Relation between roots and Coefficients

Let $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$ be a polynomial of degree n with coefficients real or complex. Then $a_0 \neq 0$. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the roots of the equation $f(x) = 0$

Then we can write

$$f(x) = a_0 (x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)$$

Then $a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$

$$= a_0 (x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)$$

$$= a_0\big[x^n - (\alpha_1+\alpha_2+\cdots+\alpha_n)x^{n-1} + (\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3 + \cdots)x^{n-2} - (\alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_2\alpha_4 + \cdots)x^{n-3} + (-1)^n(\alpha_1\alpha_2\cdots\alpha_n)\big]$$

$$= a_0\big[x^n - \Sigma\alpha_1 x^{n-1} + \Sigma\alpha_1\alpha_2 x^{n-2} - \Sigma\alpha_1\alpha_2\alpha_3 x^{n-3} + \cdots + (-1)^n(\alpha_1\alpha_2\cdots\alpha_n)\big]$$

Where $\Sigma\alpha_1$ = Sum of the roots

$\Sigma\alpha_1\alpha_2$ = Sum of the product of the roots taken two at a time

$\Sigma\alpha_1\alpha_2\alpha_3$ = Sum of the product of the roots taken three at a time,

and so on;

and finally $\alpha_1, \alpha_2 \cdots \alpha_n$ = the product of all the roots

From the equality of polynomials it follows that

$$a_1 = a_0(-\Sigma\alpha_1)$$
$$a_2 = a_0(\Sigma\alpha_1\alpha_2)$$
$$a_3 = a_0(-\Sigma\alpha_1\alpha_2\alpha_3)$$
$$\cdots\cdots$$
$$\cdots\cdots$$
$$a_n = a_0(-1)^n\alpha_1\alpha_2\cdots\alpha_n$$

$$\therefore \Sigma\alpha_1 = -\frac{a_1}{a_0},\ \Sigma\alpha_1\alpha_2 = \frac{a_2}{a_0},\ \Sigma\alpha_1\alpha_2\alpha_3 = -\frac{a_3}{a_0}$$
$$\cdots\cdots,\ \alpha_1\alpha_2\cdots\alpha_n = (-1)^n\frac{a_n}{a_0}$$

Particular cases.

(i) If α, β, γ be the roots of the equation

$$ax^3 + bx^2 + cx + d = 0$$

then $\Sigma\alpha = \alpha+\beta+\gamma = -\frac{b}{a}$, $\Sigma\alpha\beta = \alpha\beta + \alpha\gamma + \beta\gamma = c/a$

and $\alpha\beta\gamma = -\frac{d}{a}$

(ii) If $\alpha, \beta, \gamma, \delta$ are the roots of the equation

$$a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0,$$

then $\Sigma\alpha = \alpha+\beta+\gamma+\delta = -\frac{a_1}{a_0}$,

$$\Sigma\alpha\beta = \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \frac{a_2}{a_0}$$
$$\Sigma\alpha\beta\gamma = \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -\frac{a_3}{a_0}$$
and $\alpha\beta\gamma\delta = \frac{a_4}{a_0}$

~~Reciprocal equation~~ of the Second type (or Second class or Second kind)

Examples.

1. Solve the equation $2x^3 - x^2 - 18x + 9 = 0$ if two of the roots are equal in magnitude but opposite in sign.

Solⁿ. Let the roots be α, β, γ and $\alpha = -\beta$

Then $\alpha + \beta + \gamma = -\frac{-1}{2} = \frac{1}{2}$ —— ①

$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{-18}{2} = -9$ —— ②

$\alpha\beta\gamma = -\frac{9}{2}$ —— ③

Since $\alpha + \beta = 0$, from ①, $\gamma = \frac{1}{2}$

and from ③ $-\alpha^2 \cdot \frac{1}{2} = -\frac{9}{2}$ ⟨$\because \alpha = -\beta$ $\therefore \beta = -\alpha$⟩

i.e. $\alpha^2 = 9$

$\therefore \alpha = \pm 3$

$\therefore$ The roots are $3, -3, \frac{1}{2}$

C.H.-'87

2. Solve the equation $16x^4 - 64x^3 + 56x^2 + 16x - 15 = 0$ whose roots are in arithmetic progression.

Solⁿ. Let the roots be $\alpha - 3\delta, \alpha - \delta, \alpha + \delta, \alpha + 3\delta$.

