(*) If z_1, z_2, z_3 are three complex numbers and
$z_1 + z_2 + z_3 = u_1$, $z_1 + z_2\omega + z_3\omega^2 = u_2$, $z_1 + z_2\omega^2 + z_3\omega = u_3$
where ω is imaginary cube root of unity,
Then show that
$$|u_1|^2 + |u_2|^2 + |u_3|^2 = 3(|z_1|^2 + |z_2|^2 + |z_3|^2)$$

Soln: Here ω is imaginary cube root of unity

$\therefore\ 1 + \omega + \omega^2 = 0$

& $\omega^3 = 1$ i.e. $\omega.\omega^2 = 1$ i.e. $|\omega.\omega^2| = 1$

$\therefore\ \omega^2 = \bar{\omega}$ & $\bar{\omega^2} = \omega$ — (1) i.e. $|\omega||\omega^2| = 1$ i.e. $|\omega| = 1$ & $|\omega^2| = 1$

$\Rightarrow |\omega| = |\omega^2| = 1$

Now $|u_1|^2 = |z_1 + z_2 + z_3|^2$

$= (z_1 + z_2 + z_3)(\overline{z_1 + z_2 + z_3})$

$= (z_1 + z_2 + z_3)(\bar{z}_1 + \bar{z}_2 + \bar{z}_3)$

$= |z_1|^2 + z_1\bar{z}_2 + z_1\bar{z}_3 + z_2\bar{z}_1 + |z_2|^2 + z_2\bar{z}_3 + z_3\bar{z}_1$
$+ z_3\bar{z}_2 + |z_3|^2$ — (2)

Again $|u_2|^2 = |z_1 + z_2\omega + z_3\omega^2|$

$= (z_1 + z_2\omega + z_3\omega^2)(\overline{z_1 + z_2\omega + z_3\omega^2})$

$= (z_1 + z_2\omega + z_3\omega^2)(\bar{z}_1 + \overline{z_2\omega} + \overline{z_3\omega^2})$

$= (z_1 + z_2\omega + z_3\omega^2)(\bar{z}_1 + \bar{z}_2\omega^2 + \bar{z}_3\omega)$

$= |z_1|^2 + z_1\bar{z}_2\omega^2 + z_1\bar{z}_3\omega + z_2\bar{z}_1\omega + |z_2|^2 + z_2\bar{z}_3\omega^2$
$+ z_3\bar{z}_1\omega^2 + z_3\bar{z}_2\omega + |z_3|^2$ — (3)

and $|u_3|^2 = |z_1 + \omega^2 z_2 + z_3\omega|$

$= (z_1 + z_2\omega^2 + z_3\omega)(\overline{z_1 + z_2\omega^2 + z_3\omega})$

$= (z_1 + z_2\omega^2 + z_3\omega)(\bar{z}_1 + \overline{z_2\omega^2} + \overline{z_3\omega})$

$= (z_1 + z_2\omega^2 + z_3\omega)(\bar{z}_1 + \bar{z}_2\omega + \bar{z}_3\omega^2)$

$= |z_1|^2 + z_1\bar{z}_2\omega + z_1\bar{z}_3\omega^2 + z_2\bar{z}_1\omega^2 + |z_2|^2 + z_2\bar{z}_3\omega$
$+ z_3\bar{z}_1\omega + z_3\bar{z}_2\omega^2 + |z_3|^2$ — (4)

Adding ②, ③ & ④, we get

$$|u_1|^2+|u_2|^2+|u_3|^2 = 3|z_1|^2 + z_1\bar{z}_2(1+\omega^2+\omega) + z_1\bar{z}_3(1+\omega+\omega^2)$$
$$+ z_2\bar{z}_1(1+\omega+\omega^2) + 3|z_2|^2 + z_2\bar{z}_3(1+\omega^2+\omega)$$
$$+ z_3\bar{z}_1(1+\omega^2+\omega) + z_3\bar{z}_2(1+\omega+\omega^2) + 3|z_3|^2$$

$$= 3|z_1|^2 + 3|z_2|^2 + 3|z_3|^2$$
$$= 3(|z_1|^2 + |z_2|^2 + |z_3|^2)$$

(*) For any two non-zero complex numbers z_1 and z_2 prove that

$$2(|z_1|+|z_2|) \geq (|z_1+z_2|)\left|\frac{z_1}{|z_1|}+\frac{z_2}{|z_2|}\right|$$

