SEMESTER-I

HONOURS

CORE COURSE---C 1T

UNIT-I (MARKS-16)

UNIT-I

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Syllabus for Unit-I: Hyperbolic functions, higher order derivatives, Leibnitz rule and its applications to problems of type $e^{ax+b}\sin x$, $e^{ax+b}\cos x$, $(ax+b)^n\sin x$, $(ax+b)^n\cos x$, concavity and inflection points, envelopes, asymptotes, curve tracing in cartesian coordinates, tracing in polar coordinates of standard curves, L'Hospital's rule, applications in business, economics and life sciences.

HIGHER ORDER DERIVATIVES (SUCCESSIVE DIFFERENTIATION)

Let a function of x is given as follows and we are to differentiate that function.

$y = f(x) = x^3$. Then $\frac{dy}{dx} = 3x^2$. This is called differential co-efficient of **first order** of the function $y = f(x) = x^3$ with respect to x. It it is also denoted by $\frac{d}{dx}(y)$ or by $D(y)$ or by $f'(x)$ or by y_1. That is, $y_1 = f'(x) = D(y) = \frac{d}{dx}(y) = \frac{dy}{dx} = 3x^2$ ....(i). If we differentiate the function in (i) once again with respect to x, we get $y_2 = f''(x) = D^2(y) = \frac{d^2}{dx^2}(y) = \frac{d^2y}{dx^2} = 6x$ ....(ii).

Differential co-efficient given by (i) is known as **first order differential co-efficient of the given function** $y = f(x) = x^3$**.**

Differential co-efficient given by (ii) is known as **second order differential co-efficient of the given function** $y = f(x) = x^3$**.**

If we differentiate the function in (ii) once again with respect to x, we get the **third order differential co-efficient of the given function** $y = f(x) = x^3$ **and so on.**

So, the higher order differential co-efficient of a function can be obtained and all the differential co-efficients of order greater than 1 are known as **higher order differential co-efficients or higher order derivatives.**

n^{th} ORDER DERIVATIVE OF SOME STANDARD FUNCTIONS

(1) $y = x^n$, **where** n **is a positive integer.**

$$y_1 = nx^{n-1}$$

$$y_2 = n(n-1)x^{n-2}$$

$$y_3 = n(n-1)(n-2)x^{n-3}$$

. . . .

. . . . **and proceeding in a similar manner, we have**

$$y_{n-1} = n(n-1)(n-2)\cdots\cdots\{n-(n-1-1)\}x^{n-(n-1)}$$

$$y_n = n(n-1)(n-2)\cdots\cdots\{n-(n-1)\}x^{n-n}$$ **. That is,**

$$y_n = n(n-1)(n-2)\cdots\cdots 3\cdot 2\cdot 1\cdot x^{n-n} = n(n-1)(n-2)\cdots\cdots 3\cdot 2\cdot 1$$

$$\boxed{y_n = n!}$$

(2) $y = (ax+b)^m$, **where** m **is any number.**

$$y_1 = ma(ax+b)^{m-1}$$

$$y_2 = m(m-1)a^2(ax+b)^{m-2}$$

$$y_3 = m(m-1)(m-2)a^3(ax+b)^{m-3}$$

. . . .

. . . . **and proceeding in a similar manner, we have**

$$y_{n-1} = m(m-1)(m-2)\cdots\cdots\{m-(n-1-1)\}(ax+b)^{m-(n-1)}$$

$$y_n = m(m-1)(m-2)\cdots\cdots\{m-(n-1)\}\cdot a^n\cdot(ax+b)^{m-n}$$ **. That is,**

$$y_n = m(m-1)(m-2)\cdots\cdots(m-n+1)\cdot a^n\cdot(ax+b)^{m-n}.$$

$$\boxed{y_n = m(m-1)(m-2)\cdots\cdots(m-n+1)\cdot a^n\cdot(ax+b)^{m-n}}$$

(3) $y = e^{ax}$

$$y_1 = ae^{ax}$$
$$y_2 = a^2e^{ax}$$
$$y_3 = a^3e^{ax}$$

. . . .

. . . . **and proceeding in a similar manner, we have**

$$y_{n-1} = a^{n-1}e^{ax}$$

$y_n = a^n e^{ax}$ **. That is,**

$$\boxed{y_n = a^n e^{ax}}$$

(4) $y = \dfrac{1}{x+a}$, **or** $y = \dfrac{1}{x-a}$

Let us consider the first one, i.e.,

$$y = \frac{1}{x+a} = (x+a)^{-1}$$
$$y_1 = (-1)(x+a)^{-2}$$
$$y_2 = (-1)(-2)(x+a)^{-3}$$
$$y_3 = (-1)(-2)(-3)(x+a)^{-4}$$

. . . .

. . . . **and proceeding in a similar manner, we have**

$y_n = (-1)(-2)(-3)\cdots\cdots(-n)(x+a)^{-(n+1)}$ **. That is,**

$$y_n = (-1)^n 1\cdot 2\cdot 3\cdots\cdots n(x+a)^{-(n+1)}$$

$$\boxed{y_n = \frac{(-1)^n n!}{(x+a)^{(n+1)}}}$$

Similarly, if we consider the second one, i.e., $y = \dfrac{1}{x-a}$ **then we get**

$$\boxed{y_n = \frac{(-1)^n n!}{(x-a)^{(n+1)}}}$$

.

(5) $y = \log(x+a)$, **or** $y = \log(x-a)$

Let us consider the first one, i.e.,

$y = \log(x+a)$, **then**

$$y_1 = \frac{1}{x+a} = (x+a)^{-1}$$

$$y_2 = (-1)(x+a)^{-2}$$

$$y_3 = (-1)(-2)(x+a)^{-3}$$

$$y_4 = (-1)(-2)(-3)(x+a)^{-4}$$

. . . .

. . . . **and proceeding in a similar manner, we have**

$y_n = (-1)(-2)(-3)\cdots\cdots\{-(n-1)\}(x+a)^{-n}$. **That is,**

$$y_n = (-1)^{n-1} 1\cdot 2\cdot 3\cdots\cdots(n-1)(x+a)^{-n}$$

$$\boxed{y_n = \frac{(-1)^{n-1}(n-1)!}{(x+a)^n}}$$

Similarly, if we consider the second one, i.e., $y = \log(x-a)$ **then we get**

$$\boxed{y_n = \frac{(-1)^{n-1}(n-1)!}{(x-a)^n}}$$

(6) $y = \sin(ax+b)$, **or** $y = \cos(ax+b)$

Let us consider the first one, i.e.,

$y = \sin(ax+b)$, **then**

$$y_1 = a\cos(ax+b) = a\sin\left(\frac{\pi}{2}+(ax+b)\right)$$

$$y_2 = a^2\cos\left(\frac{\pi}{2}+(ax+b)\right) = a^2\sin\left(\frac{\pi}{2}+\left[\frac{\pi}{2}+ax+b\right]\right)$$

$$= a^2\sin\left(\frac{2\cdot\pi}{2}+ax+b\right)$$

. . . .

. . . .

. . . . and proceeding in a similar manner, we have

$$y_n = a^n\sin\left(\frac{n\cdot\pi}{2}+ax+b\right).$$

That is,

$$\boxed{y_n = a^n\sin\left(\frac{n\pi}{2}+ax+b\right)}$$

Similarly, if we consider the second one, i.e., $y=\cos(ax+b)$ **then we get**

$$\boxed{y_n = a^n\cos\left(\frac{n\pi}{2}+ax+b\right)}$$

(7) $y = \dfrac{1}{x^2-a^2}$

$$y = \frac{1}{x^2-a^2} = \frac{1}{(x+a)(x-a)} = \frac{1}{2a}\left[\frac{1}{x-a}-\frac{1}{x+a}\right]$$

$= \dfrac{1}{2a}\left[\dfrac{1}{x-a}\right]-\dfrac{1}{2a}\left[\dfrac{1}{x+a}\right] = \dfrac{1}{2a}U-\dfrac{1}{2a}V$ **where** $U=\dfrac{1}{x-a}$ **and** $V=\dfrac{1}{x+a}$

Therefore, $y_n = \dfrac{1}{2a}U_n-\dfrac{1}{2a}V_n = \dfrac{1}{2a}\dfrac{(-1)^n n!}{(x-a)^{(n+1)}}-\dfrac{1}{2a}\dfrac{(-1)^n n!}{(x+a)^{(n+1)}}$

$$\boxed{y_n = \frac{(-1)^n n!}{2a}\left[\frac{1}{(x-a)^{(n+1)}}-\frac{1}{(x+a)^{(n+1)}}\right]}$$

(8) $y = \dfrac{1}{x^2 + a^2}$

$$y = \frac{1}{x^2 + a^2} = \frac{1}{(x+ia)(x-ia)} = \frac{1}{2ia}\left[\frac{1}{x-ia} - \frac{1}{x+ia}\right]$$

$$= \frac{1}{2ia}\left[\frac{1}{x-ia}\right] - \frac{1}{2ia}\left[\frac{1}{x+ia}\right] = \frac{1}{2ia}U - \frac{1}{2ia}V$$

where $U = \dfrac{1}{x-ia}$ **and** $V = \dfrac{1}{x+ia}$

Therefore, $y_n = \dfrac{1}{2ia}U_n - \dfrac{1}{2ia}V_n = \dfrac{1}{2ia}\dfrac{(-1)^n n!}{(x-ia)^{(n+1)}} - \dfrac{1}{2ia}\dfrac{(-1)^n n!}{(x+ia)^{(n+1)}}$

That is, $y_n = \dfrac{(-1)^n n!}{2ia}\left[\dfrac{1}{(x-ia)^{(n+1)}} - \dfrac{1}{(x+ia)^{(n+1)}}\right]$

$$y_n = \frac{(-1)^n n!}{2ia}\left[(x-ia)^{-(n+1)} - (x+ia)^{-(n+1)}\right]$$

Let $x = r\cos\theta$ **and** $a = r\sin\theta$.

Then $y_n = \dfrac{(-1)^n n!}{2ia}\left[(r\cos\theta - ir\sin\theta)^{-(n+1)} - (r\cos\theta + ir\sin\theta)^{-(n+1)}\right]$

$$= \frac{(-1)^n n!}{2iar^{(n+1)}}\left[(\cos\theta - i\sin\theta)^{-(n+1)} - (\cos\theta + i\sin\theta)^{-(n+1)}\right]$$

$$= \frac{(-1)^n n!}{2iar^{(n+1)}}\left[\{\cos(n+1)\theta + i\sin(n+1)\theta\} - \{\cos(n+1)\theta - i\sin(n+1)\theta\}\right]$$

[applying D' Moiver's theorem, that is, $(\cos\theta \pm i\sin\theta)^n = \cos n\theta \pm i\sin n\theta$ **]**

$$= \frac{(-1)^n n!}{2iar^{(n+1)}}\left[\cos(n+1)\theta + i\sin(n+1)\theta - \cos(n+1)\theta + i\sin(n+1)\theta\right]$$

$$= \frac{(-1)^n n!}{2iar^{(n+1)}}\left[2i\sin(n+1)\theta\right]$$

$$= \frac{(-1)^n n!}{ar^{(n+1)}} \times \sin(n+1)\theta = \frac{(-1)^n n!}{a\left(\dfrac{a}{\sin\theta}\right)^{(n+1)}} \times \sin(n+1)\theta \qquad (\because a = r\sin\theta)$$

$$=\frac{(-1)^n n!}{a^{n+2}}\times\sin^{(n+1)}\theta\times\sin(n+1)\theta$$. **That is,**

$$\boxed{y_n=\frac{(-1)^n n!}{a^{n+2}}\sin^{(n+1)}\theta\sin(n+1)\theta}$$

Now let us try to find n[th] derivative of different types of functions using the experience of the above functions.

EXAMPLE (1): Find y_n when $y=\sin^2 x$.

Solution : $y=\sin^2 x=\frac{1}{2}-\frac{1}{2}\cos 2x$.

Therefore, $y_n=-\frac{1}{2}2^n\cos\left(\frac{n\pi}{2}+2x\right)=2^{n-1}\cos\left(\frac{n\pi}{2}+2x\right)$ **[using function no (6)]**

EXAMPLE (2): Find y_n when $y=\cos 2x\cos x$.

Solution : $y=\cos 2x\cos x=\frac{1}{2}\times 2\cos 2x\cos x=\frac{1}{2}\{\cos(2x+x)+\cos(2x-x)\}$

$$=\frac{1}{2}\{\cos 3x+\cos x\}=\frac{1}{2}\times\cos 3x+\frac{1}{2}\times\cos x=\frac{1}{2}U+\frac{1}{2}V$$

Where $U=\cos 3x$ **and** $V=\cos x$.

Then $y_n=\frac{1}{2}U_n+\frac{1}{2}V_n=\frac{1}{2}\cdot 3^n\cdot\cos\left(\frac{n\pi}{2}+3x\right)+\frac{1}{2}\cdot 1^n\cdot\cos\left(\frac{n\pi}{2}+x\right)$

That is, $y_n=\frac{3^n}{2}\cdot\cos\left(\frac{n\pi}{2}+3x\right)+\frac{1}{2}\cdot\cos\left(\frac{n\pi}{2}+x\right)$.

EXAMPLE (3): Find y_n when $y=\frac{x}{x-1}$.

Solution : Given function is $y=\frac{x}{x-1}=1+\frac{1}{x-1}=1+U$ **Where** $U=\frac{1}{x-1}$.

Therefore, $y_n=0+U_n=U_n=\frac{(-1)^n n!}{(x-1)^{(n+1)}}$ **[using function no (4)]**

EXAMPLE (4): Find y_n when $y=\dfrac{x^2}{x-1}$.

Solution : Given function is $y=\dfrac{x^2}{x-1}=x+1+\dfrac{1}{x-1}=x+1+U$ **Where** $U=\dfrac{1}{x-1}$.

Therefore, $y_n=0+0+U_n=U_n=\dfrac{(-1)^n n!}{(x-1)^{(n+1)}}$ **[if** $n>1$**] [using function no (4)]**

EXAMPLE (6): Find y_n when $y=\dfrac{x^n}{x-1}$.

