ADVANCED HIGHER ALGEBRA

1 COMPLEX NUMBERS

1.1. Definitions.

The system of real numbers is not sufficient to solve all the algebraic equations. There are no real numbers which satisfy the equation $x^2+1=0$ or $x^2=-1$. In order to solve such equations, we extend the system of numbers and introduce a new class of numbers known as *imaginary* or *complex numbers*.

To cope with this idea, we introduce the symbol i to denote $\sqrt{-1}$ so that $i^2=-1$.

This i is not a real number. It is called the *fundamental imaginary unit*. We suppose that it combines with itself and with real numbers.

Any expression of the form $(a+ib)$, where a and b are both real, is called a *complex number*. Here the sign '+' does not indicate addition as usually understood. It is a mere symbol.

The real part of the complex number $(a+ib)$ is a and is written as $Re(a+ib)$, while its imaginary part is b and is written as $Imag\,(a+ib)$.

If $a=0$, then the complex number $(a+ib)$ becomes ib, which is purely imaginary. If $b=0$, then the complex number $(a+ib)$ becomes purely real. If both $a=0$ and $b=0$, then the complex number becomes zero. Hence the real numbers are particular cases of complex numbers.

If the real parts of two complex numbers be the same and their imaginary parts be same but of opposite signs, then the two numbers are said to be *complex conjugate numbers*. Thus $(a+ib)$ and $(a-ib)$ are complex conjugate numbers. If z be a complex number $(a+ib)$, then its conjugate $(a-ib)$ is denoted by $\bar{z}$. It is evident that conjugate of $\bar{z}$ is z.

Note. A complex number $z = (a, b)$ may also be defined as an ordered pair of real numbers a and b such that

(i) $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$

(ii) $(a, b) + (c, d) = (a + c, b + d)$

(iii) $(a, b) . (c, d) = (ac - bd, ad + bc)$,

where (c, d) is another complex number.

The first component a of the complex number (a, b) is its real part, while the second component b is its imaginary part.

Thus the complex number $(a, 0)$ is purely real, $(0, b)$ is purely imaginary and $(0, 1)$ is the fundamental imaginary unit.

It is observed that the two conventions $a + ib$ and (a, b) are equivalent, if we simply replace i^2 by (-1) where necessary.

1.2. Properties of complex numbers.

(a) *If $a + ib = 0$, a and b being real, then $a = 0$ and $b = 0$.*

We have $a + ib = 0$

or, $a = -ib$

or, $a^2 = i^2 b^2 = -b^2$, whence $a^2 + b^2 = 0$.

Therefore $a = 0$ and $b = 0$, since a and b are real.

(b) *If $a + ib = c + id$, then $a = c$ and $b = d$.*

We have $a + ib = c + id$

or, $(a - c) + i(b - d) = 0$.

Hence, by the above property, we have

$$a - c = 0 \text{ and } b - d = 0, \text{ whence } a = c \text{ and } b = d.$$

(c) *The algebraic sum, difference, product or ratio of two complex numbers is a complex number.*

If $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$, then

$z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2)$, which is a complex number.

$z_1 - z_2 = (a_1 - a_2) + i(b_1 - b_2)$, which is a complex number.

$z_1 z_2 = (a_1a_2 - b_1b_2) + i(a_1b_2 + a_2b_1)$, which is a complex number.

$$\frac{z_1}{z_2} = \frac{a_1 + ib_1}{a_2 + ib_2} = \frac{(a_1 + ib_1)(a_2 - ib_2)}{(a_2 + ib_2)(a_2 - ib_2)}$$

$$= \frac{a_1a_2 + b_1b_2}{a_2^2 + b_2^2} + i\,\frac{a_2b_1 - a_1b_2}{a_2^2 + b_2^2},$$ which is a complex number.

(*d*) *The sum and the product of two conjugate complex numbers are both real.*

Let the two conjugate complex numbers be $(a+ib)$ and $(a-ib)$.

Then $(a+ib)+(a-ib)=2a$, which is real

and $(a+ib)(a-ib)=a^2-i^2b^2=a^2+b^2$, which is real.

Note. These properties follow directly from the definition of complex numbers as an ordered pair of real numbers.

As in the case of real numbers, the addition of complex numbers is commutative and associative and the multiplication of complex numbers is commutative, associative and distributive. If the product of two complex numbers be zero, then at least one of them is zero.

Ordinary laws of algebra on real numbers hold good for complex numbers also.

1.3. Geometrical representation of a complex number : Argand diagram.

Let $z=x+iy$ be a complex number. Let the two mutually perpendicular straight lines XOX' and YOY' be taken as x-axis and y-axis respectively, O being the origin. Let P be the point whose cartesian co-ordinates are (x, y).