Solⁿ Now we have

$$(|z_1+z_2|)\left|\frac{z_1}{|z_1|}+\frac{z_2}{|z_2|}\right|$$
$$= \left|(z_1+z_2)\left(\frac{z_1}{|z_1|}+\frac{z_2}{|z_2|}\right)\right| \text{ as } |z_1||z_2| = |z_1z_2|$$
$$= \left|\frac{z_1^2}{|z_1|}+\frac{z_1z_2}{|z_2|}+\frac{z_1z_2}{|z_1|}+\frac{z_2^2}{|z_2|}\right|$$
$$\leq \left|\frac{z_1^2}{|z_1|}\right|+\left|\frac{z_1z_2}{|z_2|}\right|+\left|\frac{z_1z_2}{|z_1|}\right|+\left|\frac{z_2^2}{|z_2|}\right| \text{ as } |z_1+z_2| \leq |z_1|+|z_2|$$
i.e. $\leq \frac{1}{|z_1|}|z_1^2| + \frac{1}{|z_2|}|z_1z_2| + \frac{1}{|z_1|}|z_1z_2| + \frac{1}{|z_2|}|z_2^2|$

i.e. $\leq \frac{|z_1||z_1|}{|z_1|} + \frac{|z_1||z_2|}{|z_2|} + \frac{|z_1||z_2|}{|z_1|} + \frac{|z_2||z_2|}{|z_2|}$

i.e. $\leq |z_1| + |z_1| + |z_2| + |z_2|$

i.e. $\leq 2|z_1| + 2|z_2|$

i.e. $\leq 2(|z_1| + |z_2|)$

Therefore $2(|z_1|+|z_2|) \geq (|z_1+z_2|)\left|\frac{z_1}{|z_1|^2}+\frac{z_2}{|z_2|^2}\right|$

(*) If $\bar{z}$ be the conjugate of z, then for any two complex numbers z_1 and z_2

Prove that $|1-z_1\bar{z}_2|^2-|z_1-z_2|^2=(1-|z_1|^2)(1-|z_2|^2)$

Soln:

$$L.H.S = |1-z_1\bar{z}_2|^2-|z_1-z_2|^2$$

$$=(1-z_1\bar{z}_2)(\overline{1-z_1\bar{z}_2})-(z_1-z_2)(\overline{z_1-z_2})$$

$$=(1-z_1\bar{z}_2)(1-\bar{z}_1z_2)-(z_1-z_2)(\bar{z}_1-\bar{z}_2)$$

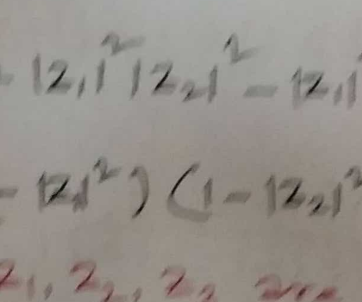
$$=1-\bar{z}_1z_2-z_1\bar{z}_2+(z_1\bar{z}_1)(z_2\bar{z}_2)-(z_1\bar{z}_1)+z_1\bar{z}_2$$
$$+\bar{z}_1z_2-(z_2\bar{z}_2)$$

$$=1+|z_1|^2|z_2|^2-|z_1|^2-|z_2|^2$$

$$=(1-|z_1|^2)(1-|z_2|^2)=R.H.S.$$

(*) Prove that z_1, z_2, z_3 are the vertices of an equilateral triangle in the Argand's plane if and only if $z_1^2+z_2^2+z_3^2-z_1z_2-z_1z_3-z_2z_3=0$

