Study Material – Sem. 1 – C2T

– Oscillations – Dr. T. Kar – Class 4

B. Velocity resonance

The steady state displacement for a system executing forced vibration is given by $x = A\cos(\omega t - \alpha)$

$\therefore$ The steady state velocity is given by $\left|\frac{dx}{dt}\right| = v = A\omega\sin(\omega t - \alpha)$

$$= \frac{(F/m)\,\omega\sin(\omega t-\alpha)}{\sqrt{(\omega_0^2-\omega^2)^2+4b^2\omega^2}}$$

$$= \frac{(F/m)\sin(\omega t-\alpha)}{\left[4b^2+\left(\frac{\omega_0^2}{\omega}-\omega\right)^2\right]^{1/2}} = \frac{F\sin(\omega t-\alpha)}{\left[4b^2m^2+\left(\frac{m\omega_0^2}{\omega}-m\omega\right)^2\right]^{1/2}}$$

$$= \frac{F\sin(\omega t-\alpha)}{\left[k^2+\left(m\omega-\frac{s}{\omega}\right)^2\right]^{1/2}} \qquad \left[\because \omega_0^2=\frac{s}{m},\ 2b=\frac{k}{m}\right] \longrightarrow (1)$$

The velocity amplitude is $V = \frac{F}{\left[k^2+\left(m\omega-\frac{s}{\omega}\right)^2\right]^{1/2}} \longrightarrow (2)$

As ω varied, V attains its maximum value, $V_m = \frac{F}{k}$ when $m\omega - \frac{s}{\omega} = 0$.

Hence, the velocity resonance occurs when,

$m\omega = \frac{s}{\omega}$; $\omega^2 = \frac{s}{m} = \omega_0^2$

$\therefore \omega = \omega_0 \longrightarrow (3)$, i.e, when the forcing frequency equals the undamped natural frequency of the forced system.

The kinetic energy of the forced system is $\frac{1}{2}mv^2$. The velocity resonance occurs simultaneously with the maximum kinetic energy of the driven system as the frequency of the driver is varied. Hence, the velocity resonance is sometimes referred to as the energy resonance. The maximum kinetic energy is $E_m = \frac{1}{2}mV_m^2 = \frac{mF^2}{2k^2} \longrightarrow (4)$

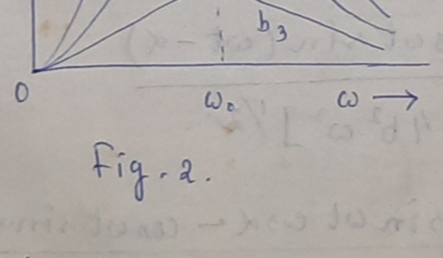

Fig. 2.

Fig. 2 gives the variation of the velocity amplitudes with the angular frequency of the driver at three different decay constants b_1, b_2, b_3 respectively. At $\omega = 0$, $V = 0$. The maximum value of V occurs at $\omega = \omega_0$ for all values of

b. The maximum velocity amplitude increases as k i.e, b decreases. For zero damping, the velocity amplitude is infinite at $\omega = \omega_0$.

Sharpness of Resonance

The power of the driver is the rate at which it works. In the steady state, the instantaneous power of the driver in forced vibration is —

$$P = (f \cos\omega t)\left(\frac{dx}{dt}\right)$$

In steady state, $x = A\cos(\omega t - \alpha)$

where, $A = \dfrac{f/m}{[(\omega_0^2 - \omega^2)^2 + 4b^2\omega^2]^{1/2}}$ & $\sin\alpha = \dfrac{2bA\omega}{(f/m)}$

$$\therefore \frac{dx}{dt} = -A\omega \sin(\omega t - \alpha)$$

$$\therefore P = -\frac{(f/m) f\omega \cos\omega t \sin(\omega t - \alpha)}{[(\omega_0^2 - \omega^2)^2 + 4b^2\omega^2]^{1/2}}$$

$$= -\frac{f^2\omega \cos\omega t\,[\sin\omega t \cos\alpha - \cos\omega t \sin\alpha]}{[(m\omega_0^2 - m\omega^2)^2 + 4b^2m^2\omega^2]^{1/2}}$$