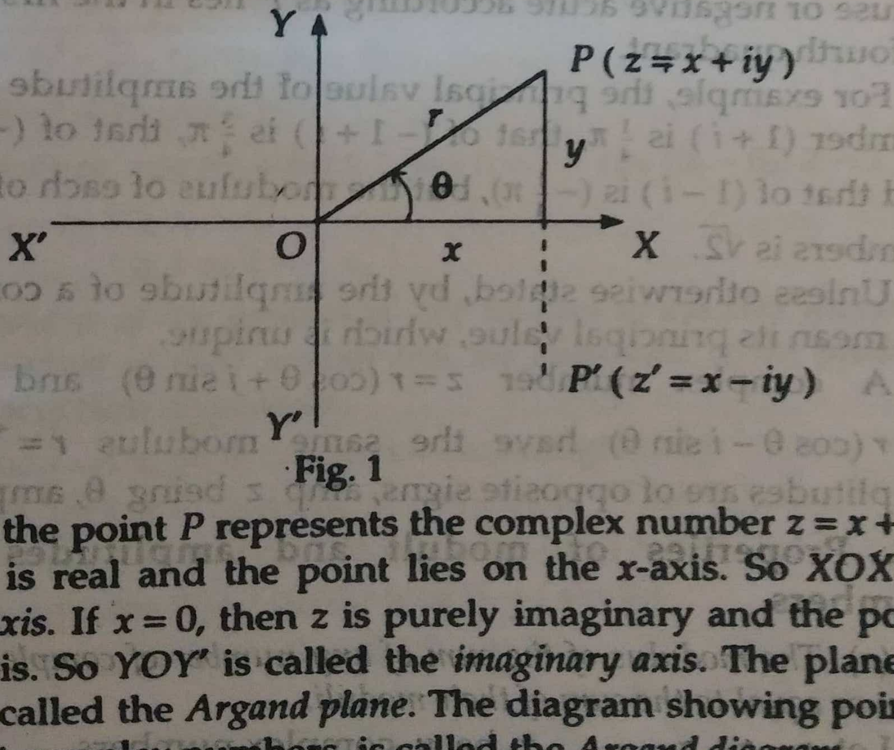

Fig. 1

We say that the point P represents the complex number $z=x+iy$. If $y=0$, then z is real and the point lies on the x-axis. So XOX' is called the *real axis*. If $x=0$, then z is purely imaginary and the point lies on the y-axis. So YOY' is called the *imaginary axis*. The plane of the two axes is called the *Argand plane*. The diagram showing points, which represent complex numbers, is called the *Argand diagram*.

The point P' whose cartesian co-ordinates are $(x, -y)$ represents the complex number $\bar{z}=x-iy$, which is the conjugate of $(x+iy)$.

1.4. Modulus and amplitude.

Let (r, θ) be the polar co-ordinates of P whose cartesian co-ordinates are (x, y) in Fig. 1.

Then $x = r\cos\theta$, $y = r\sin\theta$ and hence $r = \sqrt{x^2+y^2}$ and $\theta = \tan^{-1}\frac{y}{x}$.

Therefore $z = x + iy = r(\cos\theta + i\sin\theta)$.

This is called the *polar form* of the complex number.

Here r, the length of the line segment OP, is called the *modulus* or *magnitude* of z and is denoted by $|z|$ or by *mod* z, and is essentially positive.

Thus $|z| = |x + iy| = r = +\sqrt{x^2+y^2}$.

The angle θ, which the straight line OP makes with the x-axis, is called the *amplitude* or *argument* of z and is denoted by *amp* z. It has many values differing by multiples of 2π.

It is to be noted that the amplitude θ is to be determined from the two relations $x = r\cos\theta$ and $y = r\sin\theta$ simultaneously and not from the single relation $\theta = \tan^{-1}\frac{y}{x}$ only.

For the *principal value of* θ, we have $-\pi < \theta \leq \pi$.

The principal value of the amplitude of a complex number represented by the point P is positive acute, positive obtuse, negative obtuse or negative acute according as P lies in the first, second, third or fourth quadrant.

For example, the principal value of the amplitude of the complex number $(1 + i)$ is $\frac{1}{4}\pi$, that of $(-1 + i)$ is $\frac{3}{4}\pi$, that of $(-1 - i)$ is $(-\frac{3}{4}\pi)$ and that of $(1 - i)$ is $(-\frac{1}{4}\pi)$, but the modulus of each of these complex numbers is $\sqrt{2}$.

Unless otherwise stated, by the amplitude of a complex number, we mean its principal value, which is unique.

A complex number $z = r(\cos\theta + i\sin\theta)$ and its conjugate $\bar{z} = r(\cos\theta - i\sin\theta)$ have the same modulus $r = \sqrt{z\bar{z}}$ but their amplitudes are of opposite signs, amp z being θ, amp $\bar{z}$ is $(-\theta)$.

1.5. Properties of moduli and amplitudes of complex numbers.

(*a*) *The modulus of the sum of any number of complex numbers is less than or equal to the sum of their moduli.*

Let $z_1, z_2, z_3, \ldots\ldots, z_n$ be n complex numbers.

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, so that $|z_1| = r_1$ and $|z_2| = r_2$.

Now $| z_1 + z_2 |$

$$= | (r_1 \cos \theta_1 + r_2 \cos \theta_2) + i (r_1 \sin \theta_1 + r_2 \sin \theta_2) |$$
$$= \sqrt{(r_1 \cos \theta_1 + r_2 \cos \theta_2)^2 + (r_1 \sin \theta_1 + r_2 \sin \theta_2)^2}$$
$$= \sqrt{r_1^2 + r_2^2 + 2 r_1 r_2 \cos (\theta_1 - \theta_2)}$$
$$\leq \sqrt{r_1^2 + r_2^2 + 2 r_1 r_2 . 1}, \quad \text{since } \cos (\theta_1 - \theta_2) \leq 1.$$

Hence $| z_1 + z_2 | \leq r_1 + r_2$

or, $\quad | z_1 + z_2 | \leq | z_1 | + | z_2 |$.

Similarly, $| z_1 + z_2 + z_3 | = | z_1 + (z_2 + z_3) | \leq | z_1 | + | z_2 + z_3 |$

or, $| z_1 + z_2 + z_3 | \leq | z_1 | + | z_2 | + | z_3 |$.