Soln:



First we assume that z_1, z_2, z_3 are the vertices of an equilateral triangle in Argand's plane

$$\therefore\ |z_1-z_2|=|z_2-z_3|=|z_3-z_1| \quad \text{--- (1)}$$

Let $z_1-z_2=\alpha$, $z_2-z_3=\beta$ and $z_3-z_1=\gamma$

$\therefore$ from eqn. (1), we have

$$|\alpha|=|\beta|=|\gamma|$$

$$\text{and } \alpha+\beta+\gamma=0 \quad \text{--- (2)}$$

$$\therefore\ \overline{\alpha+\beta+\gamma}=0$$

Then $\bar{\alpha} + \bar{\beta} + \bar{\gamma} = 0$ where $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$ are the conjugate of α, β, γ ——— (3)

Now $|\alpha| = |\beta| = |\gamma|$

$\Rightarrow |\alpha|^2 = |\beta|^2 = |\gamma|^2$

$\Rightarrow \alpha\bar{\alpha} = \beta.\bar{\beta} = \gamma.\bar{\gamma} = k$ (say), $k \neq 0$

$\therefore \bar{\alpha} = \frac{k}{\alpha}, \bar{\beta} = \frac{k}{\beta}, \bar{\gamma} = \frac{k}{\gamma}$

Now from equ^n (3) we have

$$\frac{k}{\alpha} + \frac{k}{\beta} + \frac{k}{\gamma} = 0$$

$$\therefore \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = 0 \quad [\because k \neq 0]$$

$$\therefore \alpha\beta + \beta.\gamma + \gamma\alpha = 0 \text{ ——— (4)}$$

Again from equ^n (2), we have

$$\alpha + \beta + \gamma = 0$$

$$\therefore (\alpha + \beta + \gamma)^2 = 0$$

or $\alpha^2 + \beta^2 + \gamma^2 + 2\alpha\beta + 2\alpha\gamma + 2\beta\gamma = 0$

or $(\alpha^2 + \beta^2 + \gamma^2) + 2(\alpha\beta + \beta\gamma + \gamma\alpha) = 0$

or $\alpha^2 + \beta^2 + \gamma^2 = 0$ by equ^n (4)

$\therefore (z_1 - z_2)^2 + (z_2 - z_3)^2 + (z_3 - z_1)^2 = 0$ by equ^n (1)

or $z_1^2 - 2z_1z_2 + z_2^2 + z_2^2 - 2z_2z_3 + z_3^2 + z_3^2 - 2z_3z_1 + z_1^2 = 0$

or $2z_1^2 + 2z_2^2 + 2z_3^2 - 2z_1z_2 - 2z_2z_3 - 2z_3z_1 = 0$

or $z_1^2 + z_2^2 + z_3^2 - z_1z_2 - z_2z_3 - z_3z_1 = 0$

$\therefore z_1^2 + z_2^2 + z_3^2 - z_1z_2 - z_2z_3 - z_3z_1 = 0$ (Proved)

Conversely let $z_1^2 + z_2^2 + z_3^2 - z_1z_2 - z_2z_3 - z_3z_1 =$

$\Rightarrow (z_1 + z_2\omega + z_3\omega^2)(z_1 + z_2\omega^2 + z_3\omega) = 0$

$\Rightarrow z_1 + z_2\omega + z_3\omega^2 = 0$ and $z_1 + z_2\omega^2 + z_3\omega = 0$

as ω = imaginary cube root of unity

$\therefore z_1 + z_2\omega + z_3\omega^2 = 0$

$\Rightarrow z_1 + z_2\omega + z_3(-1-\omega) = 0$ as $\omega^2 + \omega + 1 = 0$