Then $\alpha - 3\delta + \alpha - \delta + \alpha + \delta + \alpha + 3\delta = -\frac{-64}{16}$

i.e. $4\alpha = 4$

i.e. $4\alpha = 4$ —— (i)

$(\alpha - 3\delta)(\alpha - \delta) + (\alpha - 3\delta)(\alpha + \delta) + (\alpha - 3\delta)(\alpha + 3\delta)$
$+ (\alpha - \delta)(\alpha + \delta) + (\alpha - \delta)(\alpha + 3\delta) + (\alpha + \delta)(\alpha + 3\delta)$

or $(\alpha - 3\delta)\{(\alpha - \delta) + (\alpha + \delta)\}$ $= \frac{56}{16}$
$+ (\alpha + 3\delta)\{(\alpha - \delta) + (\alpha + \delta)\} + (\alpha^2 - 9\delta^2) + (\alpha^2 - \delta^2) = \frac{56}{16}$

or $\{(\alpha-3\delta)+(\alpha+3\delta)\}\{(\alpha-\delta)+(\alpha+\delta)\}+(\alpha^2-9\delta^2)$
$+(\alpha^2-\delta^2)=\frac{56}{16}$

or $4\alpha^2+(\alpha^2-9\delta^2)+(\alpha^2-\delta^2)=\frac{56}{16}$

or $6\alpha^2-10\delta^2=\frac{56}{16}$ ——— (ii)

$(\alpha-3\delta)(\alpha-\delta)(\alpha+\delta)+(\alpha-3\delta)(\alpha-\delta)(\alpha+3\delta)$
$+(\alpha-3\delta)(\alpha+\delta)(\alpha+3\delta)+(\alpha-\delta)(\alpha+\delta)(\alpha+3\delta)$
$=-\frac{16}{16}$

or $(\alpha-\delta)(\alpha+\delta)\{(\alpha-3\delta)+(\alpha+3\delta)\}$
$+(\alpha-3\delta)(\alpha+3\delta)\{(\alpha-\delta)+(\alpha+\delta)\}$
$=-\frac{16}{16}$

or $(\alpha^2-\delta^2)\cdot 2\alpha+(\alpha^2-9\delta^2)\cdot 2\alpha=-\frac{16}{16}$

or $2\alpha\{(\alpha^2-\delta^2)+\alpha^2-9\delta^2\}=-\frac{16}{16}$

or $2\alpha(2\alpha^2-10\delta^2)=-\frac{16}{16}$ ——— (iii)

and $(\alpha^2-9\delta^2)(\alpha^2-\delta^2)=-\frac{15}{16}$ ——— (iv)

from (i) $\alpha=1$, from (ii) $6\alpha^2-10\delta^2=\frac{56}{16}=\frac{7}{2}$

or $6-10\delta^2=7/2$

or $6-7/2=10\delta^2$

or $5/2=10\delta^2$

or $\delta^2=\frac{5}{20}=\frac{1}{4}$

$\therefore \delta=\pm\frac{1}{2}$

[illegible] of the Second type (or Second class or Second kind)

$\therefore$ the roots are $1-\frac{3}{2}, 1-\frac{1}{2}, 1+\frac{1}{2}, 1+\frac{3}{2}$

i.e., $-\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$

3. If α be a multiple root of order 3 of the equation $x^4+bx^2+cx+d=0$, Show that $\alpha = -\frac{8d}{3c}$.

CH-'69

<u>Solⁿ:</u> Let the roots be $\alpha, \alpha, \alpha, \beta$

Then $\alpha+\alpha+\alpha+\beta = 0$.

i.e. $3\alpha+\beta=0$ ——— (i)

$\alpha^2+\alpha^2+\alpha\beta+\alpha^2+\alpha\beta+\alpha\beta = b$

i.e. $3\alpha^2+3\alpha\beta = b$ ——— (ii)

$\alpha^3+\alpha^2\beta+\alpha^2\beta+\alpha^2\beta = -c$

i.e. $\alpha^3+3\alpha^2\beta = -c$ ——— (iii)

and $\alpha^3\beta = d$ ——— (iv)

From (i), $\beta = -3\alpha$; from (iii) $\alpha^3-9\alpha^3=-c$

or $-8\alpha^3 = -c$

or $8\alpha^3 = c$;

and from (iv) $-3\alpha^4 = d$

or $3\alpha^4 = -d$.

