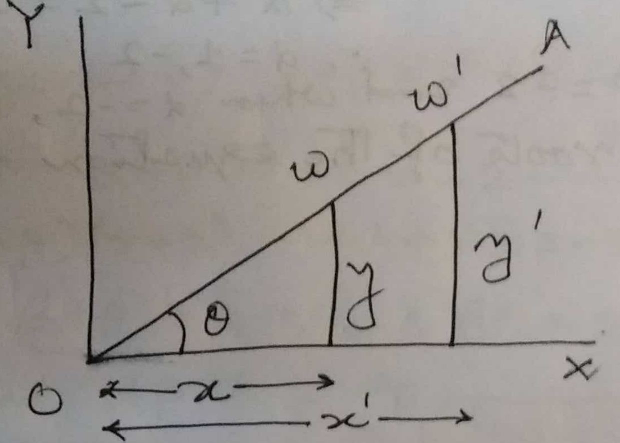
(*) If w and w' be two Complex no^s. with a non-real ratio, Then show that w and w' can not lie on a straight line through the origin.

Solution:



Let OX and OY be the real and imaginary axes. Let w and w' be on the line OA.

Let $w = x+iy$ and $w' = x'+iy'$ be two Complex no^s.

$$\therefore \tan\theta = y/x = y'/x' = m$$

$$\therefore y = mx, \; y' = mx'$$

$$\therefore \frac{w}{w'} = \frac{x+iy}{x'+iy'} = \frac{x+imx}{x'+imx'} = \frac{x(1+im)}{x'(1+im)} = \frac{x}{x'} = \text{real}$$

Therefore this a contradiction that the points w and w' can not lie on the same straight line throught the origin.

(*) Prove That if ratio $\frac{z-i}{z-1}$ is purely imaginary. then the point z lies on the circle, whose radius is $\frac{1}{\sqrt{2}}$ and the centre is the point $\frac{1}{2}(1+i)$.

Solution: Let $z = x+iy$ be a Complex no.

we have $\frac{z-i}{z-1} = \frac{x+iy-i}{x+iy-1} = \frac{x+i(y-1)}{(x-1)+iy}$

$$= \frac{\{x+i(y-1)\}\{(x-1)-iy\}}{\{(x-1)+iy\}\{(x-1)-iy\}}$$

$$= \frac{x(x-1)+y(y-1)+i\{(x-1)(y-1)-xy\}}{(x-1)^2-i^2y^2}$$

$$= \frac{x^2 - x + y^2 - y + i\{xy - x - y + 1 - xy\}}{(x-1)^2 + y^2}$$

$$= \frac{(x^2 + y^2 - x - y) + i(1 - x - y)}{(x-1)^2 + y^2} = \frac{x^2+y^2-x-y}{(x-1)^2+y^2} + i\frac{1-x-y}{(x-1)^2+y^2}$$

If it is purely imaginary, then

$$\frac{x^2+y^2-x-y}{(x-1)^2+y^2} = 0 \quad \text{or} \quad x^2+y^2-x-y=0$$

$$\text{or } x^2 - 2.x.\frac{1}{2} + \left(\frac{1}{2}\right)^2 + y^2 - 2.y.\frac{1}{2} + \left(\frac{1}{2}\right)^2 - \frac{1}{4} - \frac{1}{4} = 0$$

$$\text{or } \left(x-\frac{1}{2}\right)^2 + \left(y-\frac{1}{2}\right)^2 = \frac{1}{4} + \frac{1}{4} = \frac{2}{4} = \frac{1}{2} = \left(\frac{1}{\sqrt{2}}\right)^2$$

$$\text{i.e. } \left(x-\frac{1}{2}\right)^2 + \left(y-\frac{1}{2}\right)^2 = \left(\frac{1}{\sqrt{2}}\right)^2$$

Therefore the point (x,y) lies on the circle, whose Centre is $\left(\frac{1}{2},\frac{1}{2}\right)$ and radius $\frac{1}{\sqrt{2}}$. But the point $\left(\frac{1}{2},\frac{1}{2}\right)$ Corresponds to the Complex no. $\frac{1}{2}+\frac{1}{2}i$. Hence the point $z = x+iy$ lies on the circle whose centre is $\frac{1}{2}(1+i)$ and radius is $\frac{1}{\sqrt{2}}$.

* If c is a real, $b = p+iq$, $z = x+iy$ and $\bar{b}, \bar{z}$ are Conjugates of b, z respectively, Show that the equation $b(z+\bar{z}) + \bar{b}(\bar{z}-z) + c = 0$ represents two straight lines in xy plane and find the angle between them.

Solution: As $z = x+iy$ and $b = p+iq$ then $\bar{z} = x-iy$ and $\bar{b} = p-iq$.

With these values, the given equation takes the form

$$(p+iq)\{(x+iy)+(x-iy)\} + (p-iq)\{(x-iy)-(x+iy)\} + c = 0$$

$$\text{or } (p+iq).2x + (p-iq)(-2iy) + c = 0$$

$$\text{or } 2px + 2iqx - 2ipy - 2qy + c = 0$$

$$\text{or } (2px - 2qy + c) + i(2qx - 2py) = 0$$

So that $2px - 2qy + c = 0$

and $2qx - 2py = 0$

Hence the given equation represents the two straight lines in the xy-plane, the equations of the lines being

$$2px - 2qy + c = 0$$
$$2qx - 2py = 0$$

and the gradients of the two lines are p/q and q/p respectively.