$\Rightarrow z_1 + z_2\omega - (1+\omega)z_3 = 0$

$\Rightarrow z_1 + z_2\omega - z_3 - z_3\omega = 0$

$\Rightarrow z_1 - z_3 = \omega(z_3 - z_2)$

$\Rightarrow |z_1 - z_3| = |\omega(z_3 - z_2)| = |\omega||z_3 - z_2|$

$\Rightarrow |z_1 - z_3| = |z_3 - z_2|$ as $|\omega| = 1$. i.e. $|z_3 - z_1| = |z_2 -$ ——— (6)

Similiarly let $z_1 + z_2\omega^2 + z_3\omega = 0$

$\Rightarrow z_1 - (1+\omega)z_2 + z_3\omega = 0$

$\Rightarrow z_1 - z_2 - \omega(z_2 - z_3) = 0$

$\Rightarrow (z_1 - z_2) = \omega(z_2 - z_3)$

$\Rightarrow |z_1 - z_2| = |\omega(z_2 - z_3)| = |\omega||z_2 - z_3|$

$= |z_2 - z_3| \because |\omega| =$

$\Rightarrow |z_1 - z_2| = |z_2 - z_3|$ ——— (7)

From equn. (6) & (7), we get

$|z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1|$

$\therefore z_1, z_2, z_3$ are the vertices of an equilateral triangle (Proved)

Complex Numbers

(2) If $\left|\frac{z-i}{z+1}\right| = k$, then show that z lies on a circle if $k \neq 1$ and lies on a straight line if $k=1$.

Soln. Given that $\left|\frac{z-i}{z+1}\right| = k$

Let $z = x+iy$

Then $\left|\frac{(x+iy)-i}{(x+iy)+1}\right| = k$

i.e. $\left|\frac{x+i(y-1)}{(x+1)+iy}\right| = k$

i.e. $\left|\frac{\{x+i(y-1)\}\{(x+1)-iy\}}{\{(x+1)+iy\}\{(x+1)-iy\}}\right| = k$

i.e. $\left|\frac{x(x+1)+y(y-1)+i\{(y-1)(x+1)-xy\}}{(x+1)^2+y^2}\right| = k$

i.e. $\left|\frac{x^2+x+y^2-y+i\{(y-1)(x+1)-xy\}}{(x+1)^2+y^2}\right| = k$

i.e. $\left|\frac{x+i(y-1)}{(x+1)+iy}\right| = k$

i.e. $\frac{|x+i(y-1)|}{|(x+1)+iy|} = k$

i.e. $\frac{\sqrt{x^2+(y-1)^2}}{\sqrt{(x+1)^2+y^2}} = k$

i.e. $\sqrt{\frac{x^2+(y-1)^2}{(x+1)^2+y^2}} = k$

i.e. $\frac{x^2+(y-1)^2}{(x+1)^2+y^2} = k^2$

i.e. $x^2+(y-1)^2 = k^2\{(x+1)^2+y^2\}$

i.e. $x^2+y^2-2y+1 = k^2\{x^2+2x+1+y^2\}$

or $x^2(1-k^2)+y^2(1-k^2)-2k^2x-2y+(1-k^2)=0$

or $x^2+y^2-\frac{2k^2}{1-k^2}x-\frac{2}{1-k^2}y+1=0$

or $\left(x-\frac{k^2}{1-k^2}\right)^2+\left(y-\frac{1}{1-k^2}\right)^2 = \frac{k^4}{(1-k^2)^2}+\frac{1}{(1-k^2)^2}-1$

$$= \frac{\cancel{k^4}+\cancel{1}-\cancel{1}+2k^2-\cancel{k^4}}{(1-k^2)^2}$$

$$= \left(\frac{k\sqrt{2}}{1-k^2}\right)^2$$

Simplifying, we get

$$\left(x-\frac{k^2}{1-k^2}\right)^2+\left(y-\frac{1}{1-k^2}\right)^2 = \left(\frac{k\sqrt{2}}{1-k^2}\right)^2$$

This is a circle, if $k\neq 1$, whose centre is the point $\left(\frac{k^2}{1-k^2}, \frac{1}{1-k^2}\right)$ i.e. $\frac{k^2+i}{1-k^2}$ and radius is $\frac{k\sqrt{2}}{1-k^2}$.

