2=\theta-\alpha_1-\alpha_3$

$=-\alpha_1-\alpha_3$

$=(-1).\alpha_1+(-1).\alpha_3$.

$\alpha_3=1.\alpha_3+0.\alpha_1$.

$\therefore$ Each element of T is a linear combination of the elements of W.

$\therefore L(T)\subset L(W)$ —— ⑤

again $\alpha_3=0\alpha_2+1.\alpha_3$,

and $\alpha_1=\theta-\alpha_2-\alpha_3$

$=(-1).\alpha_2+(-1).\alpha_3$

$\therefore$ Each element of W is a linear combination of the elements of T.

$\therefore L(W) \subset L(T)$ — (6).

From (5) and (6), $L(W) = L(T)$ — (7)

From (4) and (7), $L(S) = L(T) = L(W)$.

$\therefore L\{\alpha_1, \alpha_2\} = L\{\alpha_2, \alpha_3\} = L\{\alpha_3, \alpha_1\}$.

(4) OR Determine the subspace of $\mathbb{R}^3$ spanned by the vectors $\alpha = (1,3,0), \beta = (2,1,-2)$.

Examine if $\gamma = (-1,3,2), \delta = (4,7,-2)$ are in ~~$L\{\alpha,\beta\}$~~ this subspace.

Soln:

The subspace of $\mathbb{R}^3$ spanned by the vectors α and β is

$L\{\alpha,\beta\} = \{c\alpha + d\beta : c,d \in \mathbb{R}\}$.

where $c\alpha + d\beta$

$= c(1,3,0) + d(2,1,-2)$

$= (c+2d, 3c+d, -2d)$.

2nd part:

If $\gamma \in L\{\alpha,\beta\}$, we must have real numbers c and d such that $\gamma = c\alpha + d\beta$

$\therefore (-1,3,2) = (c+2d, 3c+d, -2d)$

$\therefore -2d = 2, \quad c+2d = -1, \quad 3c+d = 3$.

$\therefore d = -1 \quad \therefore c = 1$.

The above equ^ns are inconsistent

Hence $\gamma \notin L\{\alpha,\beta\}$.

If $\delta \in L\{\alpha,\beta\}$ we must have real number c and d such that

$\delta = c\alpha + d\beta$

$\therefore (4,7,-2) = (c+2d, 3c+d, -2d)$

$\therefore c+2d = 4, \quad 3c+d = 7, \quad -2d = -2$

$\therefore d = 1$

$\therefore c = 2$,

$\therefore \delta = 2.\alpha + 1.\beta$ Hence $\delta \in L\{\alpha,\beta\}$.

10.(iii)

$S = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}_{2\times2} : \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 0 \right\}$.

Sol^n:-

Since $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in S$. then S is a non-empty subset of $\mathbb{R}_{2\times2}$

Let $\alpha = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \in S$; $\det \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} = 0$. i.e. $a_1d_1 - b_1c_1 = 0$

$\beta = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \in S$; $\det \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = 0$ i.e. $a_2d_2 - b_2c_2 = 0$

i) $\alpha + \beta = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$

$= \begin{pmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{pmatrix}$

Hence ~~det(α+β)~~), $\det \begin{pmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{pmatrix}$

$= (a_1+a_2).(d_1+d_2) - (c_1+c_2)(b_1+b_2)$

$= a_1d_1 - b_1c_1 + a_2d_2 - b_2c_2$
$+ a_2d_1 + a_1d_2 - c_1b_2 - c_2b_1$

$\neq 0$

$\therefore \alpha \in S, \beta \in S \Rightarrow \alpha+\beta \notin S$.

(ii).

2.(ii)

Examine if the sets S is a subspace of $\mathbb{R}^3$.

where $S = \{(x,y,z) \in \mathbb{R}^3 : xy = z\}$.

Soln:- Since $(0,0,0) \in S$, then S is a non-empty sub set of $\mathbb{R}^3$.

let, $\alpha = (x_1, y_1, z_1) \in S$; where $x_1 y_1 = z_1$

and $\beta = (x_2, y_2, z_2) \in S$; where $x_2 y_2 = z_2$.

Now, (i) $\alpha + \beta = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$

we have, $(x_1 + x_2)(y_1 + y_2)$

$= x_1 y_1 + x_2 y_2 + x_2 y_1 + x_1 y_2$

$= z_1 + z_2 + x_2 y_1 + x_1 y_2$

$\therefore \alpha \in S, \beta \in S \Rightarrow \alpha + \beta \notin S$

(ii) let, $c \in \mathbb{R}$.

$\therefore c\alpha = c(x_1, y_1, z_1)$

$= (cx_1, cy_1, cz_1)$

we have $cx_1 \cdot cy_1$

$= c(x_1 y_1)$

$= cz_1$

$\therefore c \in \mathbb{R}, \alpha \in S \Rightarrow c\alpha \in S$,

Hence S is not a subspace of $\mathbb{R}^3$.

(vii) Examine if the sets S is a subspace of $\mathbb{R}^3$

where $S = \{(x,y,z) \in \mathbb{R}^3 : x + 2y - z = 1, 2x - y + z = 2\}$.

Soln:- Since $(1,0,0) \in S$, then S is a non-empty subset of $\mathbb{R}^3$

let $\alpha = (x_1, y_1, z_1) \in S$, where $x_1 + 2y_1 - z_1 = 1$, $2x_1 - y_1 + z_1 = 2$.

and $\beta = (x_2, y_2, z_2) \in S$; where $x_2 + 2y_2 - z_2 = 1$, $2x_2 - y_2 + z_2 = 2$.

Now $\alpha + \beta$

$2c_1 + 2c_2 + c_3 = 0$

$3c_1 + c_2 + c_3 = 0$