Proceeding in this way, we can prove that

$$| z_1 + z_2 + z_3 + \cdots\cdots + z_n |$$
$$\leq | z_1 | + | z_2 | + | z_3 | + \cdots\cdots + | z_n |.$$

(b) *The modulus of the difference of two complex numbers is greater than or equal to the difference of their moduli.*

Let z_1 and z_2 be two complex numbers.

Let $z_1 = r_1 (\cos \theta_1 + i \sin \theta_1)$ and $z_2 = r_2 (\cos \theta_2 + i \sin \theta_2)$, so that $| z_1 | = r_1$ and $| z_2 | = r_2$.

Now $| z_1 - z_2 |$

$$= | (r_1 \cos \theta_1 - r_2 \cos \theta_2) + i (r_1 \sin \theta_1 - r_2 \sin \theta_2) |$$
$$= \sqrt{(r_1 \cos \theta_1 - r_2 \cos \theta_2)^2 + (r_1 \sin \theta_1 - r_2 \sin \theta_2)^2}$$
$$= \sqrt{r_1^2 + r_2^2 - 2 r_1 r_2 \cos (\theta_1 - \theta_2)}$$
$$\geq \sqrt{r_1^2 + r_2^2 + 2 r_1 r_2 (-1)}$$

[since $\cos (\theta_1 - \theta_2) \leq 1$, we have $- \cos (\theta_1 - \theta_2) \geq -1$].

Hence $| z_1 - z_2 | \geq r_1 \sim r_2$

$$\geq | z_1 | \sim | z_2 |.$$

(c) *The modulus of the product of any number of complex numbers is equal to the product of their moduli and the amplitude of the product is equal to the sum of their amplitudes.*

Let $z_1, z_2, z_3, \ldots\ldots, z_n$ be n complex numbers given by

$$z_1 = r_1 (\cos \theta_1 + i \sin \theta_1), \; z_2 = r_2 (\cos \theta_2 + i \sin \theta_2), \ldots\ldots,$$

so that $| z_1 | = r_1, \; | z_2 | = r_2$, amp $z_1 = \theta_1$, amp $z_2 = \theta_2, \ldots\ldots$

Now $z_1 z_2 = r_1 r_2 (\cos \theta_1 + i \sin \theta_1) (\cos \theta_2 + i \sin \theta_2)$

$$= r_1 r_2 \{(\cos\theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + i (\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2)\}$$
$$= r_1 r_2 \{\cos (\theta_1 + \theta_2) + i \sin (\theta_1 + \theta_2)\}.$$

Hence $|z_1 z_2| = r_1 r_2 = |z_1| \cdot |z_2|$

and $\quad \text{amp}\,(z_1 z_2) = \theta_1 + \theta_2 = \text{amp}\, z_1 + \text{amp}\, z_2$.

Similarly,

$$|z_1 z_2 z_3| = |z_1 (z_2 z_3)| = |z_1| \, |z_2 z_3| = |z_1| \, |z_2| \, |z_3|$$

and $\quad \text{amp}\,(z_1 z_2 z_3) = \text{amp}\,\{z_1 (z_2 z_3)\} = \text{amp}\, z_1 + \text{amp}\,(z_2 z_3)$

$$= \text{amp}\, z_1 + \text{amp}\, z_2 + \text{amp}\, z_3.$$

Proceeding in this way, we can prove that

$$|z_1 z_2 z_3 \ldots\ldots z_n| = |z_1| \, |z_2| \, |z_3| \ldots\ldots |z_n|$$

and $\quad \text{amp}\,(z_1 z_2 z_3 \ldots\ldots z_n) = \text{amp}\, z_1 + \text{amp}\, z_2 + \text{amp}\, z_3 + \ldots\ldots + \text{amp}\, z_n$.

Note. If the principal values of the amplitudes be considered, then $\text{amp}\,(z_1 z_2) \neq \text{amp}\, z_1 + \text{amp}\, z_2$, in general. For example, if $z_1 = i$ and $z_2 = -1 + i$, then $\text{amp}\,(z_1 z_2) = \text{amp}\,(-1 - i) = -\frac{3}{4}\pi$, $\text{amp}\, z_1 = \frac{1}{2}\pi$ and $\text{amp}\, z_2 = \frac{3}{4}\pi$ and therefore $\text{amp}\,(z_1 z_2) \neq \text{amp}\, z_1 + \text{amp}\, z_2$.

(d) *The modulus of the quotient of two complex numbers is equal to the quotient of their moduli and the amplitude of the quotient is equal to the difference of their amplitudes.*

Let z_1 and z_2 be two complex numbers.

Let $z_1 = r_1 (\cos \theta_1 + i \sin \theta_1)$ and $z_2 = r_2 (\cos \theta_2 + i \sin \theta_2)$, so that $|z_1| = r_1$, $|z_2| = r_2$, $\text{amp}\, z_1 = \theta_1$ and $\text{amp}\, z_2 = \theta_2$.