$\therefore \frac{3\alpha^4}{8\alpha^3} = -\frac{d}{c}$

or $\frac{3\alpha}{8} = -\frac{d}{c}$ $\therefore \alpha = -\frac{8d}{3c}$

4. Find the relation that exists among p, q, r if the roots of the equation $x^3+px^2+qx+r=0$ be in geometric progression.

Soln: Let the roots be $\alpha/\delta, \alpha, \alpha\delta$

Then $\frac{\alpha}{\delta}\cdot\alpha\cdot\alpha\delta = -r$

or $\alpha^3 = -r$

or $\alpha = (-r)^{1/3}$.

Since α is a root of the equation,

$$\alpha^3 + p\alpha^2 + q\alpha + r = 0$$

$\therefore -r + p\{(-r)^{1/3}\}^2 + q(-r)^{1/3} + r = 0$

or $-r + p(-r)^{2/3} + q\cdot(-r)^{1/3} + r = 0$

or $p\cdot(-r)^{2/3} + q(-r)^{1/3} = 0$

or $p^3(-r)^2 + q^3(-r) = 0$

or $p^3r^2 - q^3r = 0$

or $p^3r - q^3 = 0$

5. Find the relation among p, q, r, s so that the product of two roots of the equation $x^4+px^3+qx^2+rx+s=0$ is unity.

C.H.-'86

Soln: Let $\alpha, \beta, \gamma, \delta$ be the roots and $\alpha\beta = 1$

Then $\alpha+\beta+\gamma+\delta = -p$ —— (i)

$(\alpha+\beta)(\gamma+\delta) + \alpha\beta + \gamma\delta = q$ —— (ii)

$\alpha\beta(\gamma+\delta) + \gamma\delta(\alpha+\beta) = -r$ —— (iii)

$\alpha\beta\gamma\delta = s$ —— (iv)

— the second type (or Second

from (iv) $\gamma\delta = \lambda$, from (iii) $(\gamma+\delta) + \lambda(\alpha+\beta) = -r$ — (v)

From (i), $(\gamma+\delta) = -p-(\alpha+\beta)$

$\therefore$ from (v),

$$-p-(\alpha+\beta)+\lambda(\alpha+\beta) = -r$$

$$\text{or } -(\alpha+\beta)+\lambda(\alpha+\beta) = p-r$$

$$\text{or } (\alpha+\beta)-\lambda(\alpha+\beta) = r-p$$

$$\text{or } (\alpha+\beta)(1-\lambda) = r-p$$

$$\therefore \alpha+\beta = \frac{r-p}{1-\lambda}$$

Again from (i),

$$(\gamma+\delta) = -p-(\alpha+\beta)$$

$$= -p-\frac{r-p}{1-\lambda}$$

$$= \frac{\cancel{-p}+p\lambda-r+\cancel{p}}{1-\lambda}$$

$$= \frac{p\lambda-r}{1-\lambda}$$

$\therefore$ from (ii), we have

$$\left(\frac{r-p}{1-\lambda}\right)\left(\frac{p\lambda-r}{1-\lambda}\right)+1+\lambda = q$$

$$\text{or } \left(\frac{r-p}{1-\lambda}\right)\left(\frac{p\lambda-r}{1-\lambda}\right) = (q-\lambda-1)$$

$$\text{or } (r-p)(p\lambda-r) = (1-\lambda)^2(q-\lambda-1)$$

$$\text{i.e. } (r-p)(p\lambda-r) = (1-\lambda)^2(q-\lambda-1)$$

which is the required relation.