Solution : Given function is $y=\dfrac{x^2}{x-1}=x^{n-1}+x^{n-2}+\ldots\ldots+x^{n-(n-1)}+1+\dfrac{1}{x-1}$

$=x^{n-1}+x^{n-2}+\ldots\ldots+x^{n-(n-1)}+1+U$ **Where** $U=\dfrac{1}{x-1}$.

Therefore, $y_n=0+0+\ldots\ldots+0+0+U_n=U_n=\dfrac{(-1)^n n!}{(x-1)^{(n+1)}}$ **[if** $n>1$**] [using function no (4)]**

EXAMPLE (7): Find y_n when $y=\dfrac{1}{x^2-16}$.

Solution : $y=\dfrac{1}{x^2-16}=\dfrac{1}{(x+4)(x-4)}=\dfrac{1}{8}\left[\dfrac{1}{x-4}-\dfrac{1}{x+4}\right]$

$=\dfrac{1}{8}\left[\dfrac{1}{x-4}\right]-\dfrac{1}{8}\left[\dfrac{1}{x+4}\right]=\dfrac{1}{8}U-\dfrac{1}{8}V$ **where** $U=\dfrac{1}{x-4}$ **and** $V=\dfrac{1}{x+4}$

Therefore, $y_n=\dfrac{1}{8}U_n-\dfrac{1}{8}V_n=\dfrac{1}{8}\dfrac{(-1)^n n!}{(x-4)^{(n+1)}}-\dfrac{1}{8}\dfrac{(-1)^n n!}{(x+4)^{(n+1)}}$

$$\boxed{y_n=\frac{(-1)^n n!}{8}\left[\frac{1}{(x-4)^{(n+1)}}-\frac{1}{(x+4)^{(n+1)}}\right]}$$

[using function no (7)]

EXAMPLE (8): Find y_n when $y=\dfrac{1}{x^2+16}$.

Solution : Use function no (8).

EXAMPLE (9): Find y_n when $y=\tan^{-1}\frac{x}{a}$.

Solution : $y=\tan^{-1}\frac{x}{a}$. **Therefore,** $y_1=\frac{a}{x^2+a^2}=a\left(\frac{1}{x^2+a^2}\right)=aU$. **Where** $U=\frac{1}{x^2+a^2}$.

Therefore, $y_n=aU_{n-1}=a\times\frac{(-1)^{n-1}(n-1)!}{a^{n-1+2}}\sin^{(n-1+1)}\theta\sin(n-1+1)\theta$

[using function no (8)].

That is, $y_n=\frac{(-1)^{n-1}(n-1)!}{a^n}\sin^n\theta\sin(n\theta)$

In order to find the n^th derivative of product of two or more than two functions we need the following theorem which is known as LEIBNITZ'S THEOREM

LEIBNITZ'S THEOREM :

THEOREM(Leibnitz's) : If u and v be two functions of x, both derivable at least upto n times, then the n^th derivative of their product, that is, $(uv)_n$ is given by

$$(uv)_n=u_nv+n_{C_1}u_{n-1}v_1+n_{C_2}u_{n-2}v_2+\dots\dots+n_{C_r}u_{n-r}v_r+\dots\dots\dots+n_{C_n}uv_n$$

where the suffixes denote the order of differentiation.

EXAMPLE (10): Find y_n when $y=x^2e^{ax}$.

Solution : Given function is $y=x^2e^{ax}$.

Let $U=e^{ax}$ **and** $V=x^2$. **Then**

$U_1=ae^{ax}$ **and** $V_1=2x$

$U_2=a^2e^{ax}$ **and** $V_2=2$

$U_3=a^3e^{ax}$ **and** $V_3=0$

. . . .

. . . . **and proceeding in a similar manner, we have**

$$U_{n-2} = a^{n-2}e^{ax} \quad \text{and} \quad V_{n-2} = 0$$

$$U_{n-1} = a^{n-1}e^{ax} \quad \text{and} \quad V_{n-1} = 0$$

$U_n = a^n e^{ax}$ **and** $V_n = 0$. **Now by Leibnitz's theorem, we have**

$$y = x^2 e^{ax} = (UV)_n = U_n V + n_{C_1} U_{n-1} V_1 + n_{C_2} U_{n-2} V_2 + n_{C_3} U_{n-3} V_3 + \ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots + n_{C_n} U V_n$$

$$= a^n e^{ax} x^2 + n_{C_1} a^{n-1} e^{ax} \cdot 2x + n_{C_2} a^{n-2} e^{ax} \cdot 2 + n_{C_3} a^{n-3} e^{ax} \cdot 0 + \ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots + n_{C_n} e^{ax} \cdot 0$$

$$= a^n e^{ax} x^2 + 2nxa^{n-1} e^{ax} + \frac{n(n-1)}{2!} a^{n-2} e^{ax} \cdot 2$$

$$= a^n e^{ax} x^2 + 2nxa^{n-1} e^{ax} + n(n-1) a^{n-2} e^{ax}$$

EXAMPLE (11): Find y_n when $y = x^2 \sin x$.

Solution : Given function is $y = x^2 \sin x$.

Let $U = \sin x$ **and** $V = x^2$. **Then**

$$U_1 = \cos x = \sin\left(\frac{\pi}{2} + x\right) \quad \text{and} \quad V_1 = 2x$$

$$U_2 = \cos\left(\frac{\pi}{2} + x\right) = \sin\left(2 \cdot \frac{\pi}{2} + x\right) \quad \text{and} \quad V_2 = 2$$

$$U_3 = \cos\left(2 \cdot \frac{\pi}{2} + x\right) = \sin\left(3 \cdot \frac{\pi}{2} + x\right) \quad \text{and} \quad V_3 = 0$$

. . . .

. . . . **and proceeding in a similar manner, we have**

$$U_{n-2} = \cos\left((n-3) \cdot \frac{\pi}{2} + x\right) = \sin\left((n-2) \cdot \frac{\pi}{2} + x\right) \quad \text{and} \quad V_{n-2} = 0$$

$$U_{n-1} = \cos\left((n-2) \cdot \frac{\pi}{2} + x\right) = \sin\left((n-1) \cdot \frac{\pi}{2} + x\right) \quad \text{and} \quad V_{n-1} = 0$$

$$U_n = \cos\left((n-1)\cdot\frac{\pi}{2}+x\right) = \sin\left(n\cdot\frac{\pi}{2}+x\right) \quad \textbf{and} \quad V_n = 0$$

Now by Leibnitz's theorem, we have

$$y = x^2\sin x = (UV)_n = U_nV + n_{C_1}U_{n-1}V_1 + n_{C_2}U_{n-2}V_2 + n_{C_3}U_{n-3}V_3 + \ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots. + n_{C_n}UV_n$$

$$= \sin\left(n\cdot\frac{\pi}{2}+x\right)x^2 + n_{C_1}\sin\left((n-1)\cdot\frac{\pi}{2}+x\right)\cdot 2x +$$

$$n_{C_2}\sin\left((n-2)\cdot\frac{\pi}{2}+x\right)\cdot 2 + n_{C_3}\sin\left((n-3)\cdot\frac{\pi}{2}+x\right)\cdot 0 + \ldots.. + n_{C_n}\sin x\cdot 0$$

$$= \sin\left(\frac{n\pi}{2}+x\right)x^2 + n\sin\left(\frac{(n-1)\pi}{2}+x\right)\cdot 2x +$$

$$\frac{n(n-1)}{2!}\sin\left(\frac{(n-2)\pi}{2}+x\right)\cdot 2 + 0 + \ldots\ldots\ldots\ldots\ldots\ldots + 0$$

$$= \sin\left(\frac{n\pi}{2}+x\right)x^2 + 2nx\sin\left(\frac{(n-1)\pi}{2}+x\right) + n(n-1)\sin\left(\frac{(n-2)\pi}{2}+x\right)\cdot$$

EXAMPLE (12): Find y_n when $y = x^2\log x$.

Solution : Given function is $y = x^2\sin x$.

Let $U = \log x$ **and** $V = x^2$. **Then**

$U_1 = \frac{1}{x} = x^{-1}$ **and** $V_1 = 2x$

$U_2 = (-1)x^{-2}$ **and** $V_2 = 2$

$U_3 = (-1)(-2)x^{-3}$ **and** $V_3 = 0$

. . . .

. . . . **and proceeding in a similar manner, we have**

$U_{n-2} = (-1)(-2)\cdots\cdots\{-(n-3)\}x^{-(n-2)}$ **and** $V_{n-2} = 0$

$U_{n-1} = (-1)(-2)\cdots\cdots\{-(n-2)\}x^{-(n-1)}$ **and** $V_{n-1} = 0$

$$U_n = (-1)(-2)\cdots\cdots\{-(n-1)\}x^{-n} \quad \text{and} \quad V_n = 0$$

Now by Leibnitz's theorem, we have

$$y = x^2 \log x = (UV)_n = U_n V + n_{C_1} U_{n-1} V_1 + n_{C_2} U_{n-2} V_2 + n_{C_3} U_{n-3} V_3 + \ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + n_{C_n} U V_n$$

$$= (-1)(-2)\cdots\{-(n-1)\}x^{-n}x^2 + n_{C_1}(-1)(-2)\cdots\{-(n-2)\}x^{-(n-1)}2x +$$

$$n_{C_2}(-1)(-2)\cdots\{-(n-3)\}x^{-(n-2)}2 + n_{C_3}(-1)(-2)\cdots\{-(n-4)\}x^{-(n-3)}0 + .$$

$$\ldots\ldots\ldots\ldots\ldots + n_{C_n}\log x \cdot 0$$

$$= \frac{(-1)^{n-1}(n-1)!}{x^{n-2}} + 2n\frac{(-1)^{n-2}(n-2)!}{x^{n-2}} +$$

$$\frac{n(n-1)}{2!}\frac{(-1)^{n-3}(n-3)2}{x^{n-2}} + 0 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots + 0$$

$$= \frac{(-1)^{n-1}n!}{x^{n-2}}\left[\frac{1}{n} - \frac{1}{(n-1)} - \frac{1}{(n-2)}\right]$$

EXAMPLE (13): Find y_n when $y = e^x \log x$.

Solution : Given function is $y = x^2 e^{ax}$.

Let $U = e^x$ **and** $V = \log x$. **Then**

$$U_1 = e^x \quad \text{and} \quad V_1 = \frac{1}{x} = x^{-1}$$

$$U_2 = e^x \quad \text{and} \quad V_2 = (-1)x^{-2}$$

$$U_3 = e^x \quad \text{and} \quad V_3 = (-1)(-2)x^{-3}$$

. . . .

. . . . **and proceeding in a similar manner, we have**

$$U_{n-2} = e^x \quad \text{and} \quad V_{n-2} = (-1)(-2)\cdots\cdots\{-(n-3)\}x^{-(n-2)}$$

$$U_{n-1} = e^x \quad \text{and} \quad V_{n-1} = (-1)(-2)\cdots\cdots\{-(n-2)\}x^{-(n-1)}$$

$$U_n = e^x \quad \text{and} \quad V_n = (-1)(-2)\cdots\cdots\{-(n-1)\}x^{-n}$$

Now by Leibnitz's theorem, we have

$$y = e^x \log x = (UV)_n = U_n V + n_{C_1} U_{n-1} V_1 + n_{C_2} U_{n-2} V_2 + n_{C_3} U_{n-3} V_3 + \ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots + n_{C_n} U V_n$$

$$= e^x \log x + n_{C_1} e^x \cdot x^{-1} + n_{C_2} e^x \cdot (-1) x^{-2} + n_{C_3} e^x \cdot (-1)(-2) x^{-3} + \ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots + n_{C_n} e^x \cdot (-1)(-2) \cdots \cdot \{-(n-1)\} x^{-n}$$

$$= e^x \log x + \frac{ne^x}{x} - \frac{n(n-1)}{2!} \frac{e^x}{x^2} + \frac{n(n-1)(n-2)}{3!} \cdot 2! \cdot \frac{e^x}{x^3} + \cdots + (-1)^{n-1} (n-1)! \frac{e^x}{x^n}$$

$$= e^x \left(\log x + \frac{n}{x} - \frac{n(n-1)}{2x^2} + \frac{n(n-1)(n-2)}{3x^3} + \cdots + \frac{(-1)^{n-1}(n-1)!}{x^n} \right)$$

EXAMPLE (14): Find y_n when $y = e^{ax+b} \sin x$.

Solution :

Let $U = e^{ax+b}$ **and** $V = \sin x$. **Then**

$$U_1 = ae^{ax+b} \quad \text{and} \quad V_1 = \cos x = \sin\left(\frac{\pi}{2} + x\right)$$

$$U_2 = a^2 e^{ax+b} \quad \text{and} \quad V_2 = \cos\left(\frac{\pi}{2} + x\right) = \sin\left(2 \cdot \frac{\pi}{2} + x\right)$$

$$U_3 = a^3 e^{ax+b} \quad \text{and} \quad V_3 = \cos\left(2 \cdot \frac{\pi}{2} + x\right) = \sin\left(3 \cdot \frac{\pi}{2} + x\right)$$

. . . .

. . . . **and proceeding in a similar manner, we have**

$$U_{n-2} = a^{n-2} e^{ax+b} \quad \text{and} \quad V_{n-2} = \cos\left((n-3) \cdot \frac{\pi}{2} + x\right) = \sin\left((n-2) \cdot \frac{\pi}{2} + x\right)$$

$$U_{n-1} = a^{n-1} e^{ax+b} \quad \text{and} \quad V_{n-1} = \cos\left((n-2) \cdot \frac{\pi}{2} + x\right) = \sin\left((n-1) \cdot \frac{\pi}{2} + x\right)$$

$$U_n = a^n e^{ax+b} \quad \text{and} \quad V_n = \cos\left((n-1) \cdot \frac{\pi}{2} + x\right) = \sin\left(n \cdot \frac{\pi}{2} + x\right)$$

Now by Leibnitz's theorem, we have

$$y = e^{ax+b}\sin x = (UV)_n = U_nV + n_{C_1}U_{n-1}V_1 + n_{C_2}U_{n-2}V_2 + n_{C_3}U_{n-3}V_3 + \ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots + n_{C_n}UV_n$$

$$= a^n e^{ax+b}\sin x + n_{C_1}a^{n-1}e^{ax+b}\sin\left(\frac{\pi}{2}+x\right) + n_{C_2}a^{n-2}e^{ax+b}\sin\left(2\cdot\frac{\pi}{2}+x\right) +$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots + n_{C_n}e^{ax+b}\sin\left(n\cdot\frac{\pi}{2}+x\right)$$

$$= a^n e^{ax+b}\sin x + na^{n-1}e^{ax+b}\sin\left(\frac{\pi}{2}+x\right) + \frac{n(n-1)}{2!}a^{n-2}e^{ax+b}\sin\left(\frac{2\pi}{2}+x\right) +$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots + e^{ax+b}\sin\left(\frac{n\pi}{2}+x\right)$$

EXAMPLE (15): Find y_n when $y = e^{ax+b}\cos x$.