Now $\dfrac{z_1}{z_2} = \dfrac{r_1 (\cos \theta_1 + i \sin \theta_1)}{r_2 (\cos \theta_2 + i \sin \theta_2)}$

$$= \frac{r_1 (\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 - i \sin \theta_2)}{r_2 (\cos \theta_2 + i \sin \theta_2)(\cos \theta_2 - i \sin \theta_2)}$$

$$= \frac{r_1}{r_2} \cdot \frac{(\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2) + i (\sin \theta_1 \cos \theta_2 - \cos \theta_1 \sin \theta_2)}{\cos^2 \theta_2 + \sin^2 \theta_2}$$

$$= \frac{r_1}{r_2} \cdot \{\cos (\theta_1 - \theta_2) + i \sin (\theta_1 - \theta_2)\}.$$

Hence $\left|\dfrac{z_1}{z_2}\right| = \dfrac{r_1}{r_2} = \dfrac{|z_1|}{|z_2|}$

and $\quad \text{amp}\left(\dfrac{z_1}{z_2}\right) = \theta_1 - \theta_2 = \text{amp}\, z_1 - \text{amp}\, z_2$.

Note. If the principal values of the amplitudes be considered, then $\text{amp}\left(\dfrac{z_1}{z_2}\right) \neq \text{amp}\, z_1 - \text{amp}\, z_2$, in general. For example, if $z_1 = 1 - i$ and $z_2 = -1 + i$, then $\text{amp}\left(\dfrac{z_1}{z_2}\right) = \text{amp}\,(-1) = \pi$, $\text{amp}\, z_1 = -\frac{1}{4}\pi$ and $\text{amp}\, z_2 = \frac{3}{4}\pi$ and therefore $\text{amp}\left(\dfrac{z_1}{z_2}\right) \neq \text{amp}\, z_1 - \text{amp}\, z_2$.

1.6. Geometrical representation of sum, difference, product and quotient of two complex numbers.

(*a*) *Sum of two complex numbers.*

Let the points P and Q represent the two complex numbers $z_1 = x_1 + i y_1$ and $z_2 = x_2 + i y_2$ respectively in the Argand diagram [Fig. 2]. Thus the cartesian co-ordinates of P and Q with respect to the set of rectangular axes XOX' and YOY' are (x_1, y_1) and (x_2, y_2) respectively.

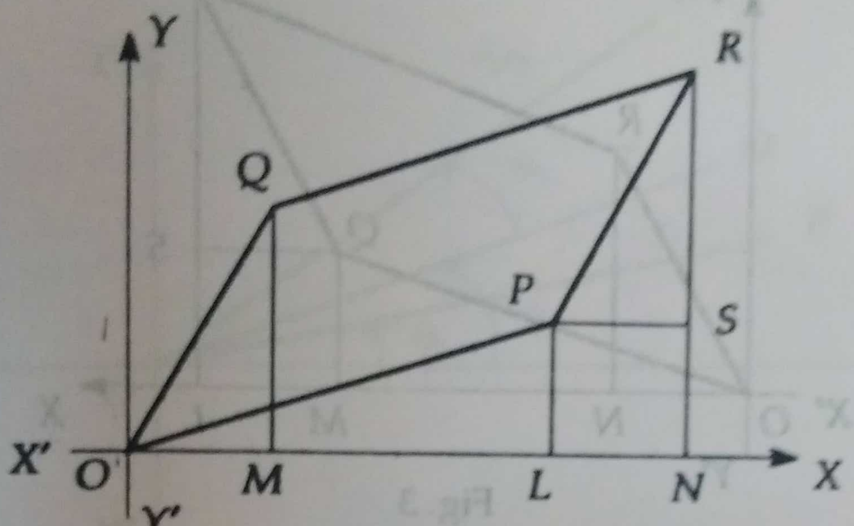

Fig. 2

The parallelogram $OPRQ$ is completed. PL, QM and RN are drawn perpendiculars to the x-axis and PS is drawn perpendicular to RN. Then $OL = x_1$, $PL = y_1$, $OM = x_2$, $QM = y_2$. Since OQ and PR are equal and parallel, their projections on the x-axis will also be equal. Hence $OM = LN$. Similarly, $QM = RS$.

Therefore $ON = OL + LN = OL + OM = x_1 + x_2$

and $RN = RS + SN = QM + PL = y_1 + y_2$.

Hence the point R represents the complex number $\{(x_1 + x_2) + i(y_1 + y_2)\}$ which is the sum of the two complex numbers $(x_1 + iy_1)$ and $(x_2 + iy_2)$, represented respectively by the points P and Q.

It is evident from the figure that $\angle XOR = \text{amp}(z_1 + z_2)$.

Also $|z_1| = OP$, $|z_2| = OQ = PR$ and $|z_1 + z_2| = OR$.

Since in the triangle OPR, $OR \leq OP + PR$, we have

$$|z_1 + z_2| = OR \leq OP + PR$$

that is, $\leq |z_1| + |z_2|$.

This result can be generalised and we can get the result of Article 1.5 (*a*).

(b) *Difference of two complex numbers.*

Let the points P and Q represent the two complex numbers $z_1 = x_1 + i\,y_1$ and $z_2 = x_2 + i\,y_2$ respectively in the Argand diagram [Fig. 3]. Thus the cartesian co-ordinates of P and Q with respect to the set of rectangular axes XOX' and YOY' are (x_1, y_1) and (x_2, y_2) respectively.