6. If the equation $x^4+ax^3+bx^2+cx+d=0$ has three equal roots, show that each of them is equal to $\frac{6c-ab}{3a^2-8b}$

<u>Soln:</u> Let the roots of the equation be α, α, α & β

Then $3\alpha+\beta=-a$ ——— (1)

$\alpha.\alpha+\alpha.\alpha+\alpha.\beta+\alpha.\alpha+\alpha\beta+\alpha\beta=b$

i.e. $3\alpha^2+3\alpha\beta=b$ - - - - - - (2)

$\alpha.\alpha.\alpha+\alpha.\alpha.\beta+\alpha.\alpha.\beta+\alpha.\alpha.\beta=-c$

i.e. $\alpha^3+3\alpha^2\beta=-c$ ——— (3)

& $\alpha^3\beta=d$ ——— (4)

Eleminating β from (1), we get $\beta=-a-3\alpha$ and putting the value of β in eqn. (2), we have

$$3\alpha^2+3\alpha(-a-3\alpha)=b$$
$$\Rightarrow 3\alpha^2-3a\alpha-9\alpha^2=b$$
$$\Rightarrow 6\alpha^2+3a\alpha+b=0 \text{ ——— (5)}$$

Eliminating β from (1) and (3), we get

$$\alpha^3+3\alpha^2(-a-3\alpha)=-c$$
$$\Rightarrow \alpha^3-3a\alpha^2-9\alpha^3=-c$$
$$\Rightarrow -8\alpha^3-3a\alpha^2=-c \Rightarrow 8\alpha^3+3a\alpha^2-c=0 \text{ ——— (6)}$$

(5) $\times 4\alpha$ = $24\alpha^3+12a\alpha^2+4b\alpha=0$

(6) $\times 3$ $24\alpha^3+9a\alpha^2-3c=0$

$3a\alpha^2+4b\alpha+3c=0$

From (5) & (6), we get

$$3a\alpha^2+4b\alpha+3c=0 \text{ ——— (7)}$$

from (5) & (7), we get, by cross multiplication,

$$\frac{\alpha^2}{9ac-4b^2}=\frac{\alpha}{3ab-18c}=\frac{1}{24b-9a^2}$$

Therefore $\alpha=\frac{3ab-18c}{24b-9a^2}=\frac{ab-6c}{8b-3a^2}=\frac{6c-ab}{3a^2-8b}$

(*) Solve the equation $4x^3+16x^2-9x-36=0$, when the sum of two roots is zero.

Solution: Since the sum of two roots of the given cubic equation is zero, let us take the roots as $\alpha, -\alpha, \beta$

The the sum of the roots $= \alpha+(-\alpha)+\beta = -\frac{16}{4} = -4$

$\therefore \beta = -4$

The product of the three roots $= \alpha(-\alpha)\beta = -\frac{(-36)}{4}$

$= \frac{36}{4} = 9$

or $-\alpha^2 \cdot (-4) = 9$ as $\beta = -4$

or $\alpha^2 = \frac{9}{4}$ $\therefore \alpha = \pm\frac{3}{2}$

Hence the roots are $3/2, -3/2, -4$

(*) Solve the equation $x^3-3x^2-6x+8=0$, if the roots are in A.P.

Solution: Since the roots of the given cubic equation are in A.P., we take the roots as $\alpha-\beta$, α and $\alpha+\beta$

Then the sum of the roots $=(\alpha-\beta)+\alpha+(\alpha+\beta) = -(-3)/1$

$= 3$

$\therefore 3\alpha = 3 \Rightarrow \alpha = 1$

and the product of the three roots $=(\alpha-\beta)\alpha(\alpha+\beta) = -\frac{8}{1}$

$= -8$

$\therefore (\alpha^2-\beta^2)\alpha = -8$

$\Rightarrow (1-\beta^2)\cdot 1 = -8$

$\Rightarrow \beta^2 = 9$ $\therefore \beta = \pm 3$

The roots are $-2, 1, 4$

(*) Solve the equation $8x^3-52x^2+78x-27=0$, when the roots are in G.P.

Solution: We assume the roots of the given cubic as α/β, α and $\alpha\beta$, which are obviously in G.P.