When $k=1$, then the above equation i.e.

$x^2+(y-1)^2 = (x+1)^2+y^2$ i.e. $y+x=0$,

Which is a straight line passing through origin.

(*) Prove that $|z_1+z_2|^2+|z_1-z_2|^2 = 2(|z_1|^2+|z_2|^2)$, where z_1 and z_2 are complex quantities.
Hence prove that for any two complex numbers a and b

$$|a+\sqrt{a^2-b^2}|+|a-\sqrt{a^2-b^2}| = |a+b|+|a-b|$$

Soln: 2nd part

Let $z_1 = a+\sqrt{a^2-b^2}$ & $z_2 = a-\sqrt{a^2-b^2}$

Now $(|z_1|+|z_2|)^2 = |z_1|^2+|z_2|^2+2|z_1||z_2|$

$$= \frac{1}{2}(|z_1+z_2|^2+|z_1-z_2|^2)+2|z_1z_2|$$

Again $z_1+z_2 = a+\sqrt{a^2-b^2}+a-\sqrt{a^2-b^2}$

$= 2a$ $\quad\therefore$ Then $|z_1+z_2|^2 = 4|a|^2$

$z_1-z_2 = \cancel{a}+\sqrt{a^2-b^2}-\cancel{a}+\sqrt{a^2-b^2}$

$= 2\sqrt{a^2-b^2}$ $\quad\therefore |z_1-z_2|^2 = 4|a^2-b^2|$

$z_1z_2 = (a+\sqrt{a^2-b^2})(a-\sqrt{a^2-b^2})$

$= a^2-(\sqrt{a^2-b^2})^2$

$= a^2-(a^2-b^2) = b^2$

$\therefore |z_1z_2| = |b|^2$

$\therefore (|z_1|+|z_2|)^2 = \frac{1}{2}(4|a|^2+4|a^2-b^2|)+2|b|^2$

$= 2(|a|^2+|a^2-b^2|)+2|b|^2$

$= 2(|a|^2+|a^2-b^2|+|b|^2)$

$= 2(|a|^2+|b|^2)+2|a^2-b^2|$

$= |a+b|^2+|a-b|^2+2|a^2-b^2|$

$= (|a+b|+|a-b|)^2$

$= |a+b|+|a-b|$

i.e. $|z_1|+|z_2| = |a+b|+|a-b|$

i.e. $|a+\sqrt{a^2-b^2}|+|a-\sqrt{a^2-b^2}| = |a+b|+|a-b|$

(*) If the complex numbers z_1, z_2, z_3 satisfy the equation $z_1^2 + z_2^2 + z_3^2 - z_1 z_2 - z_2 z_3 - z_1 z_3 = 0$, then prove that

$$|z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1|$$

Solⁿ: We have

$$z_1^2 + z_2^2 + z_3^2 - z_1 z_2 - z_2 z_3 - z_1 z_3$$
$$= (z_1 + \omega z_2 + \omega^2 z_3)(z_1 + \omega^2 z_2 + \omega z_3)$$

where ω is an imaginary cube root of unity.

Now $(z_1 + \omega z_2 + \omega^2 z_3)(z_1 + \omega^2 z_2 + \omega z_3) = 0$

$\therefore$ Either $z_1 + \omega z_2 + \omega^2 z_3 = 0$ or $z_1 + \omega^2 z_2 + \omega z_3 = 0$

from $z_1 + \omega z_2 + \omega^2 z_3 = 0$ we get $z_1 + \omega z_2 - (1+\omega) z_3 = 0$ as $\omega^2 + \omega + 1 = 0$

or $(z_1 - z_3) + \omega(z_2 - z_3) = 0$ i.e. $(z_1 - z_3) = -\omega(z_2 - z_3)$

i.e. $(z_3 - z_1) = \omega(z_2 - z_3)$

and from $z_1 + \omega^2 z_2 + \omega z_3 = 0$

we get $z_1 - (1+\omega) z_2 + \omega z_3 = 0$

or $(z_1 - z_2) + \omega(z_3 - z_2) = 0$ i.e. $(z_1 - z_2) = \omega(z_2 - z_3)$

Taking modulus we get

$$|z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1|$$

i.e. $|z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1|$

(*) If $\left|z - \frac{1}{z}\right| = 4$, then show that greatest value of $|z|$ is $2 + \sqrt{5}$.