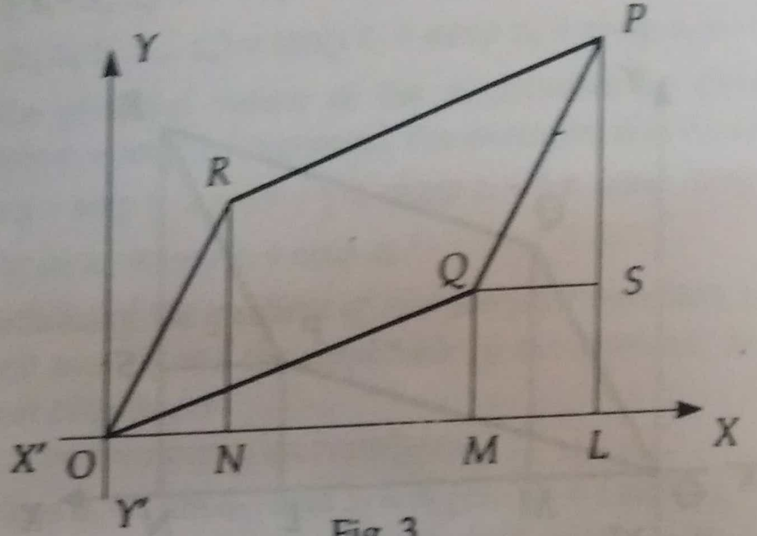

Fig. 3

The parallelogram $OQPR$ is completed. PL, QM and RN are drawn perpendiculars to the x-axis and QS is drawn perpendicular to PL.

Then $OL = x_1$, $PL = y_1$, $OM = x_2$, $QM = y_2$.

Since OR and QP are equal and parallel, their projections on the x-axis are also equal. Hence $ON = ML$. Similarly, $RN = PS$.

Therefore $ON = ML = OL - OM = x_1 - x_2$

and $RN = PS = PL - SL = PL - QM = y_1 - y_2$.

Hence the point R represents the complex number $[(x_1 - x_2) + i\,(y_1 - y_2)]$ which is the difference of the two complex numbers $(x_1 + iy_1)$ and $(x_2 + iy_2)$, represented respectively by the points P and Q.

It is evident from the figure that $|\,z_1 - z_2\,|$ = the length OR and amp $(z_1 - z_2) = \angle XOR$.

(c) *Product of two complex numbers.*

Let the points P and Q represent the two complex numbers $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ respectively in the Argand diagram [Fig. 4]. Thus the polar co-ordinates of P and Q with respect to the pole O and the initial line OX are (r_1, θ_1) and (r_2, θ_2) respectively. Hence $OP = r_1$, $OQ = r_2$, $\angle XOP = \theta_1$ and $\angle XOQ = \theta_2$.

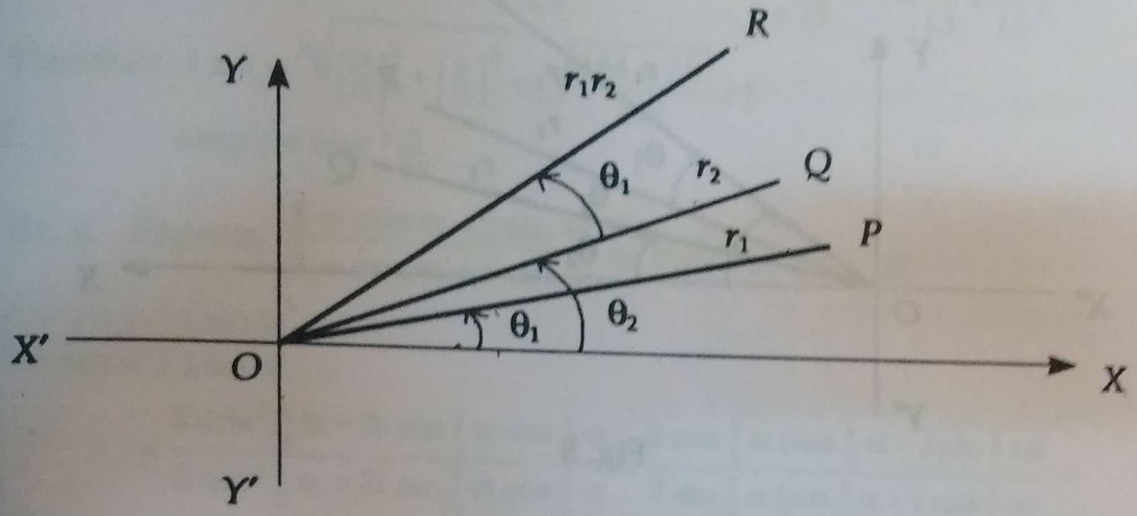

Fig. 4

Let R be a point such that $OR = r_1 r_2$ and $\angle XOR = \theta_1 + \theta_2$. Hence the point R represents the complex number $r_1 r_2 \{\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\}$ which is the product of the two complex numbers $r_1(\cos\theta_1 + i\sin\theta_1)$ and $r_2(\cos\theta_2 + i\sin\theta_2)$, represented respectively by the points P and Q.

From the figure, the results of Article 1.5 (c) can be easily obtained.

(d) *Quotient of two complex numbers.*

Let the points P and Q represent the two complex numbers $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ respectively in the Argand diagram [Fig. 5]. Thus the polar co-ordinates of P and Q with respect to the pole O and the initial line OX are (r_1, θ_1) and (r_2, θ_2) respectively. Hence $OP = r_1$, $OQ = r_2$, $\angle XOP = \theta_1$ and $\angle XOQ = \theta_2$.

Let R be a point such that $OR = \frac{r_1}{r_2}$ and $\angle XOR = \theta_1 - \theta_2$. Hence the point R represents the complex number $\frac{r_1}{r_2}\{\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)\}$ which is the quotient of the two complex numbers $r_1(\cos\theta_1 + i \sin\theta_1)$ and $r_2(\cos\theta_2 + i\sin\theta_2)$, represented respectively by the points P and Q.