Then the sum of the roots $= \frac{\alpha}{\beta}+\alpha+\alpha\beta = -\frac{-52}{8} = \frac{26}{4} = \frac{13}{2}$

i.e. $\alpha(1+\beta+\beta^2) = \frac{13}{2}\beta$

and the product of the three roots $= \alpha/\beta \cdot \alpha \cdot \alpha\beta = -\frac{-27}{8}$

$= \frac{27}{8}$

$\therefore \alpha^3 = \frac{27}{8} = \left(\frac{3}{2}\right)^3$

$\therefore \alpha = 3/2$

Since $\alpha = 3/2$, we have

$$\frac{3}{2}(1+\beta+\beta^2) = \frac{13}{2}\beta$$

$$\Rightarrow 3+3\beta+3\beta^2 = 13\beta$$

$$\Rightarrow 3\beta^2 - 10\beta + 3 = 0 \Rightarrow 3\beta^2 - 9\beta - \beta + 3 = 0$$

$$\Rightarrow 3\beta(\beta-3) - 1(\beta-3) = 0 \Rightarrow (\beta-3)(3\beta-1) = 0$$

$$\therefore \beta = 3,\ \beta = \frac{1}{3}$$

Hence the roots are $\frac{1}{2}, \frac{3}{2}, \frac{9}{2}$

* Solve the equation $6x^3 - 11x^2 - 3x + 2 = 0$, given that the roots are in H.P.

Solution: If α, β, γ be the roots of the given equation,

then $\alpha+\beta+\gamma = -\frac{-11}{6} = \frac{11}{6}$ ——— (1)

$\alpha\beta+\alpha\gamma+\beta\gamma = \frac{-3}{6} = -\frac{1}{2}$ ——— (2)

$\alpha\beta\gamma = -\frac{2}{6} = -\frac{1}{3}$ ——— (3)

Since α, β, γ are in H.P., then $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ are in A.P.

i.e. $\frac{1}{\beta} - \frac{1}{\alpha} = \frac{1}{\gamma} - \frac{1}{\beta}$

$$\Rightarrow \frac{1}{\beta} + \frac{1}{\beta} = \frac{1}{\alpha} + \frac{1}{\gamma} = \frac{\alpha+\gamma}{\alpha\gamma}$$

$$\Rightarrow \frac{2}{\beta} = \frac{\alpha+\gamma}{\alpha\gamma} \Rightarrow \beta = \frac{2\alpha\gamma}{\alpha+\gamma}$$

$$\Rightarrow \alpha\beta + \beta\gamma = 2\alpha\gamma$$

$$\therefore \alpha\beta+\beta\gamma+\gamma\alpha = 3\gamma\alpha$$

$$\Rightarrow -\frac{1}{2} = 3\gamma\alpha \text{ from (2)} \quad \therefore \gamma\alpha = -\frac{1}{6}$$

Again from (3), $\alpha\beta\gamma = -\frac{1}{3} \Rightarrow -\frac{1}{6}\beta = -\frac{1}{3} \Rightarrow \beta = 2$

Thus 2 is a root of the given equation.

Now we have

$$6x^3 - 11x^2 - 3x + 2$$
$$= (x-2)(6x^2+x-1)$$
$$= (x-2)(3x-1)(2x+1) = 0$$

$$\therefore x = 2,\ x = \frac{1}{3},\ x = -\frac{1}{2}$$

Hence the roots are $-\frac{1}{2}, \frac{1}{3}, 2$

Find the Condition that the equation x^3+px^2+qx+r may have two roots equal but of opposite signs.

Solution: Let the roots be $\alpha, (-\alpha)$ and β

Then $\alpha+(-\alpha)+\beta=-p$ ——— (1)

$\alpha.(-\alpha)+\alpha\beta+(-\alpha)\beta=q$ ——— (2)

$\alpha.(-\alpha)\beta=-r$ ——— (3)

From (1), $\beta=-p$

Substituting this value of β in (3), we get

$\alpha^2 p=-r$

Then from (2), we have $-\alpha^2=q \to \alpha^2=-q$

$\therefore -q.p=-r \Rightarrow pq=r$

Hence $pq=r$ is the required Condition.

Find the Condition that the roots of the equation $x^3-px^2+qx-r=0$ may be in A.P.