Solution :

Let $U = e^{ax+b}$ **and** $V = \cos x$. **Then**

$U_1 = ae^{ax+b}$ **and** $V_1 = -\sin x = \cos\left(\frac{\pi}{2}+x\right)$

$U_2 = a^2e^{ax+b}$ **and** $V_2 = -\sin\left(\frac{\pi}{2}+x\right) = \cos\left(2\cdot\frac{\pi}{2}+x\right)$

$U_3 = a^3e^{ax+b}$ **and** $V_3 = -\sin\left(2\cdot\frac{\pi}{2}+x\right) = \cos\left(3\cdot\frac{\pi}{2}+x\right)$

. . . .

. . . . **and proceeding in a similar manner, we have**

$U_{n-2} = a^{n-2}e^{ax+b}$ **and** $V_{n-2} = -\sin\left((n-3)\cdot\frac{\pi}{2}+x\right) = \cos\left((n-2)\cdot\frac{\pi}{2}+x\right)$

$U_{n-1} = a^{n-1}e^{ax+b}$ **and** $V_{n-1} = -\sin\left((n-2)\cdot\frac{\pi}{2}+x\right) = \cos\left((n-1)\cdot\frac{\pi}{2}+x\right)$

$U_n = a^ne^{ax+b}$ **and** $V_n = -\sin\left((n-1)\cdot\frac{\pi}{2}+x\right) = \cos\left(n\cdot\frac{\pi}{2}+x\right)$

Now by Leibnitz's theorem, we have

$$y = e^{ax+b}\cos x = (UV)_n = U_nV + n_{C_1}U_{n-1}V_1 + n_{C_2}U_{n-2}V_2 + n_{C_3}U_{n-3}V_3 + \ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots + n_{C_n}UV_n$$

$$= a^n e^{ax+b}\cos x + n_{C_1}a^{n-1}e^{ax+b}\cos\left(\frac{\pi}{2}+x\right) + n_{C_2}a^{n-2}e^{ax+b}\cos\left(2\cdot\frac{\pi}{2}+x\right) +$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots + n_{C_n}e^{ax+b}\cos\left(n\cdot\frac{\pi}{2}+x\right)$$

$$= a^n e^{ax+b}\cos x + na^{n-1}e^{ax+b}\cos\left(\frac{\pi}{2}+x\right) + \frac{n(n-1)}{2!}a^{n-2}e^{ax+b}\cos\left(\frac{2\pi}{2}+x\right) +$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots + e^{ax+b}\cos\left(\frac{n\pi}{2}+x\right)$$

EXAMPLE (16): Find y_n when $y = (ax+b)^n \sin x$.

Solution :

Let $U = (ax+b)^n$ **and** $V = \sin x$. **Then**

$U_1 = an(ax+b)^{n-1}$ **and** $V_1 = \cos x = \sin\left(\frac{\pi}{2}+x\right)$

$U_2 = a^2n(n-1)(ax+b)^{n-2}$ **and** $V_2 = \cos\left(\frac{\pi}{2}+x\right) = \sin\left(2\cdot\frac{\pi}{2}+x\right)$

$U_3 = a^3n(n-1)(n-2)(ax+b)^{n-3}$ **and** $V_3 = \cos\left(2\cdot\frac{\pi}{2}+x\right) = \sin\left(3\cdot\frac{\pi}{2}+x\right)$

. . . .

. . . .

. . . . **and proceeding in a similar manner, we have**

$U_{n-2} = a^{n-2}n(n-1)\cdots\{n-(n-3)\}(ax+b)^2$ **and**

$V_{n-2} = \cos\left((n-3)\cdot\frac{\pi}{2}+x\right) = \sin\left((n-2)\cdot\frac{\pi}{2}+x\right)$

$U_{n-1} = a^{n-1}n(n-1)\cdots\{n-(n-2)\}(ax+b)^1$ **and**

$V_{n-1} = \cos\left((n-2)\cdot\frac{\pi}{2}+x\right) = \sin\left((n-1)\cdot\frac{\pi}{2}+x\right)$

$$U_n = a^n n(n-1)\cdots\{n-(n-1)\}$$ **and**

$$V_n = \cos\left((n-1)\cdot\frac{\pi}{2}+x\right) = \sin\left(n\cdot\frac{\pi}{2}+x\right)$$

Now by Leibnitz's theorem, we have

$$y = (ax+b)^n \sin x = (UV)_n = U_nV + n_{C_1}U_{n-1}V_1 + n_{C_2}U_{n-2}V_2 + n_{C_3}U_{n-3}V_3 + \ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots + n_{C_n}UV_n$$

$$= a^n n!\sin x + na^{n-1}n!(ax+b)^1\sin\left(\frac{\pi}{2}+x\right) + \frac{n(n-1)}{2!}a^{n-2}\frac{n!}{2}(ax+b)^2\sin\left(2\cdot\frac{\pi}{2}+x\right)+$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots + n_{C_n}(ax+b)^n\sin\left(n\cdot\frac{\pi}{2}+x\right)$$

EXAMPLE (17): Find y_n when $y = (ax+b)^n\cos x$.

Solution :

Let $U = (ax+b)^n$ **and** $V = \cos x$. **Then**

$U_1 = an(ax+b)^{n-1}$ **and** $V_1 = -\sin x = \cos\left(\frac{\pi}{2}+x\right)$

$U_2 = a^2n(n-1)(ax+b)^{n-2}$ **and** $V_2 = -\sin\left(\frac{\pi}{2}+x\right) = \cos\left(2\cdot\frac{\pi}{2}+x\right)$

$U_3 = a^3n(n-1)(n-2)(ax+b)^{n-3}$ **and** $V_3 = -\sin\left(2\cdot\frac{\pi}{2}+x\right) = \cos\left(3\cdot\frac{\pi}{2}+x\right)$

. . . .

. . . . **and proceeding in a similar manner, we have**

$$U_{n-2} = a^{n-2}n(n-1)\cdots\{n-(n-3)\}(ax+b)^2$$ **and**

$$V_{n-2} = -\sin\left((n-3)\cdot\frac{\pi}{2}+x\right) = \cos\left((n-2)\cdot\frac{\pi}{2}+x\right)$$

$$U_{n-1} = a^{n-1}n(n-1)\cdots\{n-(n-2)\}(ax+b)^1$$ **and**

$$V_{n-1} = -\sin\left((n-2)\cdot\frac{\pi}{2}+x\right) = \cos\left((n-1)\cdot\frac{\pi}{2}+x\right)$$

$$U_n = a^n n(n-1)\cdots\{n-(n-1)\}$$ **and**

$$V_n = -\sin\left((n-1)\cdot\frac{\pi}{2}+x\right) = \cos\left(n\cdot\frac{\pi}{2}+x\right)$$

Now by Leibnitz's theorem, we have

$$y=(ax+b)^n\cos x=(UV)_n=U_nV+n_{C_1}U_{n-1}V_1+n_{C_2}U_{n-2}V_2+n_{C_3}U_{n-3}V_3+\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots+n_{C_n}UV_n$$

$$=a^n n!\cos x+na^{n-1}n!(ax+b)^1\cos\left(\frac{\pi}{2}+x\right)+\frac{n(n-1)}{2!}a^{n-2}\frac{n!}{2}(ax+b)^2\cos\left(2\cdot\frac{\pi}{2}+x\right)+$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots+n_{C_n}(ax+b)^n\cos\left(n\cdot\frac{\pi}{2}+x\right)$$

APPLICATIONS OF LIEBNITZ'S THEOREM

EXAMPLE (18): If $y=\tan^{-1}x$, then show that
$(1+x^2)y_{n+2}+2(n+1)xy_{n+1}+n(n+1)y_n=0$

Solution : Given function is $y=\tan^{-1}x$. Then differentiating both sides with respect to x, we get $y_1=\dfrac{1}{1+x^2}$. Therefore, $y_1\left(1+x^2\right)=1$. Differentiating both sides once again, we get $\left(1+x^2\right)y_2+2xy_1=0$. Then applying Leibnitz's theorem, we get

$$\left(y_{n+2}\left(1+x^2\right)+n_{C_1}y_{n+1}\cdot 2x+n_{C_2}y_n\cdot 2+n_{C_3}y_{n-1}\cdot 0+0+0+\cdots+0\right)+$$
$$2\left(y_{n+1}x+n_{C_1}y_n\cdot 1+0+0+\cdots\cdots+0\right)=0.$$

$$\Rightarrow y_{n+2}\left(1+x^2\right)+2nxy_{n+1}+\frac{n(n-1)}{2!}y_n\cdot 2+2xy_{n+1}+2ny_n=0$$

$$\Rightarrow y_{n+2}\left(1+x^2\right)+2(n+1)xy_{n+1}+n(n+1)y_n=0$$

$$\boxed{\left(1+x^2\right)y_{n+2}+2(n+1)xy_{n+1}+n(n+1)y_n=0}$$

EXAMPLE (19): If $y=\left(\sin^{-1}x\right)^2$, then show that $\left(1-x^2\right)y_{n+2}-(2n+1)xy_{n+1}-n^2y_n=0$

Solution : Given function is $y=\left(\sin^{-1}x\right)^2$. Then differentiating both sides with respect to x, we get $y_1=\dfrac{2\left(\sin^{-1}x\right)}{\sqrt{1-x^2}}$. Therefore, $y_1\sqrt{1-x^2}=2\left(\sin^{-1}x\right)$. Squaring both sides, we get $y_1^2\left(1-x^2\right)=4\left(\sin^{-1}x\right)^2=4y$. That is, $y_1^2\left(1-x^2\right)=4y$

Differentiating both sides once again, we get $2y_1y_2\left(1-x^2\right)-2xy_1^2=4y_1$. That is, $y_2\left(1-x^2\right)-xy_1-2=0$. Then applying Leibnitz's theorem, we get

$$\left(y_{n+2}\left(1-x^2\right)+n_{C_1}y_{n+1}\cdot(-2x)+n_{C_2}y_n\cdot(-2)+n_{C_3}y_{n-1}\cdot 0+\cdots+0\right)-\left(y_{n+1}x+n_{C_1}y_n\cdot 1+0+0+\cdots\cdots+0\right)-0=0$$

$$\Rightarrow y_{n+2}\left(1-x^2\right)-2nxy_{n+1}-\frac{n(n-1)}{2!}y_n\cdot(2)-y_{n+1}x-ny_n=0$$

$$\Rightarrow \left(1-x^2\right)y_{n+2}-(2n+1)xy_{n+1}-n^2y_n=0$$

$$\boxed{\left(1-x^2\right)y_{n+2}-(2n+1)xy_{n+1}-n^2y_n=0}$$

EXAMPLE (20): If $y=\sin\left(m\sin^{-1}x\right)$, then show that $\left(1-x^2\right)y_{n+2}-(2n+1)xy_{n+1}+(m^2-n^2)y_n=0$

Solution : Given function is $y=\sin\left(m\sin^{-1}x\right)$. Then differentiating both sides with respect to x, we get $y_1=\cos\left(m\sin^{-1}x\right)\cdot\dfrac{m}{\sqrt{1-x^2}}$ Differentiating both sides once again, we get

$$y_2=\frac{-m\sqrt{1-x^2}\sin\left(m\sin^{-1}x\right)\cdot\dfrac{m}{\sqrt{1-x^2}}-m\cos\left(m\sin^{-1}x\right)\dfrac{-2x}{2\sqrt{1-x^2}}}{1-x^2}$$

$$\Rightarrow y_2(1-x^2) = -m^2\sin\left(m\sin^{-1}x\right) + x\cos\left(m\sin^{-1}x\right)\cdot\frac{m}{\sqrt{1-x^2}}$$

$$\Rightarrow y_2(1-x^2) = -m^2 y + xy_1$$

$\Rightarrow y_2(1-x^2) - xy_1 + m^2 y = 0$. **Then applying Leibnitz's theorem, we get**

$$\left(y_{n+2}\left(1-x^2\right) + n_{C_1} y_{n+1}\cdot(-2x) + n_{C_2} y_n\cdot(-2) + n_{C_3} y_{n-1}\cdot 0 + \cdots + 0\right) - \left(y_{n+1}x + n_{C_1} y_n\cdot 1 + 0 + 0 + \cdots\cdots + 0\right) + m^2 y_n = 0$$

$$\Rightarrow y_{n+2}\left(1-x^2\right) - 2nxy_{n+1} - \frac{n(n-1)}{2!}y_n\cdot(2) - y_{n+1}x - ny_n + m^2 y_n = 0$$

$$\Rightarrow \left(1-x^2\right)y_{n+2} - (2n+1)xy_{n+1} + (m^2 - n^2)y_n = 0$$

$$\boxed{\left(1-x^2\right)y_{n+2} - (2n+1)xy_{n+1} + (m^2 - n^2)y_n = 0}$$

EXAMPLE (21): If $y = \cos\left(m\sin^{-1}x\right)$**, then show that** $\left(1-x^2\right)y_{n+2} - (2n+1)xy_{n+1} + (m^2 - n^2)y_n = 0$

Solution : Left for exercise. (Hints – same as example 20).