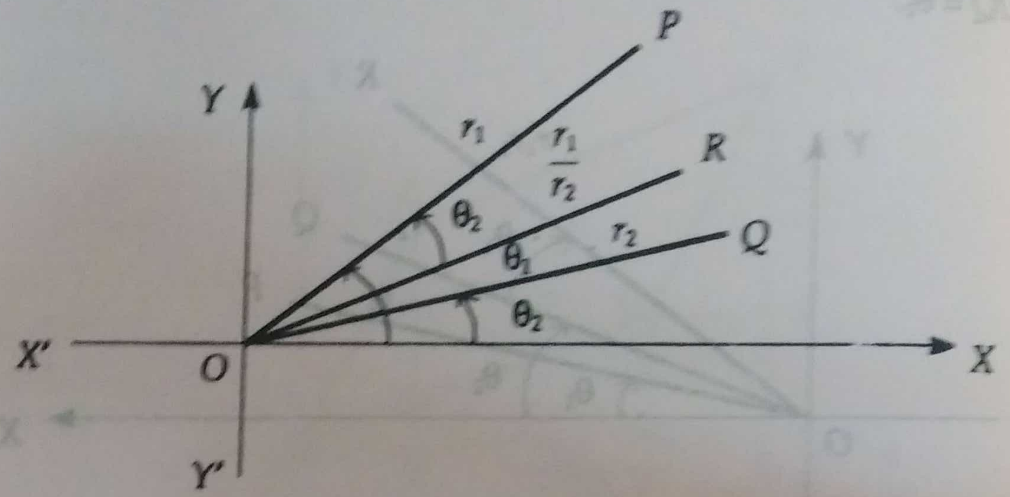

Fig. 5

From the figure, the results of Article 1.5 (*d*) can be easily obtained.

1.7. Imaginary cube roots of unity.

We usually use ω as an imaginary cube root of unity.

Thus $\omega^3 = 1$

or, $\omega^3 - 1 = 0$

or, $(\omega - 1)(\omega^2 + \omega + 1) = 0.$

Since ω is imaginary, $\omega \neq 1$, therefore $\omega^2 + \omega + 1 = 0.$

Solving, $\omega = \frac{1}{2}(-1 \pm \sqrt{1-4}) = \frac{1}{2}(-1 \pm i\sqrt{3}).$

If we take $\omega = \frac{1}{2}(-1 + i\sqrt{3}),$

then $\omega^2 = \frac{1}{4}(1 - 3 - 2i\sqrt{3}) = \frac{1}{2}(-1 - i\sqrt{3})$, which is the other root.

On the other hand, if we take $\omega = \frac{1}{2}(-1 - i\sqrt{3}),$

then also $\omega^2 = \frac{1}{2}(-1 + i\sqrt{3}).$

Thus the imaginary cube roots of unity are ω and ω^2.

1.8. Illustrative Examples.

Ex. 1. *Find the modulus and the amplitude of the complex number*

$$\frac{(1+i)(2+3i)}{(i-1)(2-3i)}.$$

Let $z=\frac{(1+i)(2+3i)}{(i-1)(2-3i)}=\frac{2+2i+3i+3i^2}{2i-2+3i-3i^2}=\frac{2+5i-3}{-2+5i+3}=\frac{-1+5i}{1+5i}$

$$=\frac{-(1-5i)^2}{(1+5i)(1-5i)}=\frac{-(1-10i+25i^2)}{1-25i^2}=\frac{-1+10i+25}{1+25}=\frac{12}{13}+\frac{5}{13}i.$$

Therefore $|z|=\sqrt{\left(\frac{12}{13}\right)^2+\left(\frac{5}{13}\right)^2}=\sqrt{\frac{144+25}{169}}=1$

and $\quad \text{amp } z=\tan^{-1}\frac{5}{12}.$

Ex. 2. *Express* $\frac{1+\cos\alpha-i\sin\alpha}{1-\cos\alpha+i\sin\alpha}$ *in the form* $(A+iB)$.

$$\frac{1+\cos\alpha-i\sin\alpha}{1-\cos\alpha+i\sin\alpha}$$

$$=\frac{2\cos^2\frac{1}{2}\alpha-2i\sin\frac{1}{2}\alpha\cos\frac{1}{2}\alpha}{2\sin^2\frac{1}{2}\alpha+2i\sin\frac{1}{2}\alpha\cos\frac{1}{2}\alpha}=\frac{2\cos\frac{1}{2}\alpha(\cos\frac{1}{2}\alpha-i\sin\frac{1}{2}\alpha)}{2\sin\frac{1}{2}\alpha(\sin\frac{1}{2}\alpha+i\cos\frac{1}{2}\alpha)}$$

$$=\cot\frac{\alpha}{2}\cdot\frac{-i\sin\frac{1}{2}\alpha-i^2\cos\frac{1}{2}\alpha}{\sin\frac{1}{2}\alpha+i\cos\frac{1}{2}\alpha}=\cot\frac{\alpha}{2}\cdot\frac{-i(\sin\frac{1}{2}\alpha+i\cos\frac{1}{2}\alpha)}{\sin\frac{1}{2}\alpha+i\cos\frac{1}{2}\alpha}$$

$$=\cot\tfrac{1}{2}\alpha.(-i)=0+i(-\cot\tfrac{1}{2}\alpha).$$

Ex. 3. *Find the conjugate of the complex number* $\frac{2-i}{(1-2i)^2}$.