Solution: Let the roots be $\alpha-\beta$, α and $\alpha+\beta$, which are obviously in A.P.

Then $(\alpha-\beta)+\alpha+(\alpha+\beta)=-\frac{-p}{1}=p$ ——— (1)

$(\alpha-\beta)\alpha+(\alpha-\beta)(\alpha+\beta)+\alpha(\alpha+\beta)=q$ ——— (2)

and $(\alpha-\beta)\alpha(\alpha+\beta)=-\frac{-r}{1}=r$ ——— (3)

From (1), $3\alpha=p \quad \therefore \alpha=p/3$

From (2), $3\alpha^2-\beta^2=q$

$\Rightarrow 3.(p/3)^2-\beta^2=q$

$\Rightarrow 3.\frac{p^2}{9}-\beta^2=q \Rightarrow \frac{p^2}{3}-\beta^2=q \Rightarrow \beta^2=p^2/3-q$

Using these values of α and β^2 in (3), we have

$\alpha(\alpha^2-\beta^2)=r$

$\Rightarrow p/3\left(\frac{p^2}{9}-p^2/3+q\right)=r$

$\Rightarrow \frac{p^3}{27}-\frac{p^3}{9}+\frac{pq}{3}=r$

$\Rightarrow p^3-3p^3+9pq=27r$

$\Rightarrow -2p^3+9pq=27r$

Hence $9pq-2p^3=27r$ is the required condition.

(*) If the roots of the equation $x^4 + ax^3 + bx^2 + cx + d = 0$ are in G.P., then prove that $c^2 = a^2 d$.

Solution: Let the roots of the equation be $\alpha/r^3, \alpha/r, \alpha r, \alpha r^3$ which are in G.P.

Then $\frac{\alpha}{r^3} + \frac{\alpha}{r} + \alpha r + \alpha r^3 = -a$ ——— (1)

$\frac{\alpha^2}{r^4} + \frac{\alpha^2}{r^2} + \alpha^2 + \alpha^2 + \alpha^2 r^2 + \alpha^2 r^4 = b$ ——— (2)

$\frac{\alpha^3}{r^3} + \frac{\alpha^3}{r} + \alpha^3 r + \alpha^3 r^3 = -c$ ——— (3)

$\frac{\alpha}{r^3} \cdot \frac{\alpha}{r} \cdot \alpha r \cdot \alpha r^3 = d$ ——— (4)

From (4), $\alpha^4 = d$ ——— (5)

From (1), $\alpha\left(\frac{1}{r^3} + \frac{1}{r} + r + r^3\right) = -a$

and from (3), $\alpha^3\left(\frac{1}{r^3} + \frac{1}{r} + r + r^3\right) = -c$

on division, $\alpha^2 = c/a$

$\Rightarrow \alpha^4 = c^2/a^2$

$\Rightarrow d = c^2/a^2 \Rightarrow a^2 d = c^2 \Rightarrow c^2 = a^2 d$

(*) If the equation $x^3 - rx^2 + rx - 4 = 0$ has two roots reciprocal to each other, find the third root and the value of r.

Solution: Let the roots be $\alpha, \frac{1}{\alpha}$ and β.

Then $\alpha + \frac{1}{\alpha} + \beta = r$ ——— (1)

$\alpha \cdot \frac{1}{\alpha} + \alpha\beta + \frac{1}{\alpha}\beta = r$ ——— (2)

$\alpha \cdot \frac{1}{\alpha}\beta = 4$ ——— (3)

From (3), $\beta = 4$

using this value of β in (1), $\alpha + \frac{1}{\alpha} + 4 = r$

$\Rightarrow \alpha + \frac{1}{\alpha} = r - 4$

Now from (2), $1 + \beta\left(\alpha + \frac{1}{\alpha}\right) = r$

$\Rightarrow 1 + \beta(r-4) = r$ as $\alpha + \frac{1}{\alpha} = r - 4$

$\Rightarrow 1 + 4(r-4) = r$

$\Rightarrow 1 + 4r - 16 = r$

$\Rightarrow 3r - 15 = 0$

Hence the third root is 4 $\Rightarrow r = \frac{15}{3}$ and the value of r is 5.