EXAMPLE (22): If $y = a\cos(\log x) + b\sin(\log x)$**, then show that** $x^2 y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0$

Solution : Given function is $y = a\cos(\log x) + b\sin(\log x)$. **Then differentiating both sides with respect to** x **, we get**

$y_1 = \dfrac{-a\sin(\log x) + b\cos(\log x)}{x}$. **Differentiating both sides once again, we get**

$$y_2 = \frac{x\cdot\dfrac{-a\cos(\log x) - b\sin(\log x)}{x} + a\sin(\log x) - b\cos(\log x)}{x^2}$$

$$\Rightarrow y_2 x^2 = -(a\cos(\log x) + b\sin(\log x)) - (-a\sin(\log x) + b\cos(\log x))$$

$$\Rightarrow y_2 x^2 = -y - xy_1$$

$\Rightarrow y_2 x^2 + xy_1 + y = 0$. **Then applying Leibnitz's theorem, we get**

$$\left(y_{n+2}x^2 + n_{C_1}y_{n+1}\cdot 2x + n_{C_2}y_n\cdot 2 + n_{C_3}y_{n-1}\cdot 0 + \cdots + 0\right) + \left(y_{n+1}x + n_{C_1}y_n\cdot 1 + n_{C_2}y_{n-1}\cdot 0 + \cdots + 0\right) + y_n$$

$$\Rightarrow \left(x^2y_{n+2} + 2nxy_{n+1} + \frac{n(n-1)}{2!}y_n\cdot 2\right) + \left(xy_{n+1} + ny_n\right) + y_n$$

$$\Rightarrow x^2y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0$$

$$\boxed{x^2y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0}$$

EXAMPLE (23): If $y = e^{m\sin^{-1}x}$ **, then show that**
$\left(1-x^2\right)y_{n+2} - (2n+1)xy_{n+1} - (m^2+n^2)y_n = 0$

Solution : Left for exercise.

EXAMPLE (24): If $y^{\frac{1}{m}} + y^{-\frac{1}{m}} = 2x$**, then show that**
$\left(x^2-1\right)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0$

Solution : Given that $y^{\frac{1}{m}} + y^{-\frac{1}{m}} = 2x$.

Now $\left(y^{\frac{1}{m}} - y^{-\frac{1}{m}}\right)^2 = \left(y^{\frac{1}{m}} + y^{-\frac{1}{m}}\right)^2 - 4y^{\frac{1}{m}}y^{-\frac{1}{m}} = (2x)^2 - 4 = 4x^2 - 4$

Therefore, $y^{\frac{1}{m}} - y^{-\frac{1}{m}} = \pm 2\sqrt{x^2-1}$

$y^{\frac{1}{m}} + y^{-\frac{1}{m}} = 2x$ **(given). Adding we get** $y^{\frac{1}{m}} = x \pm \sqrt{x^2-1}$. **Hence**

$$\frac{1}{m}\log y = \log\left(x \pm \sqrt{x^2-1}\right) \Rightarrow \frac{1}{m}\frac{y_1}{y} = \frac{1}{x \pm \sqrt{x^2-1}} \times \left(1 \pm \frac{2x}{2\sqrt{x^2-1}}\right)$$

$$\Rightarrow \frac{1}{m}\frac{y_1}{y} = \frac{1}{x \pm \sqrt{x^2-1}} \times \left(\frac{\sqrt{x^2-1} \pm x}{\sqrt{x^2-1}}\right) = \frac{\pm 1}{\sqrt{x^2-1}}$$

$$\Rightarrow \frac{y_1^2}{m^2y^2}=\frac{1}{(x^2-1)} \Rightarrow y_1^2(x^2-1)-m^2y^2=0.$$

That is, $y_1^2(x^2-1)-m^2y^2=0$. **Differentiating both sides with respect to we get**

$$2y_1y_2(x^2-1)+2xy_1^2-2m^2yy_1=0$$

$$\Rightarrow y_2(x^2-1)+xy_1-m^2y=0$$ **Then applying Leibnitz's theorem, we get**

$$\left(y_{n+2}\left(x^2-1\right)+n_{C_1}y_{n+1}\cdot(2x)+n_{C_2}y_n\cdot(2)+n_{C_3}y_{n-1}\cdot 0+\cdots+0\right)-$$
$$\left(y_{n+1}x+n_{C_1}y_n\cdot 1+n_{C_2}y_{n-1}\cdot 0+0+\cdots\cdots+0\right)-m^2y_n=0$$

$$\Rightarrow y_{n+2}\left(x^2-1\right)+2nxy_{n+1}+\frac{n(n-1)}{2!}y_n\cdot 2+y_{n+1}x+ny_n-m^2y_n$$

$$\Rightarrow \left(1-x^2\right)y_{n+2}+(2n+1)xy_{n+1}+(n^2-m^2)y_n=0$$

$$\boxed{\left(1-x^2\right)y_{n+2}+(2n+1)xy_{n+1}+(n^2-m^2)y_n=0}$$

EXAMPLE (25): If $\cos^{-1}\left(\frac{y}{b}\right)=\log\left(\frac{x}{n}\right)^n$, **then prove that**

$$x^2y_{n+2}+(2n+1)xy_{n+1}+2n^2y_n=0$$

Solution : Given that $\cos^{-1}\left(\frac{y}{b}\right)=\log\left(\frac{x}{n}\right)^n$ **So,** $y=b\left\{\cos\left(n\log\left(\frac{x}{n}\right)\right)\right\}$.

Then differentiating both sides with respect to x **, we get**

$$y_1=-b\sin\left(n\log\left(\frac{x}{n}\right)\right)\times n\times\frac{n}{x}\times\frac{1}{n}=\frac{-bn\sin\left(n\log\left(\frac{x}{n}\right)\right)}{x}$$

$$\Rightarrow y_2=\frac{-bnx\cos\left(n\log\left(\frac{x}{n}\right)\right)\times n\times\frac{n}{x}\times\frac{1}{n}+bn\sin\left(n\log\left(\frac{x}{n}\right)\right)}{x^2}$$

$$\Rightarrow y_2=\frac{-bn^2\cos\left(n\log\left(\frac{x}{n}\right)\right)+bn\sin\left(n\log\left(\frac{x}{n}\right)\right)}{x^2}$$

$$\Rightarrow y_2 = \frac{-n^2 y - xy_1}{x^2}$$

$\Rightarrow y_2 x^2 + xy_1 + n^2 y = 0$. **Then applying Leibnitz's theorem, we get**

$$\left(y_{n+2}x^2 + n_{C_1} y_{n+1}\cdot 2x + n_{C_2} y_n \cdot 2 + n_{C_2} y_{n-1}\cdot 0 + \cdots + 0\right) +$$
$$\left(y_{n+1}x + n_{C_1} y_n \cdot 1 + n_{C_1} y_{n-1}\cdot 0 + \cdots + 0\right) + n^2 y_n = 0$$

$$\Rightarrow x^2 y_{n+2} + (2n+1)xy_{n+1} + 2n^2 y_n = 0$$

$$\boxed{x^2 y_{n+2} + (2n+1)xy_{n+1} + 2n^2 y_n = 0}$$

EXAMPLE (26): If $x + y = 1$**, prove that the n^th^ derivative of** $x^n y^n$ **is**
$$n!\left\{y^n - \left(n_{C_1}\right)^2 y^{n-1}x + \left(n_{C_2}\right)^2 y^{n-2}x^2 - \left(n_{C_3}\right)^2 y^{n-3}x^3 + \cdots\cdots + (-1)^n x^n\right\}$$

Solution : The n^th^ derivative of $x^n y^n$ **is** $D^n\left\{x^n y^n\right\} = D^n\left\{x^n (1-x)^n\right\}$.

Let $U = x^n$ **and** $V = y^n = (1-x)^n$ $\left(\because x + y = 1\right)$. **Then**

$$U_1 = nx^{n-1}$$
$$U_2 = n(n-1)x^{n-2}$$
$$U_3 = n(n-1)(n-2)x^{n-3}$$

. . . .

. . . . **and proceeding in a similar manner, we have**

$$U_{n-2} = n(n-1)(n-2)\cdots\cdots\cdots\{n-(n-2-1)\}x^{n-(n-2)} = \frac{n!}{2}x^2$$
$$U_{n-1} = n(n-1)(n-2)\cdots\cdots\cdots\{n-(n-1-1)\}x^{n-(n-1)} = n!x$$
$$U_n = n(n-1)(n-2)\cdots\cdots\cdots\{n-(n-1)\}x^{n-n}.$$

That is

$$U_n = n(n-1)(n-2)\cdots\cdots\cdots 3\cdot 2\cdot 1\cdot x^{n-n} = n(n-1)(n-2)\cdots\cdots\cdots 3\cdot 2\cdot 1 = n!$$

Similarly,

$$V_1 = (-1)n(1-x)^{n-1}$$

$$V_2 = (-1)^2 n(n-1)(1-x)^{n-2}$$
$$V_3 = (-1)^3 n(n-1)(n-2)(1-x)^{n-3}$$

. . . .

. . . . **and proceeding in a similar manner, we have**

$$V_{n-1} = (-1)^{n-1} n(n-1)(n-2)\cdots\cdots\{n-(n-1-1)\}(1-x)^{n-(n-1)}$$
$$V_n = (-1)^n n(n-1)(n-2)\cdots\cdots\{n-(n-1)\}(1-x)^{n-n}.$$

That is $V_n = (-1)^n n(n-1)(n-2)\cdots\cdots 3\cdot 2\cdot 1\cdot(1-x)^{n-n}$

$$= (-1)^n n(n-1)(n-2)\cdots\cdots 3\cdot 2\cdot 1 = (-1)^n n!$$

Therefore, the n^th derivative of $x^n y^n$ **is**

$$D^n\{x^n y^n\} = D^n\{x^n(1-x)^n\} = D^n\{UV\}.$$

$= U_n V + n_{C_1} U_{n-1} V_1 + n_{C_2} U_{n-2} V_2 + \cdots\cdots + UV_n$ **(by Liebnitz's theorem)**

$$= n!(1-x)^n + n_{C_1} n!\, x \times (-1)n(1-x)^{n-1} + n_{C_2}\frac{n!}{2}x^2 \times (-1)^2 n(n-1)(1-x)^{n-2}$$
$$+\cdots\cdots\cdots\cdots + x^n(-1)^n n!$$
$$= n!\left\{(1-x)^n - (n_{C_1})^2(1-x)^{n-1}x + (n_{C_2})^2(1-x)^{n-2}x^2 - \cdots\cdots + (-1)^n x^n\right\}$$
$$= n!\left\{y^n - (n_{C_1})^2 y^{n-1}x + (n_{C_2})^2 y^{n-2}x^2 - (n_{C_3})^2 y^{n-3}x^3 + \cdots\cdots + (-1)^n x^n\right\}$$

EXAMPLE (27): Show that the n^th derivative of $\dfrac{\log x}{x}$ **is**

$$\frac{(-1)^n n!}{x^{n+1}}\left[\log x - 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \cdots\cdots\cdots - \frac{1}{n}\right]$$

Solution : Left as an exercise for students.

EXAMPLE (28): If $y = x^{n-1}\log x$, then prove that $y_n = \dfrac{(n-1)!}{x}$ is

Solution : $y_n = D^n(y) = D^n(x^{n-1}\log x) = D^{n-1}\left(D(x^{n-1}\log x)\right)$

$$= D^{n-1}\left((n-1)x^{n-2}\log x + \frac{x^{n-1}}{x}\right) = (n-1)D^{n-1}\left(x^{n-2}\log x\right) + D^{n-1}\left(x^{n-2}\right)$$

$$= (n-1)D^{n-1}\left(x^{n-2}\log x\right) + 0 \quad \left(\because D^{n-1}\left(x^{n-2}\right) = 0\right)$$

$$= (n-1)D^{n-2}\left(D\left(x^{n-2}\log x\right)\right)$$

$$= (n-1)D^{n-2}\left((n-2)x^{n-3}\log x + \frac{x^{n-2}}{x}\right)$$

$$= (n-1)(n-2)D^{n-2}\left(x^{n-3}\log x\right) + D^{n-2}\left(x^{n-3}\right)$$

$$= (n-1)(n-2)D^{n-2}\left(x^{n-3}\log x\right) + 0 \quad \left(\because D^{n-2}\left(x^{n-3}\right) = 0\right)$$

$$= (n-1)(n-2)D^{n-2}\left(x^{n-3}\log x\right)$$

. . . .