$$= \frac{F^2\omega\cos\omega t\,[\cos\omega t\sin\alpha - \sin\omega t\cos\alpha]}{\left[\left(\frac{s}{m}m - m\omega^2\right)^2 + \frac{k^2}{m^2}m^2\omega^2\right]^{1/2}}$$

$$= \frac{F^2\omega\,[\cos^2\omega t\sin\alpha - \sin\omega t\cos\omega t\cos\alpha]}{[k^2\omega^2 + (m\omega^2 - s)^2]^{1/2}}$$

$$= \frac{F^2\omega\,[\cos^2\omega t\sin\alpha - \frac{1}{2}\sin 2\omega t\cos\alpha]}{\omega\left[k^2 + \left(m\omega - \frac{s}{\omega}\right)^2\right]^{1/2}}$$

$$= \frac{F^2\,[\sin\alpha\cos^2\omega t - \frac{1}{2}\sin 2\omega t\cos\alpha]}{[k^2 + (m\omega - s/\omega)^2]^{1/2}}$$

The average power over a complete cycle is —

$P_{av} = \frac{1}{T}\int_0^T P\,dt$, where, $T = \frac{2\pi}{\omega}$ is the period of the driving force.

$$\therefore P_{av} = \left(\frac{1}{T}\right)\frac{F^2\sin\alpha\,(T/2)}{\left[k^2 + \left(m\omega - \frac{s}{\omega}\right)^2\right]^{1/2}} = \frac{F^2\sin\alpha}{2\left[k^2 + \left(m\omega - \frac{s}{\omega}\right)^2\right]^{1/2}}$$

Now, $\sin\alpha = \left(\frac{2b\omega m}{F}\right)\frac{F/m}{[(\omega_0^2 - \omega^2)^2 + 4b^2\omega^2]^{1/2}}$

$$= \frac{k\omega}{m\,[(\omega_0^2 - \omega^2)^2 + 4b^2\omega^2]^{1/2}} = \frac{k\omega}{[(m\omega_0^2 - m\omega^2)^2 + k^2\omega^2]^{1/2}}$$

or, $\sin\alpha = \dfrac{k\omega}{\left[\left(m\frac{s}{m} - m\omega^2\right)^2 + k^2\omega^2\right]^{1/2}}$

$= \dfrac{k\omega}{\left[(m\omega^2 - s)^2 + k^2\omega^2\right]^{1/2}}$

$= \dfrac{k\omega}{\omega\left[\left(m\omega - \frac{s}{\omega}\right)^2 + k^2\right]^{1/2}}$

$= \dfrac{k}{\left[k^2 + \left(m\omega - \frac{s}{\omega}\right)^2\right]^{1/2}}$

$\therefore P_{av} = \dfrac{F^2 \sin\alpha}{2\left[k^2 + \left(m\omega - \frac{s}{\omega}\right)^2\right]^{1/2}}$

$= \dfrac{kF^2}{2\left[k^2 + \left(m\omega - \frac{s}{\omega}\right)^2\right]} \longrightarrow (1)$

At resonance, $m\omega - \frac{s}{\omega} = 0$ or, $\omega^2 = \frac{s}{m} = \omega_0^2$

$\therefore \omega = \omega_0$

Therefore, $(P_{av})_{resonance} = \dfrac{F^2}{2k} \longrightarrow (2)$

If P_{av} is plotted against the angular frequency ω of the driver, clearly, P_{av} will attain its maximum value of $\frac{F^2}{2k}$ at the resonance frequency

$\omega = \omega_0 = \sqrt{\frac{s}{m}}$ and will fall on either side of this frequency (fig. 1). When k is small, the resonance peak of P_{av} is high and vice-versa. The smaller the damping, the greater the drop of P_{av} as ω differs a little from ω_0.

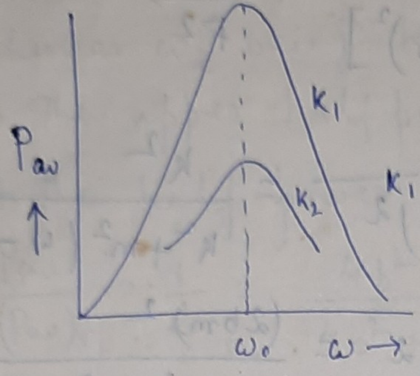

fig. 1.