Let $z=\frac{2-i}{(1-2i)^2}=\frac{2-i}{1-4i+4i^2}=\frac{2-i}{1-4i-4}=-\frac{2-i}{3+4i}$

$$=-\frac{(2-i)(3-4i)}{(3+4i)(3-4i)}=-\frac{6-3i-8i-4}{9+16}=-\frac{1}{25}(2-11i).$$

Hence the conjugate of z is $-\frac{1}{25}(2+11i)$.

Ex. 4. *Simplify* : $(1-i)\left(1-\frac{1}{i}\right)$.

$$(1-i)\left(1-\frac{1}{i}\right)=(1-i)\left(1+\frac{i^2}{i}\right)$$

$$=(1-i)(1+i)=1-i^2=1+1=2.$$

Ex. 5. *Find the square root of* $(a^2-1-2ia)$.

Let $\sqrt{a^2-1-2ia} = x-iy$, where x and y are real.

Therefore $(a^2-1)-2ia = (x-iy)^2 = x^2-y^2-2ixy$.

Equating the real and the imaginary parts from both sides, we get

$$x^2-y^2 = a^2-1 \quad \cdots \quad (1)$$

and $$2xy = 2a, \text{ whence } xy = a. \quad \cdots \quad (2)$$

Now $(x^2+y^2)^2 = (x^2-y^2)^2+4x^2y^2 = (a^2-1)^2+4a^2 = (a^2+1)^2$

or, $$x^2+y^2 = \pm(a^2+1). \quad \cdots \quad (3)$$

Hence, by adding (1) and (3), we get

$$2x^2 = 2a^2, -2$$

or, $$x^2 = a^2; -1.$$

Since x is real, we have $x^2 = a^2$, giving $x = \pm a$.

From (1), we have $y^2 = x^2-a^2+1 = a^2-a^2+1 = 1$

or, $$y = \pm 1.$$

Therefore $\sqrt{a^2-1-2ia} = \pm(a-i)$.

Second method :

We may write $a^2-1-2ia = a^2+i^2-2ia = (a-i)^2$.

Hence $\sqrt{a^2-1-2ia} = \pm(a-i)$.

Ex. 6. *If* $(a+ib)(c+id) = A+iB$, *then prove that*

$$(a-ib)(c-id) = A-iB.$$

Here $A+iB = (a+ib)(c+id) = (ac-bd)+i(ad+bc)$.

Equating the real and the imaginary parts from both sides, we get

$$A = ac-bd \quad \text{and} \quad B = ad+bc.$$

Hence $A-iB = (ac-bd)-i(ad+bc) = (a-ib)(c-id)$.

Ex. 7. *If* $\sqrt[3]{x+iy} = a+ib$, *then show that* $\frac{x}{a}+\frac{y}{b} = 4(a^2-b^2)$.

We have $\sqrt[3]{x+iy} = a+ib$

or, $$x+iy = (a+ib)^3 = a^3+3a^2bi+3ab^2i^2+b^3i^3$$
$$= (a^3-3ab^2)+i(3a^2b-b^3).$$

Equating the real and the imaginary parts from both sides, we get

$$x = a^3-3ab^2 \quad \text{and} \quad y = 3a^2b-b^3.$$

Hence $\frac{x}{a}+\frac{y}{b} = \frac{a^3-3ab^2}{a}+\frac{3a^2b-b^3}{b} = 4(a^2-b^2)$.

Ex. 8. *If* $x = 2+3i$, *then find the value of* $(x^3 - 4x^2 + 13x + 1)$.

We have $x = 2+3i$, whence $x-2 = 3i$.

Squaring both sides, we get

$$(x-2)^2 = 9i^2$$

or, $x^2 - 4x + 4 = -9$

or, $x^2 - 4x + 13 = 0$.

Therefore $x^3 - 4x^2 + 13x + 1 = x(x^2 - 4x + 13) + 1 = x \times 0 + 1 = 1$.

Ex. 9. *If* z_1 *and* z_2 *be two complex numbers, then show that*

(*i*) $| z_1 \pm z_2 | \geq | \, |z_1| - |z_2| \, |$.

(*ii*) $z_1 \bar{z}_2 + \bar{z}_1 z_2 \leq 2 \, |z_1| \, |z_2|$, *where* $\bar{z}$ *is the conjugate of* z.

[C. H. 1977]

Let $z_1 = r_1 (\cos \theta_1 + i \sin \theta_1)$ and $z_2 = r_2 (\cos \theta_2 + i \sin \theta_2)$

so that $|z_1| = r_1$ and $|z_2| = r_2$.

(*i*) Now $|z_1 \pm z_2| = | r_1 (\cos \theta_1 + i \sin \theta_1) \pm r_2 (\cos \theta_2 + i \sin \theta_2) |$

$$= | (r_1 \cos \theta_1 \pm r_2 \cos \theta_2) + i (r_1 \sin \theta_1 \pm r_2 \sin \theta_2) |$$

$$= \sqrt{(r_1 \cos \theta_1 \pm r_2 \cos \theta_2)^2 + (r_1 \sin \theta_1 \pm r_2 \sin \theta_2)^2}$$

$$= \sqrt{r_1^2 + r_2^2 \pm 2r_1 r_2 \cos (\theta_1 - \theta_2)}$$

$$\geq \sqrt{r_1^2 + r_2^2 + 2r_1 r_2 (-1)}$$

[since $\cos (\theta_1 - \theta_2) \leq 1$, we have $-\cos (\theta_1 - \theta_2) \geq -1$.
Also $+\cos (\theta_1 - \theta_2) \geq -1$. Hence $\pm \cos (\theta_1 - \theta_2) \geq -1$.]