. . . .

$$= (n-1)(n-2)\cdots\cdots\cdots(n-(n-1))D^{n-(n-1)}\left(x^{n-(n-1+1)}\log x\right)$$

$$= (n-1)(n-2)\cdots\cdots\cdots 1\cdot D(\log x)$$

$$= (n-1)(n-2)\cdots\cdots\cdots 1\cdot\frac{1}{x}$$

$$= \frac{(n-1)!}{x}$$

EXAMPLE (29): If $P_n = D^n(x^n\log x)$, then prove that $P_n = n\cdot P_{n-1} + (n-1)!$

Hence show that $P_n = n!\cdot\left[\log x + 1 + \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4} + \cdots\cdots\cdots\cdots + \dfrac{1}{n}\right]$

Solution: Given $P_n = D^n(x^n\log x) = D^{n-1}\{D(x^n\log x)\}$

$$= D^{n-1}\left\{x^n\cdot\frac{1}{x} + nx^{n-1}.\log x\right\} = D^{n-1}(x^{n-1}) + n\cdot D^{n-1}\left(x^{n-1}.\log x\right)$$

Therefore, $P_n = (n-1)! + n \cdot P_{n-1}$ **or,** $P_n = n \cdot P_{n-1} + (n-1)!$ **(first part proved)**

Next from first part, we have $P_n = n \cdot P_{n-1} + (n-1)!$**. Dividing both sides by** $n!$ **we get,**

$$\frac{P_n}{n!} = \frac{n \cdot P_{n-1}}{n!} + \frac{(n-1)!}{n!} \quad \text{or,} \quad \frac{P_n}{n!} - \frac{P_{n-1}}{(n-1)!} = \frac{1}{n}.$$

Replacing successively, n **by** $n-1, n-2, n-3, \cdots\cdots, 3, 2$ **we get**

$$\frac{P_{n-1}}{(n-1)!} - \frac{P_{n-2}}{(n-2)!} = \frac{1}{n-1}$$

$$\frac{P_{n-2}}{(n-2)!} - \frac{P_{n-3}}{(n-3)!} = \frac{1}{n-2}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{P_3}{3!} - \frac{P_2}{2!} = \frac{1}{3}$$

$$\frac{P_2}{2!} - \frac{P_1}{1!} = \frac{1}{2}$$

Adding these we get $\frac{P_n}{n!} - \frac{P_1}{1!} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots\cdots\cdots\cdots + \frac{1}{n-1} + \frac{1}{n}$

Or, $\frac{P_n}{n!} = P_1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots\cdots\cdots\cdots + \frac{1}{n-1} + \frac{1}{n}$

Or, $\frac{P_n}{n!} = \log x + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots\cdots\cdots\cdots + \frac{1}{n-1} + \frac{1}{n}$

$$\left(\because P_1 = D(x \log x) = \log x + x \cdot \frac{1}{x} = \log x + 1 \right)$$

$$P_n = n! \cdot \left[\log x + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots\cdots\cdots\cdots + \frac{1}{n} \right].$$

EXAMPLE (30): If $y = e^{\tan^{-1} x} = a_0 + a_1 x + a_2 x^2 + \cdots\cdots + a_n x^n + \cdots\cdots\cdots$ **then prove that** $a_{n+2} = \frac{a_{n+1} - n a_n}{n+2}$ **and hence show that**

$$e^{\tan^{-1} x} = 1 + x + \frac{1}{2} \cdot x^2 - \frac{1}{6} x^3 - \frac{7}{24} x^4 - \cdots\cdots\cdots\cdots$$

Solution: Given $y = e^{\tan^{-1} x}$ **. Then** $y_1 = e^{\tan^{-1} x} \cdot \dfrac{1}{1+x^2}$ **.**

Then $y_2 = \dfrac{(1+x^2)e^{\tan^{-1} x} \cdot \dfrac{1}{1+x^2} - e^{\tan^{-1} x} \cdot 2x}{\left(1+x^2\right)^2} = \dfrac{e^{\tan^{-1} x}(1-2x)}{\left(1+x^2\right)^2}$

Or, $\left(1+x^2\right) \cdot y_2 = (1-2x) \cdot \dfrac{e^{\tan^{-1} x}}{\left(1+x^2\right)} = (1-2x) \cdot y_1$

Or, $\left(1+x^2\right) \cdot y_2 = (1-2x) \cdot y_1$ **.................................(1)**

Now from the given reation we have

$$y = e^{\tan^{-1} x} = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + a_{n+1} x^{n+1} + a_{n+2} x^{n+2} \cdots$$

Or,

$$y_1 = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + na_n x^{n-1} + (n+1)a_{n+1} x^n + (n+2)a_{n+2} x^{n+1} \cdots$$

Or,

$$y_2 = a_1 + 2a_2 + 6a_3 x + \cdots + n(n-1)a_n x^{n-2} + n(n+1)a_{n+1} x^{n-1} + (n+1)(n+2)a_{n+2} x^n \cdots$$

Putting the values of y_1 **&** y_2 **in (1), we get**

$$(1+x^2)(a_1 + 2a_2 + 6a_3 x + \cdots + n(n-1)a_n x^{n-2} + n(n+1)a_{n+1} x^{n-1} + (n+1)(n+2)a_{n+2} x^n \cdots$$
$$= (1-2x)(a_1 + 2a_2 x + 3a_3 x^2 + \cdots + na_n x^{n-1} + (n+1)a_{n+1} x^n + (n+2)a_{n+2} x^{n+1} \cdots)$$

Comparing the co-efficients of x^n **from both sides of the above relation, we get**

$$(n+1)(n+2)a_{n+2} + n(n-1)a_n = (n+1)a_{n+1} - 2na_n$$

Or, $a_{n+2} = \dfrac{a_{n+1} - na_n}{n+2}$ **. (first part proved)......................(2)**

Next from the given relation $y = e^{\tan^{-1} x} = a_0 + a_1 x + a_2 x^2 + \cdots\cdots + a_n x^n + \cdots\cdots\cdots$

we get $(y)_0 = e^{\tan^{-1} 0} = a_0$ **. So** $a_0 = 1$.

Again, $y_1 = e^{\tan^{-1} x} \cdot \dfrac{1}{1+x^2}$

Therefore, $(y_1)_0 = 1$

Also from the relation

$$y_1 = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + na_n x^{n-1} + (n+1)a_{n+1} x^n + (n+2)a_{n+2} x^{n+1} \cdots$$

We have $(y_1)_0 = a_1$ **So,** $a_1 = 1$

Now putting $n = 0,1,2,3,\ldots\ldots.$ **in the relation (2), we get**

$$a_2 = \frac{a_1 - 0}{2} = \frac{1}{2}, \quad a_3 = \frac{a_2 - 1 \cdot a_1}{1+2} = \frac{\frac{1}{2} - 1}{3} = -\frac{1}{6},$$

$$a_4 = \frac{a_3 - 2a_2}{4} = \frac{-\frac{1}{6} - 2 \cdot \frac{1}{2}}{4} = \frac{\frac{-7}{6}}{4} = -\frac{7}{24},$$ **and so on.**

Hence from the given reation we have

$$e^{\tan^{-1} x} = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + a_{n+1} x^{n+1} + a_{n+2} x^{n+2} \cdots$$

$$= 1 + 1 \cdot x + \frac{1}{2}.x^2 - \frac{1}{6} x^3 - \frac{7}{24} x^4 - \cdots\cdots\cdots$$ **(proved)**

CONCAVITY AND INFLECTION POINTS

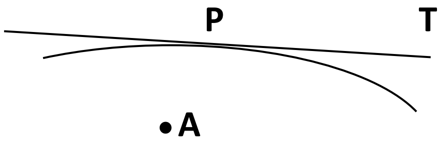


Fig-1

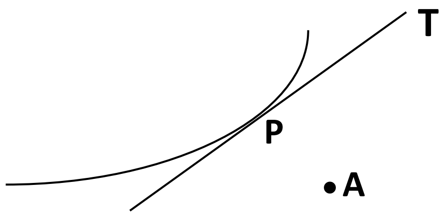


Fig-2

DEFINITION (Concavity & Convexity of a curve with respect to a point) : Let $\overline{PT}$ be the tangent to a curve at P . Then the curve at P is said to be *concave* or *convex* with respect to a point A not lying on $\overline{PT}$, according as a small portion of the curve in the immediate neighbourhood of P (on both sides of P) lies entirely on the same side of $\overline{PT}$ as A or on opposite sides of $\overline{PT}$ with respect to the point A .

Figure (1) shows that the curve at P is concave with respect to the point A . Whereas Figure (2) shows that the curve at P is convex with respect to the point A .

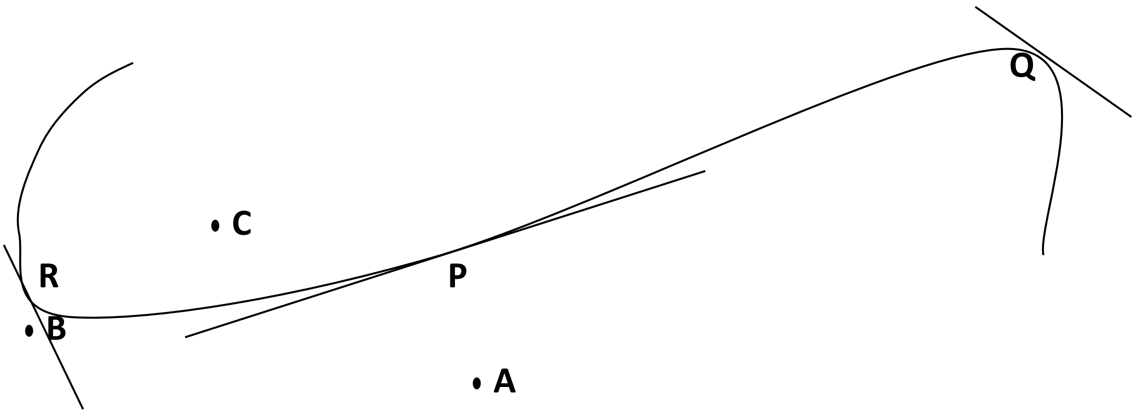


Fig-3

Figure (3) shows that the curve at P is convex with respect to the point A and concave with respect to the points B and C . The curve at R is convex with respect to the point B and concave with respect to the points C and A.

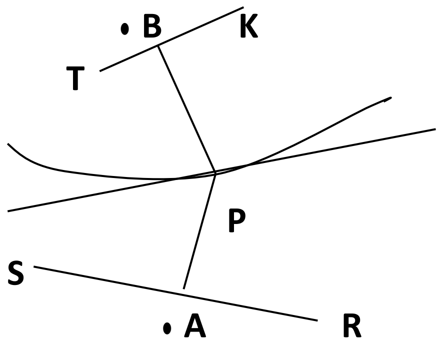
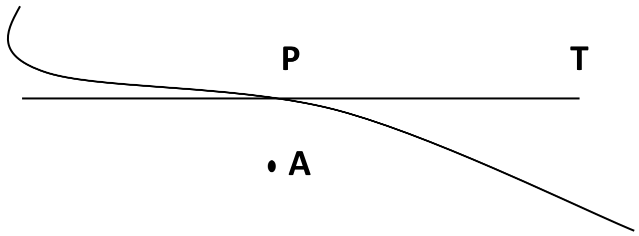


Fig-4

DEFINITION (Concavity & Convexity of a curve with respect to a line) : A curve at a point P on it is convex or concave with respect to a given line according as it is convex or concave with respect to the foot of the perpendicular from the point P to the given line.

Figure (4) shows that the curve at P is convex with respect to the line $\overline{SR}$ and is convex with respect to the line $\overline{TK}$.

POINT OF INFLEXION



DEFINITION (Point of inflexion) : If at any point P on a curve the tangent crosses the curve, then with respect to any point A not lying on the tangent line the curve, on one side of P is convex and on the other side of P it is concave. Such a point P on the curve is defined to be a *point of inflexion*. Point of inflexion is also known as *point of contrary flexure.*

TEST OF CONVEXITY AND CONCAVITY

If $y\frac{d^2y}{dx^2} > 0$ at P, the curve is convex to the $x-axis$. That is, the curve is convex to the foot of the ordinate of the point P.

If $y\frac{d^2y}{dx^2} < 0$ at P, the curve is concave to the $x-axis$. That is, the curve is concave to the foot of the ordinate of the point P.

NOTE : At a point where the tangent is parallel to the $y-axis$, $\frac{dy}{dx}$ is infinite. At such a point, instead of considering with respect to the $x-axis$, we investigate convexity or concavity of the curve with respect to the $y-axis$.

If $x\frac{d^2x}{dy^2} > 0$ at P, the curve is convex to the $y-axis$.

If $x\frac{d^2x}{dy^2} < 0$ at P, the curve is concave to the $y-axis$.

TEST OF POINT OF INFLEXION

The condition that the point P is a point of inflexion on the curve $y = f(x)$ is that, at P,

$\frac{d^2y}{dx^2} = 0$ **and** $\frac{d^3y}{dx^3} \neq 0$.

NOTE : If $\frac{dy}{dx}$ is infinite then the condition that the point P is a point of inflexion on the curve $y = f(x)$ is that, at P,

$\frac{d^2x}{dy^2} = 0$ **and** $\frac{d^3x}{dy^3} \neq 0$.

IN GENERALIZED FORM :

A curve $y = f(x)$ be such that $f''(x) = f'''(x) = \cdots\cdots = f^{n-1}(x) = 0$ and $f^{n}(x) \neq 0$ for $x = c$ then if

(1) if n is odd, the curve has a point of inflexion at $x = c$.

(2) if n is even, then

(i) if $f^{n}(x) > 0$ the curve is concave upwards (that is, convex downwards).

(ii) if $f^{n}(x) < 0$ the curve is concave downwards (that is, convex upwards).

PROBLEMS

EXAMPLE-1 : Examine the convexity and concavity to the axis of the curve $y = \sin x$. Find the points of inflexion(if any).

SOLUTION : Given curve is $y = \sin x$. Then $\dfrac{dy}{dx} = \cos x$ and $\dfrac{d^2y}{dx^2} = -\sin x$.

Hence $y\dfrac{d^2y}{dx^2} = -\sin^2 x < 0$ for all values of x except those which make $\sin x = 0$, ie., for $x = k\pi$, where k is an integer.

Thus the curve is concave to the $x-axis$ at every point except at points where it crosses the $x-axis$. At these points, ie., at $x = k\pi$, $\dfrac{d^2y}{dx^2} = 0$ and $\dfrac{d^3y}{dx^3} = -\cos x \neq 0$.

Hence the points where the curve intersects the $x-axis$ are the points of inflexion.

EXAMPLE-2 : Prove that the curve $y = \log x$ is convex with respect to $x-axis$ if $0 < x < 1$ and concave with respect to $x-axis$ if $x > 1$.

SOLUTION : Given curve is $y = \log x$. Then $\dfrac{dy}{dx} = \dfrac{1}{x}$ and $\dfrac{d^2y}{dx^2} = -\dfrac{1}{x^2}$. We know that $\log x < 0$ if $0 < x < 1$ and $\log x > 0$ if $x > 1$. Therefore, $y\dfrac{d^2y}{dx^2} = -\dfrac{\log x}{x^2} > 0$ if $0 < x < 1$ and $y\dfrac{d^2y}{dx^2} = -\dfrac{\log x}{x^2} < 0$ if $x > 1$. So, the given curve is convex with respect to $x-axis$ if $0 < x < 1$ and concave with respect to $x-axis$ if $x > 1$.

EXAMPLE-3 : Prove that the curve $y = e^x$ is convex to the $x - axis$ at every point.

OR

Show that the curve $y = e^x$ is at every point convex to the foot of the ordinate of the point.

OR

Show that the curve $y = e^x$ is everywhere concave upward.

SOLUTION : Given curve is $y = e^x$. Then $\frac{dy}{dx} = e^x$ and $\frac{d^2y}{dx^2} = e^x$. Therefore, $y\frac{d^2y}{dx^2} = e^{2x}$. Clearly, $y\frac{d^2y}{dx^2} > 0$ for all values of x.