(*) If one of the roots of the equation $x^3+px^2+qx+r=0$ is equal to the sum of the other two roots, then prove that $p^3+8r=4pq$.

Solution: Let the roots of the equation be α, β and $(\alpha+\beta)$.

$\alpha+\beta+(\alpha+\beta)=-p$ i.e. $\alpha+\beta=-p/2$ ——— ①

Then $\alpha.\beta+\alpha(\alpha+\beta)+\beta(\alpha+\beta)=q$

$\Rightarrow \alpha\beta+(\alpha+\beta)^2=q$ ——— ②

$\Rightarrow \alpha\beta+(-p/2)^2=q$ using ①

$\Rightarrow \alpha\beta+p^2/4=q$

and $\alpha\beta(\alpha+\beta)=-r$

So $(q-p^2/4).(-p/2)=-r$

$\Rightarrow \left(\frac{4q-p^2}{4}\right).p/2=r$

$\Rightarrow p(4q-p^3)=8r \Rightarrow 4pq-p^3=8r$

$\Rightarrow p^3+8r=4pq$

(*) Find the relation between the coefficients of the equation $x^3+ax^2+bx+c=0$, when the roots α,β are connected by the relation $1+\alpha\beta=0$

Solution: Let the roots of the equation be α,β and γ.

Then $\alpha+\beta+\gamma=-a$ ——— ①

$\alpha\beta+\alpha\gamma+\beta\gamma=b$ ——— ②

$\alpha\beta\gamma=-c$ ——— ③

and $\alpha\beta=-1$ ——— ④

From ③ & ④, $\gamma=c$

From ①, $\alpha+\beta=-a-c$

Also from ②, $-1+\gamma(\alpha+\beta)=b$

$\Rightarrow -1+c(-a-c)=b$

$\Rightarrow -1-ac-c^2=b$.

$\Rightarrow c^2+ca+b+1=0$ is the required condition

(*) Find the condition if the sum of two roots of the equation $x^4-px^3+qx^2-rx+s=0$ is equal to the sum of the other two.

Solution: Let $\alpha,\beta,\gamma,\delta$ be the roots of the equation.

Then $\alpha+\beta+\gamma+\delta=p$ ——— ①

$\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta=q$ ——— ②

$\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta=r$ ——— ③

$\alpha\beta\gamma\delta=s$ ——— ④

Also $\alpha+\beta=\gamma+\delta$ ——— (5)

from (1) & (5), $\alpha+\beta=\gamma+\delta=p/2$

from (2), $(\alpha+\beta)(\gamma+\delta)+\alpha\beta+\gamma\delta=q$ ——— (6)

from (3), $\alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta)=r$ ——— (7)

Putting, $\alpha+\beta=\gamma+\delta=p/2$ in (6) & (7), we get

$$\frac{p}{2}\cdot\frac{p}{2}+\alpha\beta+\gamma\delta=q \Rightarrow \alpha\beta+\gamma\delta=q-p^2/4$$

and $\alpha\beta\cdot p/2+\gamma\delta\cdot p/2=r$

$$\Rightarrow p/2(\alpha\beta+\gamma\delta)=r \Rightarrow \alpha\beta+\gamma\delta=\frac{2r}{p}$$

Finally $q-\frac{p^2}{4}=\frac{2r}{p}$

$$\Rightarrow \frac{4q-p^2}{4}=\frac{2r}{p} \Rightarrow 4pq-p^3=8r$$

$\Rightarrow p^3-4pq+8r=0$ is the required condition.

(*) The product of two roots of the equation $x^4+x^3+2x^2+2x+4=0$ is equal to the product of the other two roots. Find the roots.

Solution: Let the roots of the given equation be $\alpha,\beta,\gamma,\delta$.