In other words, resonance is sharp when damping is small. For large damping, the resonance is broad or flat.

The sharpness of resonance gives the rapidity with which the average power P_{av} supplied by the drivers drops off as ω differs from its value at resonance.

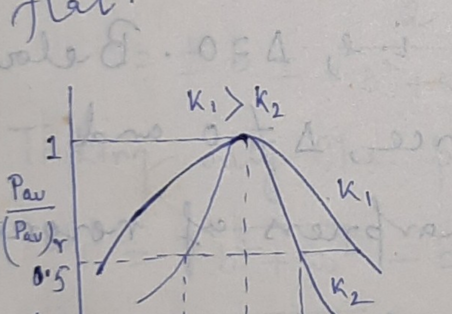

fig. 2.

The higher sharpness of resonance for low damping is clearly revealed in fig. 2, where the normalised power $P_{av}/(P_{av})_r$ is plotted against ω for two different

values of k.
We have,

$$\frac{P_{av}}{(P_{av})_r} = \frac{F^2 k}{2\left[k^2+\left(m\omega-\frac{s}{\omega}\right)^2\right]} \times \frac{2k}{F^2}$$

$$= \frac{k^2}{k^2+\left(m\omega-\frac{s}{\omega}\right)^2} = \frac{k^2}{k^2+m^2\left(\omega-\frac{s}{m\omega}\right)^2}$$

$$= \frac{k^2}{k^2+m^2\left(\omega-\frac{\omega_0^2}{\omega}\right)^2} = \frac{(2bm)^2}{(2bm)^2+m^2\omega_0^2\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right)^2}$$

$$= \frac{4b^2}{4b^2+\omega_0^2\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right)^2} = \frac{4b^2}{4b^2+\omega_0^2\Delta^2} \longrightarrow (3)$$

where, $\Delta = \frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} \longrightarrow (4)$

$\frac{P_{av}}{(P_{av})_r} = 1$ at $\omega = \omega_0$ i.e, $\Delta = 0$. Below or above resonance, $\Delta \neq 0$ and

$\frac{P_{av}}{(P_{av})_r} < 1$. The sharpness of resonance is quantitatively defined as reciprocal of $|\Delta|$ at which $P_{av}/(P_{av})_r = 1/2$. fig. 2 shows that for a given value of k or b, there are two values of ω, namely ω_1 and ω_2 for which P_{av} is half its value at resonance. These frequencies are called the half-power frequencies. ω_1 being less than ω_0, is termed the lower half power frequency, whereas ω_2 being higher than ω_0, is known as the upper half power frequency.

At the half-power frequencies,

$$\frac{P_{av}}{(P_{av})_r} = \frac{1}{2} = \frac{4b^2}{4b^2 + \omega_0^2 \Delta^2}$$

or, $4b^2 + \omega_0^2 \Delta^2 = 8b^2$ or, $\omega_0^2 \Delta^2 = 4b^2$

or, $\boxed{\omega_0 \Delta = \pm 2b}$ or, $\omega_0 \left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} \right) = \pm 2b$

or, $\omega - \frac{\omega_0^2}{\omega} = \pm 2b$ or, $\omega^2 \mp 2b\omega - \omega_0^2 = 0$

or, $$\omega = \frac{-[\mp 2b] \pm \sqrt{4b^2 + 4\omega_0^2}}{2}$$

$$= \pm b \pm \sqrt{b^2 + \omega_0^2}$$

Taking only the positive roots, we get

$$\omega_1 = \sqrt{b^2 + \omega_0^2} - b \longrightarrow (5)$$

$$\omega_2 = \sqrt{b^2 + \omega_0^2} + b \longrightarrow (6)$$

Therefore, the sharpness of resonance,

$$S_r = \frac{1}{|\Delta|} = \frac{\omega_0}{2b} \quad \text{as } \Delta = \pm \frac{2b}{\omega_0}$$

from eq. (5) & (6), we get

$$2b = \omega_2 - \omega_1$$

$$\therefore S_r = \frac{\omega_0}{\omega_2 - \omega_1} \longrightarrow (7)$$

Therefore, the sharpness of resonance, S_r is directly proportional to the natural frequency of the system and inversely to the damping constant b.