Hence the given curve $y = e^x$ is convex to the $x - axis$ at every point.

EXAMPLE-4 : Show that the curve $y = x^3$ has a point of inflexion at the origin.

SOLUTION : Given curve is $y = x^3$. Then $\frac{dy}{dx} = 3x^2$, $\frac{d^2y}{dx^2} = 6x$ and $\frac{d^3y}{dx^3} = 6 \neq 0$ Therefore, at origin $\frac{dy}{dx} = 0$, $\frac{d^2y}{dx^2} = 0$ but $\frac{d^3y}{dx^3} \neq 0$. At origin an odd differential co-efficient is non-zero. Hence $(0,0)$ is the point of inflexion of the given curve.

EXAMPLE-5 : Show that the curve $y = x^4$ is concave upward at the origin.

SOLUTION : Given curve is $y = x^3$. Then $\frac{dy}{dx} = 4x^3$, $\frac{d^2y}{dx^2} = 12x^2$, $\frac{d^3y}{dx^3} = 24x$ and $\frac{d^4y}{dx^4} = 24 \neq 0$. So, at origin $\frac{dy}{dx} = 0$, $\frac{d^2y}{dx^2} = 0$, $\frac{d^3y}{dx^3} = 0$, $\frac{d^4y}{dx^4} \neq 0$ and > 0. Therefore, at origin all the differential co-efficients upto order 3 are zero and the even differential co-efficient of order 4 is non-zero and > 0. Hence the given curve $y = x^4$ is concave upward at the origin.

EXAMPLE-6: **Find the ranges of the values of x for which $y = x^4 - 10x^3 + 36x^2 + 5x + 3$ is concave upward or downward. Fin also its points of inflexion, if any.**

SOLUTION : Given curve is $y = x^4 - 10x^3 + 36x^2 + 5x + 3$.

$$\frac{dy}{dx} = 4x^3 - 30x^2 + 72x + 5$$

$$\frac{d^2y}{dx^2} = 12x^2 - 60x + 72 = 12(x^2 - 5x + 6) = 12(x-2)(x-3)$$

For $-\infty < x < 2$ and $3 < x < \infty$ $\frac{d^2y}{dx^2} > 0$ and for $2 < x < 3$ $\frac{d^2y}{dx^2} < 0$, ie., $\neq 0$.

Hence the curve is concave upward for all $x \in (-\infty, 2) \cup (3, \infty)$ and concave downward for all $x \in (2,3)$.

SECOND PART : We have $\frac{d^2y}{dx^2} = 12x^2 - 60x + 72$. So, $\frac{d^3y}{dx^3} = 24x - 60$.

At $x = 2$, $\frac{d^2y}{dx^2} = 0$ and $\frac{d^3y}{dx^3} = -12 \neq 0$. So at the points whose abscissa are 2 and 3 , odd differential co-efficients are non-zero. Hence the points of inflexions are $(2,93)$ and $(3,153)$.

EXAMPLE-7 : **Show that the curve $y = 3x^5 - 40x^3 + 3x - 20$ is concave upwards in $-2 < x < 0$ and $2 < x < \infty$ but concave downwards in $-\infty < x < -2$ and $0 < x < 2$ and at $x = -2$, $x = 0$, $x = 2$ there are points of inflexion.**

SOLUTION : Given curve is $y = 3x^5 - 40x^3 + 3x - 20$

$\frac{dy}{dx} = 15x^4 - 120x^2 + 3$, $\frac{d^2y}{dx^2} = 60x^3 - 240x = 60x(x^2 - 4)$. Therefore, $\frac{d^2y}{dx^2} = 60x(x+2))(x-2)$. Then for $-2 < x < 0$ and $2 < x < \infty$, $\frac{d^2y}{dx^2} > 0$

and for $-\infty < x < -2$ and $0 < x < 2$, $\frac{d^2y}{dx^2} < 0$. Hence the given curve is concave upwards in $-2 < x < 0$ and $2 < x < \infty$ and concave downwards in $-\infty < x < -2$ and $0 < x < 2$.

Now $\frac{d^2y}{dx^2} = 60x^3 - 240x$. $\therefore \frac{d^3y}{dx^3} = 180x^2 - 240$.

At $x = -2$, $\frac{d^2y}{dx^2} = 0$ but $\therefore \frac{d^3y}{dx^3} = 720 - 240 \neq 0$

At $x = 0$, $\frac{d^2y}{dx^2} = 0$ but $\therefore \frac{d^3y}{dx^3} = -240 \neq 0$

At $x = 2$, $\frac{d^2y}{dx^2} = 0$ but $\therefore \frac{d^3y}{dx^3} = 720 - 240 \neq 0$. Thus the given curve has points of inflexion at $x = -2$, $x = 0$, $x = 2$.

EXAMPLE-8 : Show that the curve $y = e^{-x^2}$ has points of inflexion at $x = \pm\frac{1}{\sqrt{2}}$.

SOLUTION : Given curve is $y = e^{-x^2}$.

Therefore, $\frac{dy}{dx} = -2xe^{-x^2}$, $\frac{d^2y}{dx^2} = -2e^{-x^2} + 4x^2e^{-x^2} = 2e^{-x^2}\left(2x^2 - 1\right)$. That is, $\frac{d^2y}{dx^2} = 2e^{-x^2}\left(2x^2 - 1\right) = 2e^{-x^2}\left(\sqrt{2}x + 1\right)\left(\sqrt{2}x - 1\right)$. So,

$$\frac{d^3y}{dx^3} = -4xe^{-x^2}\left(2x^2 - 1\right) + 8xe^{-x^2} = 12xe^{-x^2} - 8x^3e^{-x^2} = 4xe^{-x^2}\left(3 - 2x^2\right).$$

So, at $x = \pm\frac{1}{\sqrt{2}}$,

$\frac{d^2y}{dx^2} = 0$ but $\therefore \frac{d^3y}{dx^3} = \pm 4 \times \frac{1}{\sqrt{2}} \times e^{-\frac{1}{2}}(3 - 2) = \pm\frac{4}{\sqrt{2e}} \neq 0$. Hence the given curve has points of inflexion at $x = \pm\frac{1}{\sqrt{2}}$.

EXAMPLE-9 : **Show that origin is a point of inflexion of the curve** $a^2y^2 = x^2\left(a^2 - x^2\right)$

SOLUTION : **Given curve is** $a^2y^2 = x^2\left(a^2 - x^2\right)$. **That is,** $ay = x\sqrt{a^2 - x^2}$ **. Then**

$$a\frac{dy}{dx} = \frac{a^2 - x^2 - x^2}{\sqrt{a^2 - x^2}} = \frac{a^2 - 2x^2}{\sqrt{a^2 - x^2}}. \quad \Rightarrow a\frac{d^2y}{dx^2} = \frac{\sqrt{a^2 - x^2}(-4x) + (a^2 - 2x^2)\frac{x}{\sqrt{a^2 - x^2}}}{(a^2 - x^2)}$$

$$\Rightarrow a\frac{d^2y}{dx^2} = \frac{(a^2 - x^2)(-4x) + a^2x - 2x^3}{(a^2 - x^2)^{\frac{3}{2}}}$$

$$\Rightarrow a\frac{d^2y}{dx^2} = \frac{2x^3 - 3xa^2}{(a^2 - x^2)^{\frac{3}{2}}}.$$

Also $$a\frac{d^3y}{dx^3} = \frac{(a^2 - x^2)^{\frac{3}{2}}(6x^2 - 3a^2) + \frac{3}{2}(2x^3 - 3xa^2)(a^2 - x^2)^{\frac{1}{2}} \cdot 2x}{(a^2 - x^2)^3}$$

$$\Rightarrow a\frac{d^3y}{dx^3} = \frac{(a^2 - x^2)^{\frac{1}{2}}\left\{(a^2 - x^2)(6x^2 - 3a^2) + 6x^4 - 9x^2a^2\right\}}{(a^2 - x^2)^3}$$

$$\Rightarrow a\frac{d^3y}{dx^3} = \frac{6x^2a^2 - 3a^4 - 6x^4 + 3x^2a^2 + 6x^4 - 9x^2a^2}{(a^2 - x^2)^{\frac{5}{2}}}$$

$$\Rightarrow a\frac{d^3y}{dx^3} = \frac{-3a^4}{(a^2 - x^2)^{\frac{5}{2}}}.$$ **Clearly, at** $x = 0$, $\frac{d^2y}{dx^2} = 0$ **but** $\frac{d^3y}{dx^3} = \frac{-3}{a^2} \neq 0$

Hence at $x = 0$**, the curve has its point of inflexion.**

EXAMPLE-10 : **Find if there is any point of inflexion of the curve** $y - 3 = 6(x-2)^5$.

SOLUTION : **Given curve is** $y - 3 = 6(x-2)^5$. **Then** $\frac{dy}{dx} = 30(x-2)^4$,

$$\frac{d^2y}{dx^2} = 120(x-2)^3, \qquad \frac{d^3y}{dx^3} = 360(x-2)^2, \qquad \frac{d^4y}{dx^4} = 720(x-2),$$

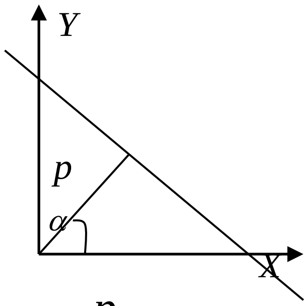
$\frac{d^5y}{dx^5} = 720 \neq 0$. **At** $x = 2$, $\frac{dy}{dx} = 0$, $\frac{d^2y}{dx^2} = 0$, $\frac{d^3y}{dx^3} = 0$, $\frac{d^4y}{dx^4} = 0$ **and**

$\frac{d^5y}{dx^5} = 720 \neq 0$. **So, all the differential co-effcients upto order 4 are zero but the 5th order (odd) differential co-effcient is non-zero. Hence** $x = 2$**, i.e., the point** $(2,3)$ **is the point of inflexion of the given curve.**

TASK:

EXAMPLE-11 : Show that the curve $(1+x^2)y = (1-x)$ **has three points of inflexion and that they lie on a straight line.**

ENVELOPE

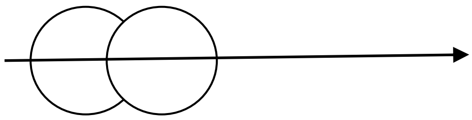


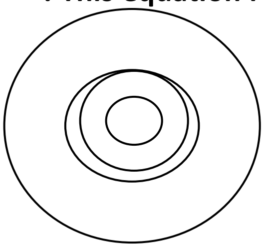
DEFINITION (FAMILY OF CURVES) :

Let us consider the equation $x\cos\alpha + y\sin\alpha = p$.

This equation represents a straight line. By giving different values to α , we shall obtain the equations of different straight lines having one characteristic feature common to them. The common feature is ---- each line is at same distance p from the origin. On account of this common property these straight lines are said to form a family, called, *"family of straight lines"*. Here α , which is constant for one line but different for different lines, and whose different values give the different members of the family, called the "*parameter*" of the family. The position of any straight line member varies with α .

Similarly, let us consider the equation $(x-\alpha)^2 + y^2 = r^2$. This equation represents a family of circles.





For the moment fixed, if we hold r and allow α to take a series of values, then we have a series of circles of equal radii r . In this case α is the parameter and r is fixed .

Again if we hold α and allow r to take a series of values, then we have a system of circles with common centre $(\alpha,0)$. In this case r is the parameter and α is fixed .

In both the cases we get *families of circles*.

A system of geometric figures (straight lines or curves) formed in this way is called *a family of curves*

In the above cases, we find a family of one-parameter curve,

Similarly, a family of two or more than two-parameters curves can be described.

The equation of one-parameter family of curves cab be expressed as $f(x,y,\alpha)=0$ where α is parameter.

The equation of two-parameters family of curves cab be expressed as $f(x,y,\alpha,\beta)=0$ where α and β are parameters.

DEFINITION (ENVELOPE) : If each of the members of the family of curves $f(x,y,\alpha)=0$ touches a fixed curve E , then that fixed curve E is called the *envelope* of the given family of curves.

DEFINITION (SINGULAR POINT) : A point $P(a,b)$ is said to be a singular point of a curve $f(x,y,\alpha)=0$ (α is fixed) if it satisfies, the equation of the given curve and other two equations $\frac{\partial f}{\partial x}=0$ and $\frac{\partial f}{\partial y}=0$.

DEFINITION (ORDINARY POINT) : A point $P(a,b)$ is said to be an ordinary point of a curve $f(x,y,\alpha)=0$ if at least one of the quantities $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ is not equal to zero at (a,b).

DEFINITION (CHARACTERISTIC POINTS) : Characteristic points are the ordinary points of a family of curves $f(x,y,\alpha)=0$ at which $f(x,y,\alpha)=0$ and $\frac{\partial f}{\partial \alpha}=0$.

MATHEMATICAL DEFINITION OF ENVELOPE : The locus of the characteristic points of a family of curves $f(x,y,\alpha)=0$ is called the envelope of that family.

NOTE : (1) Characteristic points may not exist. For example, the family of concentric circles $x^2+y^2=\alpha^2$, there is no characteristic point and hence there is no envelope.

(2) If $f(x,y,\alpha)=0$ and $\frac{\partial f}{\partial \alpha}=0$ both holds for a point where $\frac{\partial f}{\partial x}=0$ and $\frac{\partial f}{\partial y}=0$, then the point is a singular point and therefore, not a characteristic point

METHOD OF FINDING THE EQUATION OF AN ENVELOPE WHEN IT EXISTS.

CASE OF SINGLE PARAMETER :

If there exists an envelope, its equation may be obtained by either of the following process.

(1) Eliminating α from $f(x,y,\alpha)=0$ and $\frac{\partial f}{\partial \alpha}=0$, we obtain the equation of the envelope.

(2) Solving for x and y in terms of α fro the equations $f(x,y,\alpha)=0$ and $\frac{\partial f}{\partial \alpha}=0$, we get the parametric representation of the envelope.

(3) If $f(x,y,\alpha)=0$ can be expressed as

$f(x,y,\alpha)=A(x,y)\alpha^2+B(x,y)\alpha+C(x,y)$ and if two values of α are equal then the equation of the envelope is given by $B^2-4AC=0$.