If θ be the angle between the two lines then we have

$$\tan\theta = \frac{p/q - q/p}{1 + p/q \cdot q/p} = \frac{\frac{p^2 - q^2}{pq}}{2} = \frac{p^2 - q^2}{2pq}.$$

$$\therefore\ \theta = \tan^{-1}\left(\frac{p^2 - q^2}{2pq}\right)$$

1.9. De Moivre's theorem.

For all integral values of n, the value of $(\cos\theta + i\sin\theta)^n$ *is* $(\cos n\theta + i\sin n\theta)$ *and for all fractional values of n, one of the values of* $(\cos\theta + i\sin\theta)^n$ *is* $(\cos n\theta + i\sin n\theta)$.

Case. I. *When n is a positive integer.*

This is proved by the *method of induction.*

We have $(\cos\theta + i\sin\theta)^1 = \cos\theta + i\sin\theta$

and

$$(\cos\theta + i\sin\theta)^2 = \cos^2\theta - \sin^2\theta + 2i\sin\theta\cos\theta$$

$$= \cos 2\theta + i\sin 2\theta.$$

Therefore the theorem is true when $n = 1$ and 2.

Let us assume that the theorem is true for a particular value m of n (m being a positive integer).

Then we have $(\cos\theta + i\sin\theta)^m = \cos m\theta + i\sin m\theta$.

Multiplying both sides by $(\cos\theta + i\sin\theta)$, we get

$$(\cos\theta + i\sin\theta)^{m+1} = (\cos m\theta + i\sin m\theta)(\cos\theta + i\sin\theta)$$
$$= (\cos m\theta\cos\theta - \sin m\theta\sin\theta)$$
$$+ i(\sin m\theta\cos\theta + \cos m\theta\sin\theta)$$
$$= \cos(m+1)\theta + i\sin(m+1)\theta.$$

Thus, if the theorem be true for $n = m$, it is also true for $n = m+1$. But it is proved that the theorem is true for $n = 2$. So it is also true for $n = 2+1 = 3$, then for $n = 3+1 = 4$ and so on. Hence the theorem is true for all positive integral values of n.

Case II. *When n is a negative integer.*

Let $n = -m$, where m is a positive integer.

Then $(\cos\theta + i\sin\theta)^n = (\cos\theta + i\sin\theta)^{-m}$

$$= \frac{1}{(\cos\theta + i\sin\theta)^m} = \frac{1}{\cos m\theta + i\sin m\theta}, \qquad [\text{by case I}]$$

$$= \frac{\cos m\theta - i\sin m\theta}{(\cos m\theta + i\sin m\theta)(\cos m\theta - i\sin m\theta)}$$

$$= \frac{\cos m\theta - i\sin m\theta}{\cos^2 m\theta + \sin^2 m\theta} = \cos m\theta - i\sin m\theta$$

$$= \cos(-n\theta) - i\sin(-n\theta) = \cos n\theta + i\sin n\theta.$$

Thus the theorem is true for all negative integral values of n.

Case III. *When n is a fraction, positive or negative.*

Let $n = \frac{p}{q}$, where q is a positive integer and p is any integer, positive or negative.

Then, by case I,

$$\left(\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}\right)^q = \cos q\cdot\frac{\theta}{q} + i\sin q\cdot\frac{\theta}{q} = \cos\theta + i\sin\theta.$$

Extracting the q-th root of both sides, we see that one of the values of $(\cos\theta + i\sin\theta)^{\frac{1}{q}}$ is $\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}$.

Raising both sides to p-th power, we can say that one of the values of $\{(\cos\theta + i\sin\theta)^{\frac{1}{q}}\}^p$, that is, of $(\cos\theta + i\sin\theta)^{\frac{p}{q}}$,
that is, of $(\cos\theta + i\sin\theta)^n$ is

$$\left(\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}\right)^p = \cos\frac{p}{q}\theta + i\sin\frac{p}{q}\theta = \cos n\theta + i\sin n\theta,$$

[by case I and case II].

Hence one of the values of $(\cos\theta + i\sin\theta)^n$ is $(\cos n\theta + i\sin n\theta)$.

Thus the theorem holds good for all rational values of n.

This theorem is known as *De Moivre's theorem*, named after its discoverer Prof. Abraham de Moivre.

Cor. 1. $(\cos\theta - i\sin\theta)^n = \{\cos(-\theta) + i\sin(-\theta)\}^n$

$$= \{(\cos\theta + i\sin\theta)^{-1}\}^n = (\cos\theta + i\sin\theta)^{-n}$$

$$= \cos(-n\theta) + i\sin(-n\theta) = \cos n\theta - i\sin n\theta.$$

Cor. 2. $(\cos m\theta + i\sin m\theta)^n = \{(\cos\theta + i\sin\theta)^m\}^n$

$$= (\cos\theta + i\sin\theta)^{mn}$$

$$= \cos mn\theta + i\sin mn\theta$$

$$= (\cos n\theta + i\sin n\theta)^m.$$

Note. The theorem actually holds for all real values of n. If n be irrational, then the total number of values of $(\cos\theta + i\sin\theta)^n$ will be infinite, but one of the values of $(\cos\theta + i\sin\theta)^n$ is $(\cos n\theta + i\sin n\theta)$.