Hence $\quad |z_1 \pm z_2| \geq |r_1 - r_2|$

that is, $\quad \geq | \, |z_1| - |z_2| \, |$.

(*ii*) Here $\bar{z}_1 = r_1 (\cos \theta_1 - i \sin \theta_1)$ and $z_2 = r_2 (\cos \theta_2 - i \sin \theta_2)$.

Hence $\quad z_1 \bar{z}_2 + \bar{z}_1 z_2 = r_1 r_2 (\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 - i \sin \theta_2)$

$$+ r_1 r_2 (\cos \theta_1 - i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2)$$

$$= r_1 r_2 \{\cos (\theta_1 - \theta_2) + i \sin (\theta_1 - \theta_2)\}$$

$$+ r_1 r_2 \{\cos (\theta_2 - \theta_1) + i \sin (\theta_2 - \theta_1)\}$$

$$= 2r_1 r_2 \cos (\theta_1 - \theta_2) \leq 2r_1 r_2$$

or, $\quad z_1 \bar{z}_2 + \bar{z}_1 z_2 \leq 2 \, |z_1| \, |z_2|$.

Ex. 10. *Show that the points* $(2-i), (4+i)$ *and* $(2+3i)$ *in Argand diagram are the vertices of a right-angled isosceles triangle.*

In Argand diagram, the cartesian co-ordinates of the points representing the complex numbers $(2-i)$, $(4+i)$ and $(2+3i)$ are $A(2,-1)$, $B(4,1)$ and $C(2,3)$ respectively. Hence the length AB is

$$\sqrt{(2-4)^2+(-1-1)^2}=\sqrt{4+4}=\sqrt{8}.$$

Similarly, the length of BC is $\sqrt{8}$ and that of AC is 4.

Since the sum of any two of the lengths AB, BC, AC is greater than the third, A, B, C form a triangle.

Also $AB^2+BC^2=8+8=16=AC^2$ and $AB=\sqrt{8}=BC$.

Therefore the triangle ABC is right-angled as well as isosceles.

Thus the points A, B, C are the vertices of a right-angled isosceles triangle.

[C.H. 1977]

Ex. 11. *If z be a complex number and* $\dfrac{z+1}{z-i}$ *be purely imaginary, then show that z lies on the circle whose centre is at the point* $\frac{1}{2}(-1+i)$ *and the radius is* $\dfrac{1}{\sqrt{2}}$.

Let $z=x+iy$.

Then $$\frac{z+1}{z-i}=\frac{(x+1)+iy}{x+i(y-1)}=\frac{\{(x+1)+iy\}\{x-i(y-1)\}}{\{x+i(y-1)\}\{x-i(y-1)\}}$$

$$=\frac{x^2+x+ixy-i(xy+y-x-1)+y^2-y}{x^2+(y-1)^2}$$

$$=\frac{x^2+y^2+x-y}{x^2+(y-1)^2}+i\frac{x-y+1}{x^2+(y-1)^2}$$

Since $\dfrac{z+1}{z-i}$ is purely imaginary, therefore $\dfrac{x^2+y^2+x-y}{x^2+(y-1)^2}=0.$

This gives $x^2+y^2+x-y=0$

or, $$\left(x+\frac{1}{2}\right)^2+\left(y-\frac{1}{2}\right)^2=\frac{1}{4}+\frac{1}{4}=\frac{1}{2}=\left(\frac{1}{\sqrt{2}}\right)^2$$

It is a circle of radius $\dfrac{1}{\sqrt{2}}$ and the centre of the circle is at the point

$$\left(-\frac{1}{2},\frac{1}{2}\right).$$

Therefore z lies on the circle whose centre is at the point $\frac{1}{2}(-1+i)$ and the radius is $\dfrac{1}{\sqrt{2}}$.

Ex. 12. (*a*) *Factorise* : $a^2 - ab + b^2$. [*N. B. H.* 1991]

(*b*) *Show that* $(x + y + z)(x + y\omega + z\omega^2)(x + y\omega^2 + z\omega)$

$$= x^3 + y^3 + z^3 - 3xyz,$$

where ω *is an imaginary cube root of unity.*

(*a*) We have $1 + \omega + \omega^2 = 0$, where ω is an imaginary cube root of unity

or, $-1 = \omega + \omega^2$.

Therefore $a^2 - ab + b^2 = a^2 + (\omega + \omega^2)\,ab + b^2$

$= a^2 + \omega ab + \omega^2 ab + \omega^3 b^2$, since $\omega^3 = 1$

$= a(a + \omega b) + \omega^2 b(a + \omega b) = (a + \omega b)(a + \omega^2 b)$.

(*b*) $(x + y + z)(x + y\omega + z\omega^2)(x + y\omega^2 + z\omega)$

$= (x + y + z)\,\{x^2 + y^2\omega^3 + z^2\omega^3 + xy(\omega + \omega^2) + yz(\omega^4 + \omega^2) + zx(\omega + \omega^2)\}$

$= (x + y + z)\,(x^2 + y^2 + z^2 - xy - yz - zx)$,

since $\omega^3 = 1$ and $\omega + \omega^2 = -1$

$= x^3 + y^3 + z^3 - 3xyz$.