CASE OF TWO PARAMETERS :

For a fixed point (x,y) of the envelope, we have from the equations $f(x,y,\alpha,\beta)=0$ …………(i) and $\phi(\alpha,\beta)=0$ ………….(ii) by differentiation $\frac{\partial f}{\partial \alpha}+\frac{\partial f}{\partial \beta}\cdot\frac{d\beta}{d\alpha}=0$ and $\frac{\partial \phi}{\partial \alpha}+\frac{\partial \phi}{\partial \beta}\cdot\frac{d\beta}{d\alpha}=0$. Eliminating $\frac{d\beta}{d\alpha}$ from the above two relations we get $\frac{\partial f}{\partial \alpha}\Big/\frac{\partial \phi}{\partial \alpha}=\frac{\partial f}{\partial \beta}\Big/\frac{\partial \phi}{\partial \beta}$ ………….(iii). Then eliminating α and β from equations (i), (ii), (iii), we obtain the equation of the envelope.

PROBLEMS

PROBLEM OF SINGLE PARAMETER

EXAMPLE 1 : Find the equation of the envelope of the family of straight lines $y=mx+\sqrt{a^2m^2+b^2}$. Where m is the parameter.

SOLUTION : Given family of straight lines is $y=mx+\sqrt{a^2m^2+b^2}$. Then

$(y-mx)^2=a^2m^2+b^2$

Or, $m^2(x^2-a^2)-2xym+y^2-b^2=0\ \left(\approx f(x,y,m)=0\right)$…………………(i).

Differentiating with respect to m, we get $\frac{\partial f}{\partial m}=2m(x^2-a^2)-2xy$. Let $\frac{\partial f}{\partial m}=0$. That is, $2m(x^2-a^2)-2xy$ …………..(ii). From (ii) we get $m=\frac{xy}{x^2-a^2}$. Putting the value of m in equation (i), we get $\frac{x^2y^2}{(x^2-a^2)}-\frac{2x^2y^2}{(x^2-a^2)}+y^2-b^2=0$, or $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Hence the require d envelope is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

EXAMPLE 2 : **Find the equation of the envelope of the family of straight lines** $x\cos\alpha + y\sin\alpha = 4$, **where** α **is the parameter.**

SOLUTION : **Given family of straight lines is** $x\cos\alpha + y\sin\alpha = 4\{\approx f(x,y,\alpha)=0\}$**..............(i). Differentiating with respect to** α **, we get** $\frac{\partial f}{\partial \alpha} = -x\sin\alpha + y\cos\alpha$**. Let** $\frac{\partial f}{\partial m} = 0$**. That is,** $-x\sin\alpha + y\cos\alpha = 0$ **............(ii). Squaring (i) and (ii) , we get**

$x^2\cos^2\alpha + y^2\sin^2\alpha + 2xy\sin\alpha\cos\alpha = 16$ **and**

$x^2\sin^2\alpha + y^2\cos^2\alpha - 2xy\sin\alpha\cos\alpha = 0$**. Adding these two equations, i.e., eliminating** α **, we get the required envelope as** $x^2 + y^2 = 16$

PROBLEM OF TWO OR MORE THAN TWO PARAMETERS :

EXAMPLE 3 : **Find the equation of the envelope of the family of straight lines** $\frac{x}{a} + \frac{y}{b} = 1$**, where the parameters** a **and** b **are connected by the relation** $a + b = k$**.**

SOLUTION : **Given family of straight lines is** $\frac{x}{a} + \frac{y}{b} = 1$**.................(i)**

Given relation is $a + b = k$ **.................(ii)**

From (ii) we have $b = k - a$**. Putting the value of** b **in (i) we get** $\frac{x}{a} + \frac{y}{k-a} - 1 = 0$ **.......(iii) Here we see that the given family of straight lines becomes a family of straight lines of single parameter** a **i.e.,** $\left(f(x,y,a)=0\right)$**. Differentiating both sides of (iii) with respect to** a **we get** $\frac{\partial f}{\partial a} = -\frac{x}{a^2} + \frac{y}{(k-a)^2} = 0 \Rightarrow \frac{x}{a} = \frac{ya}{(k-a)^2}$ **...(iv). Putting this value in (iii) we get** $\frac{ya}{(k-a)^2} + \frac{y}{k-a} = 1 \Rightarrow (k-a) = \sqrt{y}\sqrt{k}$**. From (iv)** $\frac{x}{a} = \frac{ya}{yk} \Rightarrow a = \sqrt{x}\sqrt{k}$ **. Putting the values of** $(k-a)$ **and** a **in (iii) we get**

$\frac{x}{\sqrt{x}\sqrt{k}} + \frac{y}{\sqrt{y}\sqrt{k}} - 1 = 0$ **or** $\sqrt{x} + \sqrt{y} = \sqrt{k}$**. Hence the required envelope is** $\sqrt{x} + \sqrt{y} = \sqrt{k}$**.**

ALTERNATIVE METHOD(TWO PARAMETERS METHOD)

Given family of straight lines is $\frac{x}{a}+\frac{y}{b}=1$**..................(i)**

Given relation is $a+b=k$ **..................(ii) Differentiating both sides of (i) with respect to** a **assuming** b **as function of** a **, we get** $-\frac{x}{a^2}-\frac{y}{b^2}\cdot\frac{db}{da}=0$ **.........(iii) . Similarly, differentiating both sides of (ii) with respect to** a **assuming** b **as function of** a **, we get** $1+\frac{db}{da}=0$ **..............(iv). Eliminating** $\frac{db}{da}$ **from (iii) and (iv) we get** $-\frac{x}{a^2}-\frac{y}{b^2}\times(-1)=0$ **or** $\frac{\sqrt{x}}{a}=\frac{\sqrt{y}}{k-a}=\frac{\sqrt{x}+\sqrt{y}}{k}$ **or** $a=\frac{k\sqrt{x}}{\sqrt{x}+\sqrt{y}}$ **. Therefore,** $b=k-a=k-\frac{k\sqrt{x}}{\sqrt{x}+\sqrt{y}}=\frac{k\sqrt{y}}{\sqrt{x}+\sqrt{y}}$ **. Now putting the values of** a **and** b **in (i) we get** $\frac{x}{\frac{k\sqrt{x}}{\sqrt{x}+\sqrt{y}}}+\frac{y}{\frac{k\sqrt{y}}{\sqrt{x}+\sqrt{y}}}=1$

or $\sqrt{x}+\sqrt{y}=\sqrt{k}$ **. Hence the required envelope is** $\sqrt{x}+\sqrt{y}=\sqrt{k}$ **.**

EXAMPLE 4 : Find the equation of the envelope of the family of straight lines $\frac{x}{a}+\frac{y}{b}=1$**, where the parameters** a **and** b **are connected by the relation** $ab=k^2$**.**

SOLUTION : Given family of straight lines is $\frac{x}{a}+\frac{y}{b}=1$**..................(i)**

Given relation is $ab=k^2$**..................(ii) Differentiating both sides of (i) & (ii) with respect to** a **assuming** b **as function of** a **, we get** $-\frac{x}{a^2}-\frac{y}{b^2}\cdot\frac{db}{da}=0$ **.........(iii) and** $b+a\frac{db}{da}=0$ **..............(iv). Eliminating** $\frac{db}{da}$ **from (iii) and (iv) we get** $-\frac{x}{a^2}-\frac{y}{b^2}\times\left(-\frac{b}{a}\right)=0$ **or** $\frac{x}{a^2}=\frac{y}{ab}$ **or** $\frac{x}{a^2}=\frac{y}{k^2}$ **or** $a=k\sqrt{\frac{x}{y}}$ **. From**

(ii) $b=\dfrac{k^2}{a}$ **or** $b=\dfrac{k^2}{k\sqrt{\dfrac{x}{y}}}$ **or** $b=k\sqrt{y/x}$. **Putting the values of** a **&** b **in (i) we get** $\dfrac{x}{k\sqrt{\dfrac{x}{y}}}+\dfrac{y}{k\sqrt{\dfrac{y}{x}}}=1$ **or** $\sqrt{xy}+\sqrt{xy}=k$ **or** $4xy=c^2$**. Hence the required envelope is** $4xy=c^2$**.**

EXAMPLE 5 : Find the equation of the envelope of the family of straight lines $\dfrac{x}{a}+\dfrac{y}{b}=1$**, where the parameters** a **and** b **are connected by the relation** $a^n+b^n=k^n$**.**

SOLUTION : Given family of straight lines is $\dfrac{x}{a}+\dfrac{y}{b}=1$**..................(i)**

Given relation is $a^n+b^n=k^n$ **..................(ii) Differentiating both sides of (i) & (ii) with respect to** a **assuming** b **as function of** a **, we get** $-\dfrac{x}{a^2}-\dfrac{y}{b^2}\cdot\dfrac{db}{da}=0$**.........(iii) and** $na^{n-1}+nb^{n-1}\dfrac{db}{da}=0$**..............(iv). Eliminating** $\dfrac{db}{da}$ **from (iii) and (iv) we get** $-\dfrac{x}{a^2}-\dfrac{y}{b^2}\times\left(-\dfrac{a^{n-1}}{b^{n-1}}\right)=0$ **or** $\dfrac{x}{a^{n+1}}=\dfrac{y}{b^{n+1}}$ **or** $\dfrac{x^{\frac{1}{n+1}}}{a}=\dfrac{y^{\frac{1}{n+1}}}{b}$

or $\dfrac{x^{\frac{n}{n+1}}}{a^n}=\dfrac{y^{\frac{n}{n+1}}}{b^n}=\dfrac{x^{\frac{n}{n+1}}+y^{\frac{n}{n+1}}}{a^n+b^n}=\dfrac{x^{\frac{n}{n+1}}+y^{\frac{n}{n+1}}}{k^n}$ **or** $a^n=\dfrac{k^n x^{\frac{n}{n+1}}}{x^{\frac{n}{n+1}}+y^{\frac{n}{n+1}}}$ **or**

$a=\dfrac{kx^{\frac{1}{n+1}}}{\left(x^{\frac{n}{n+1}}+y^{\frac{n}{n+1}}\right)^{\frac{1}{n}}}$ **and** $b^n=\dfrac{k^n y^{\frac{n}{n+1}}}{x^{\frac{n}{n+1}}+y^{\frac{n}{n+1}}}$ **or** $b=\dfrac{ky^{\frac{1}{n+1}}}{\left(x^{\frac{n}{n+1}}+y^{\frac{n}{n+1}}\right)^{\frac{1}{n}}}$ **Putting the values of** a **and** b **in (i) we get** $\dfrac{x}{\dfrac{kx^{\frac{1}{n+1}}}{\left(x^{\frac{n}{n+1}}+y^{\frac{n}{n+1}}\right)^{\frac{1}{n}}}}+\dfrac{y}{\dfrac{ky^{\frac{1}{n+1}}}{\left(x^{\frac{n}{n+1}}+y^{\frac{n}{n+1}}\right)^{\frac{1}{n}}}}=1$

Or $x^{\frac{n}{n+1}}\left(x^{\frac{n}{n+1}}+y^{\frac{n}{n+1}}\right)^{\frac{1}{n}}+y^{\frac{n}{n+1}}\left(x^{\frac{n}{n+1}}+y^{\frac{n}{n+1}}\right)^{\frac{1}{n}}=k$

Or $\left(x^{\frac{n}{n+1}}+y^{\frac{n}{n+1}}\right)^{\frac{1}{n}}\left(x^{\frac{n}{n+1}}+y^{\frac{n}{n+1}}\right)=k$

Or $\left(x^{\frac{n}{n+1}}+y^{\frac{n}{n+1}}\right)^{\frac{n+1}{n}}=k$

Or $\left(x^{\frac{n}{n+1}}+y^{\frac{n}{n+1}}\right)=k^{\frac{n}{n+1}}$. **Hence the required envelope is** $\left(x^{\frac{n}{n+1}}+y^{\frac{n}{n+1}}\right)=k^{\frac{n}{n+1}}$

EXAMPLE 6 : Find the equation of the envelope of the family of curves $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where the parameters a and b are connected by the relation $a+b=k$.

SOLUTION : Given family of straight lines is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$**..................(i)**

Given relation is $a+b=k$ **..................(ii) Differentiating both sides of (i) & (ii) with respect to** a **assuming** b **as function of** a **, we get** $-\frac{2x^2}{a^3}-\frac{2y^2}{b^3}\cdot\frac{db}{da}=0$ **.........(iii) and** $1+\frac{db}{da}=0$ **..............(iv). Eliminating** $\frac{db}{da}$ **from (iii) and (iv) we get** $-\frac{2x^2}{a^3}-\frac{2y^2}{b^3}\cdot(-1)=0$ **or,** $\frac{x^2}{a^3}=\frac{y^2}{b^3}$ **or,** $\frac{x^{\frac{2}{3}}}{a}=\frac{y^{\frac{2}{3}}}{b}$.

or, $\frac{x^{\frac{2}{3}}}{a}=\frac{y^{\frac{2}{3}}}{b}=\frac{x^{\frac{2}{3}}+y^{\frac{2}{3}}}{a+b}=\frac{x^{\frac{2}{3}}+y^{\frac{2}{3}}}{k}$ **or,** $a=\frac{kx^{\frac{2}{3}}}{x^{\frac{2}{3}}+y^{\frac{2}{3}}}$ **and** $b=\frac{ky^{\frac{2}{3}}}{x^{\frac{2}{3}}+y^{\frac{2}{3}}}$.