Also $\alpha+\beta+\gamma+\delta=-1$ ——— (1)

$(\alpha+\beta)(\gamma+\delta)+\alpha\beta+\gamma\delta=2$ ——— (2)

$(\alpha+\beta)\gamma\delta+(\gamma+\delta)\alpha\beta=-2$ ——— (3)

$\alpha\beta\gamma\delta=4$ ——— (4)

and $\alpha\beta=\gamma\delta$ ——— (5)

from (4) & (5), $\alpha\beta=\gamma\delta=2$ ——— (6)

from (2) & (6), $(\alpha+\beta)(\gamma+\delta)=2-4=-2$ ——— (7)

now from (1) & (7), we can say that $(\alpha+\beta)$ and $(\gamma+\delta)$ are the roots of the equation $y^2+y-2=0$ i.e, $y=1,-2$

Then $\alpha+\beta=-2$, $\gamma+\delta=1$

Again since $\alpha+\beta=-2$ and $\alpha\beta=2$ then $\alpha=-1+i$, $\beta=-1-i$

Also $\gamma+\delta=1$ and $\gamma\delta=2$ gives $\gamma=\frac{1}{2}+\frac{\sqrt{7}}{2}i$ and $\delta=\frac{1}{2}-\frac{\sqrt{7}}{2}i$

Hence the roots are $-1\pm i$, $\frac{1}{2}\pm\frac{\sqrt{7}}{2}i$

(*) Solve the equation $x^4+2x^3-21x^2-22x+40=0$, if the sum of two roots be equal to the sum of the other two.

Solution: Let the roots of the given equation be $\alpha, \beta, \gamma, \delta$.

It is given by $\alpha+\beta=\gamma+\delta$ ——— (1)

Again $\alpha+\beta+\gamma+\delta=-2$ ——— (2)

$(\alpha+\beta)(\gamma+\delta)+\alpha\beta+\gamma\delta=-21$ ——— (3)

$(\alpha+\beta)\gamma\delta+(\gamma+\delta)\alpha\beta=22$ ——— (4)

and $\alpha\beta\gamma\delta=40$ ——— (5)

from (1) & (2), $\alpha+\beta=\gamma+\delta=-1$ ——— (6)

and so from (3), $\alpha\beta+\gamma\delta=22$ ——— (7)

Let $\alpha\beta=x$ and $\gamma\delta=y$

Then from (7) & (5), $x+y=22$ and $xy=40$

So $x=-2$, $y=-20$

i.e. $\alpha\beta=-2$, $\gamma\delta=-20$

Now $(\alpha-\beta)^2=(\alpha+\beta)^2-4\alpha\beta=(-1)^2-4\cdot(-2)=1+8=9$

$\therefore \alpha-\beta=3$ and $\alpha+\beta=-1$

i.e. $\alpha=1$, $\beta=-2$

Again $(\gamma-\delta)^2=(\gamma+\delta)^2-4\gamma\delta$

$=(-1)^2-4\cdot(-20)=1+80=81$

$\therefore \gamma-\delta=9$

Then $\gamma-\delta=9$ and $\gamma+\delta=-1$ gives $\gamma=4$, $\delta=-5$

Hence the roots are $-5, -2, 1, 4$

(*) If the roots of the equation $x^4-3x^3-5x^2+9x-2=0$ is $2-\sqrt{3}$, find the other roots.

Solution: The coefficients of the given equation are all rational. So irrational roots will occur in conjugate pairs. Since $2-\sqrt{3}$ is one root of the equation, $2+\sqrt{3}$ will be another root. The given equation being of degree 4, has four roots.

Let the roots be $2-\sqrt{3}, 2+\sqrt{3}, \alpha$ & β.

Then $2-\sqrt{3}+2+\sqrt{3}+\alpha+\beta=3$

$\therefore \alpha+\beta=-1 \quad \therefore \beta=-1-\alpha$

Again $(2-\sqrt{3})(2+\sqrt{3})\alpha\beta = -2$

$\Rightarrow (4-3)\alpha\beta = -2$

$\Rightarrow \alpha\beta = -2$

$\Rightarrow \alpha(-1-\alpha) = -2 \Rightarrow -\alpha-\alpha^2 = -2$

$\Rightarrow \alpha^2+\alpha-2 = 0$

$\therefore \alpha = 1, -2$

If $\alpha = 1$, $\beta = -2$ and when $\alpha = -2$, $\beta = 1$

Hence the roots of the equation are $2 \pm \sqrt{3}, 1, -2$