Solⁿ: Let $|z| = \left|z - \frac{1}{z} + \frac{1}{z}\right| \le \left|z - \frac{1}{z}\right| + \left|\frac{1}{z}\right| = 4 + \frac{1}{|z|}$

$\Rightarrow |z|^2 - 4|z| - 1 \le 0$

$\Rightarrow |z|^2 - 4|z| + 4 - 4 - 1 \le 0$

$\Rightarrow (|z| - 2)^2 \le 5$

$\Rightarrow |z| - 2 \le \sqrt{5}$

$\Rightarrow |z| \le 2 + \sqrt{5}$

Hence the greatest value of $|z|$ is $2 + \sqrt{5}$

(*) If $(1+x+x^2)^n = a_0 + a_1x + a_2x^2 + \cdots + a_{2n}x^{2n}$, then show that

$$a_0 + a_3 + a_6 + \cdots = a_1 + a_4 + a_7 + \cdots = a_2 + a_5 + a_8 + \cdots = 3^{n-1}.$$

Soln: Putting $x = \omega$ in both sides, we get

$$(1+\omega+\omega^2)^n = a_0 + a_1\omega + a_2\omega^2 + a_3\omega^3 + a_4\omega + a_5\omega^2 + a_6\omega^3 + \cdots$$

i.e. $0 = (a_0 + a_3 + a_6 + \cdots)$

$+ a_1 \cdot \frac{1}{2}(-1 + i\sqrt{3}) + a_2 \cdot \frac{1}{2}(-1 - i\sqrt{3})$

$+ a_4 \cdot \frac{1}{2}(-1 + i\sqrt{3})$

$+ a_5 \cdot \frac{1}{2}(-1 - i\sqrt{3}) + \cdots$

We know that $\omega^2 + \omega + 1 = 0$ and $\omega = \frac{1}{2}(-1 + i\sqrt{3})$, $\omega^2 = \frac{1}{2}(-1 - i\sqrt{3})$

$= (a_0 + a_3 + a_6 + \cdots) - \frac{1}{2}(a_1 + a_2 + a_4 + a_5 + \cdots)$

$+ \frac{i\sqrt{3}}{2}(a_1 - a_2 + a_4 - a_5 + \cdots)$

$= (a_0 + a_3 + a_6 + \cdots) - \frac{1}{2}(a_1 + a_2 + a_4 + a_5) \quad \text{--- (1)}$

Again putting $x = 1$ in both sides, we get

$$3^n = a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + \cdots$$

$$= (a_0 + a_3 + a_6 + \cdots) + (a_1 + a_2 + a_4 + a_5 + \cdots) \quad \text{--- (2)}$$

From eqn (1), we have

$$a_0 + a_3 + a_6 + \cdots = \frac{1}{2}(a_1 + a_2 + a_4 + a_5 + \cdots)$$

$$\Rightarrow (a_1 + a_2 + a_4 + a_5 + \cdots) = 2(a_0 + a_3 + a_6 + \cdots)$$

Again from eqn (2), we get

$$3^n = (a_0 + a_3 + a_6 + \cdots) + 2(a_0 + a_3 + a_6 + \cdots)$$

$$= 3(a_0 + a_3 + a_6 + \cdots)$$

$$\Rightarrow a_0 + a_3 + a_6 + \cdots = 3^n \cdot 3^{-1} = 3^{n-1}$$