Putting the values of a **and** b **in (i) we get** $\frac{x^2}{\frac{k^2x^{\frac{4}{3}}}{\left(x^{\frac{2}{3}}+y^{\frac{2}{3}}\right)^2}}++\frac{y^2}{\frac{k^2y^{\frac{4}{3}}}{\left(x^{\frac{2}{3}}+y^{\frac{2}{3}}\right)^2}}=1$

or, $x^{\frac{2}{3}}\left(x^{\frac{2}{3}}+y^{\frac{2}{3}}\right)^2+y^{\frac{2}{3}}\left(x^{\frac{2}{3}}+y^{\frac{2}{3}}\right)^2=k^2$ or, $\left(x^{\frac{2}{3}}+y^{\frac{2}{3}}\right)^2\left(x^{\frac{2}{3}}+y^{\frac{2}{3}}\right)=k^2$

or, $\left(x^{\frac{2}{3}}+y^{\frac{2}{3}}\right)^3=k^2$ or, $\left(x^{\frac{2}{3}}+y^{\frac{2}{3}}\right)=k^{\frac{2}{3}}$. Hence the required envelope is $\left(x^{\frac{2}{3}}+y^{\frac{2}{3}}\right)=k^{\frac{2}{3}}$.

EXAMPLE 7 : Find the equation of the envelope of the family of curves $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where the parameters a and b are connected by the relation $\frac{a^2}{l^2}+\frac{b^2}{m^2}=1$.

SOLUTION : Given family of straight lines is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$..................(i)

Given relation is $\frac{a^2}{l^2}+\frac{b^2}{m^2}=1$ ....(ii) Differentiating both sides of (i) & (ii) with respect to a assuming b as function of a, we get $-\frac{2x^2}{a^3}-\frac{2y^2}{b^3}\cdot\frac{db}{da}=0$ ....(iii) and $\frac{2a}{l^2}+\frac{2b}{m^2}\frac{db}{da}=0$ ....(iv). Eliminating $\frac{db}{da}$ from (iii) and (iv) we get $-\frac{x^2}{a^3}-\frac{y^2}{b^3}\cdot(-\frac{a}{b}\frac{m^2}{l^2})=0$ or, $\frac{x^2l^2}{a^4}=\frac{y^2m^2}{b4}$ or, $\frac{\frac{x^2}{a^2}}{\frac{a^2}{l^2}}=\frac{\frac{y^2}{b^2}}{\frac{b^2}{m^2}}=\frac{\frac{x^2}{a^2}+\frac{y^2}{a^2}}{\frac{a^2}{l^2}+\frac{b^2}{m^2}}=\frac{1}{1}=1$ or, $a^4=x^2l^2$, $b^4=y^2m^2$ or, $a^2=\pm xl$, $b^2=\pm ym$.

Putting the values of a^2 and b^2 in (i) we get $\frac{x^2}{\pm xl}+\frac{y^2}{\pm ym}=1$. Hence the required envelope is $\pm\frac{x}{l}\pm\frac{y}{m}=1$.

EXAMPLE 8 : Find the equation of the envelope of the family of curves $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where the parameters a and b are connected by the relation $ab=c^2$.

SOLUTION : Given family of straight lines is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$..................(i)

Given relation is $ab=c^2$ ....(ii) Differentiating both sides of (i) & (ii) with respect to a assuming b as function of a, we get $-\frac{2x^2}{a^3}-\frac{2y^2}{b^3}\cdot\frac{db}{da}=0$ ....(iii) and $b+a\frac{db}{da}=0$ ....(iv). Eliminating $\frac{db}{da}$ from (iii) and (iv) we get $-\frac{x^2}{a^3}+\frac{y^2}{b^2a}=0$ or, $\frac{x^2}{a^2}=\frac{y^2}{b^2}$ or, $\frac{x^2}{a^2}=\frac{y^2a^2}{c^4}$

or, $a=c\frac{\sqrt{x}}{\sqrt{y}}$. Putting this value of a in (ii) we get $b=c\frac{\sqrt{y}}{\sqrt{x}}$. Putting the values of a and b in (i) we get . $\frac{x^2}{\frac{c^2x}{y}}+\frac{y^2}{\frac{c^2y}{x}}=1$. or, $xy+xy=c^2$ or, $2xy=c^2$. Hence the required envelope is $2xy=c^2$.

EXAMPLE 9 : Find the equation of the envelope of the family of curves $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where the parameters a and b are connected by the relation $\sqrt{a}+\sqrt{b}=\sqrt{c}$.

SOLUTION : Given family of straight lines is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$..................(i)

Given relation is $\sqrt{a}+\sqrt{b}=\sqrt{c}$ ....(ii) Differentiating both sides of (i) & (ii) with respect to a assuming b as function of a, we get $-\frac{2x^2}{a^3}-\frac{2y^2}{b^3}\cdot\frac{db}{da}=0$ ....(iii) and $\frac{1}{2\sqrt{a}}+\frac{1}{2\sqrt{b}}\frac{db}{da}=0$ ....(iv). Eliminating $\frac{db}{da}$ from (iii) and (iv) we get $-\frac{x^2}{a^3}+\frac{y^2}{b^3}\times\frac{\sqrt{b}}{\sqrt{a}}=0$ or, $\frac{x^2}{a^2\sqrt{a}}=\frac{y^2}{b^2\sqrt{b}}$

or, $\dfrac{\left(\dfrac{x^2}{a^2}\right)}{\sqrt{a}} = \dfrac{\left(\dfrac{y^2}{b^2}\right)}{\sqrt{b}} = \dfrac{\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}}{\sqrt{a}+\sqrt{b}} = \dfrac{1}{\sqrt{c}}$. or, $a^{\frac{5}{2}} = \sqrt{c}x^2$ and $b^{\frac{5}{2}} = \sqrt{c}y^2$

or, $a = c^{\frac{1}{5}}x^{\frac{4}{5}}$ and $b = c^{\frac{1}{5}}y^{\frac{4}{5}}$. Putting the values of a and b in (i) we get $\dfrac{x^2}{c^{\frac{2}{5}}x^{\frac{8}{5}}} + \dfrac{y^2}{c^{\frac{2}{5}}y^{\frac{8}{5}}} = 1$ or, $\left(x^{\frac{2}{5}} + y^{\frac{2}{5}}\right) = c^{\frac{2}{5}}$. Hence the required envelope is $x^{\frac{2}{5}} + y^{\frac{2}{5}} = c^{\frac{2}{5}}$.

EXAMPLE 10 : Find the equation of the envelope of the family of curves $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, where the parameters a and b are connected by the relation $a^m + b^m = c^m$.

SOLUTION : Given family of straight lines is $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$..................(i)

Given relation is $a^m + b^m = c^m$ ....(ii) Differentiating both sides of (i) & (ii) with respect to a assuming b as function of a , we get $-\dfrac{2x^2}{a^3} - \dfrac{2y^2}{b^3}\cdot\dfrac{db}{da} = 0$ ....(iii) and $ma^{m-1} + mb^{m-1}\dfrac{db}{da} = 0$ ....(iv). Eliminating $\dfrac{db}{da}$ from (iii) and (iv) we get $-\dfrac{x^2}{a^3}+\dfrac{y^2}{b^3}\times\dfrac{a^{m-1}}{b^{m-1}} = 0$ or, $\dfrac{x^2}{a^{m+2}} = \dfrac{y^2}{b^{m+2}}$

or, $\dfrac{\left(\dfrac{x^2}{a^2}\right)}{a^m} = \dfrac{\left(\dfrac{y^2}{b^2}\right)}{b^m} = \dfrac{\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}}{a^m+b^m} = \dfrac{1}{c^m}$. or, $a^{m+2} = c^m x^2$ and $b^{m+2} = c^m y^2$

or, $a = c^{\frac{m}{m+2}}x^{\frac{2}{m+2}}$ and $b = c^{\frac{m}{m+2}}y^{\frac{2}{m+2}}$. Putting the values of a and b in (i) we get $\dfrac{x^2}{c^{\frac{2m}{m+2}}x^{\frac{4}{m+2}}} + \dfrac{y^2}{c^{\frac{2m}{m+2}}y^{\frac{4}{m+2}}} = 1$ or, $\left(x^{\frac{2m}{m+2}} + y^{\frac{2m}{m+2}}\right) = c^{\frac{2m}{m+2}}$. Hence the required envelope is $x^{\frac{2m}{m+2}} + y^{\frac{2m}{m+2}} = c^{\frac{2m}{m+2}}$.

SOME PROBLEMS WHERE ENVELOPE OF A FAMILY OF CURVE OF TWO PARAMETERS IS GIVEN AND WE ARE TO FIND THE RELATION BETWEEN THE PARAMETERS

EXAMPLE 11 : The envelope of the family of straight lines $\frac{x}{a}+\frac{y}{b}=1$, where a and b are parameters, is given by $\sqrt{x}+\sqrt{y}=\sqrt{k}$. Find the relation between a and b.

SOLUTION : Given family of straight lines is $\frac{x}{a}+\frac{y}{b}=1$..................(i)

Envelope of the given family of straight lines is $\sqrt{x}+\sqrt{y}=\sqrt{k}$ ....(ii)
Differentiating both sides of (i) & (ii) with respect to x, we get
$\frac{1}{a}+\frac{1}{b}\cdot\frac{dy}{dx}=0$ ....(iii) and $\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}\cdot\frac{dy}{dx}=0$............(iv). Eliminating $\frac{dy}{dx}$ from (iii) and (iv) we get $\frac{\sqrt{y}}{\sqrt{x}}=\frac{b}{a}$. Let $\sqrt{x}=\lambda a$ and $\sqrt{y}=\lambda b$. Then from (ii) we get $\lambda a+\lambda b=\sqrt{k}$ or $\lambda=\frac{\sqrt{k}}{(a+b)}$. From (i) $\frac{\lambda^2 a^2}{a}+\frac{\lambda^2 b^2}{b}=1$ or, $\lambda^2(a+b)=1$ or, $\frac{k}{(a+b)^2}(a+b)=1$ or $a+b=k$. Hence the relation between the parameters a and b is $a+b=k$.

EXAMPLE 12 : The envelope of the family of straight lines $\frac{x}{a}+\frac{y}{b}=1$, where a and b are parameters, is given by $x^{\frac{2}{3}}+y^{\frac{2}{3}}=c^{\frac{2}{3}}$. Find the relation between a and b.

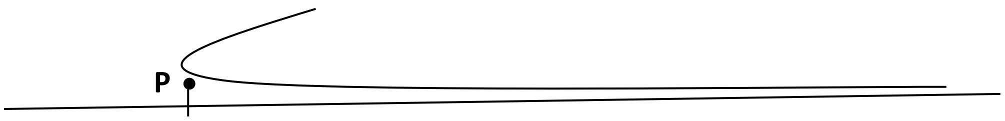
SOLUTION : Given family of straight lines is $\frac{x}{a}+\frac{y}{b}=1$..................(i)

Envelope of the given family of straight lines is $x^{\frac{2}{3}} + y^{\frac{2}{3}} = c^{\frac{2}{3}}$ **………………(ii) Differentiating both sides of (i) & (ii) with respect to** x**, we get** $\frac{1}{a} + \frac{1}{b} \cdot \frac{dy}{dx} = 0$ **………….(iii) and** $\frac{2}{3} x^{-\frac{1}{3}} + \frac{2}{3} y^{-\frac{1}{3}} \cdot \frac{dy}{dx} = 0$ **…………(iv). Eliminating** $\frac{dy}{dx}$ **from (iii) and (iv) we get** $\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}} = \frac{b}{a}$. **Let** $x^{\frac{1}{3}} = \lambda a$ **and** $y^{\frac{1}{3}} = \lambda b$**. Then from (ii) we get**

$\lambda^2 a^2 + \lambda^2 b^2 = c^{\frac{2}{3}}$ **or,** $\lambda^2 (a^2 + b^2) = c^{\frac{2}{3}}$ **……(v) Again from (i), we get** $\frac{\lambda^3 a^3}{a} + \frac{\lambda^3 b^3}{b} = 1$ **or,** $\lambda^3 (a^2 + b^2) = 1$ **……..(vi). Cubing (v) & squaring (vi) and then dividing, we get** $\frac{\lambda^6 (a^2 + b^2)^3}{\lambda^6 (a^2 + b^2)^2} = \frac{c^2}{1}$ **or,** $a^2 + b^2 = c^2$**. Hence the relation between the parameters** a **and** b **is** $a^2 + b^2 = c^2$

ASYMPTOTES

A straight line is said to be a *rectilinear asymptote* of an infinite branch of a curve if as a point P of the curve tends to infinity along the infinite branch, the perpendicular distance of the point P from that straight line tends to zero.



NOTE :- (i) Asymptotes may be parallel either to $x-axis$ or to $y-axis$. Asymptotes parallel to $x-axis$ are called *horizontal* asymptotes and asymptotes parallel to $y-axis$ are called *vertical* asymptotes otherwise they will be called *oblique* asymptotes.

(ii) For a curve lying wholly in a finite region, asymptotes cannot obviously exist. A circle or an ellipse has no asymptote.

But it does not necessarily mean that a curve having an infinite branch must have asymptote. Asymptote may or may not exist. For example, parabola is a curve extending to infinity but it has no asymptote.

RULES OF FINDING ASYMPTOTES OF AN ALGEBRAIC CURVE

An algebraic curve of the n^{th} degree can have at most n asymptotes.

(1) RULES OF FINDING HORIZONTAL/VERTICAL ASYMPTOTES OF AN ALGEBRAIC CURVE

Asymptotes parallel to $x-axis$ exist only when the co-efficient of the highest power of x is zero and in this case equating the co-efficient of the next highest available power of x is to zero, we get the equation of horizontal asymptote. Similarly, Asymptotes parallel

to $y-axis$ exist only when the co-efficient of the highest power of y is zero and in this case equating the co-efficient of the next highest available power of y is to zero, we get the equation of vertical asymptote.

(2) RULES OF FINDING OBLIQUE ASYMPTOTES OF AN ALGEBRAIC CURVE

The most general form of the equation of an algebraic curve of the n[th] degree can be written as

$$(a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \cdots + a_ny^n) + (b_0x^{n-1} + b_1x^{n-2}y + b_2x^{n-3}y^2 + \cdots + b_ny^{n-1}) +$$
$$(c_0x^{n-2} + c_1x^{n-3}y + c_2x^{n-4}y^2 + \cdots + c_ny^{n-2}) + \cdots\cdots = 0$$

Or,

$$x^n(a_0 + a_1\frac{y}{x} + a_2\frac{y^2}{x^2} + \cdots + a_n\frac{y^n}{x^n}) + x^{n-1}(b_0 + b_1\frac{y}{x} + b_2\frac{y^2}{x^2} + \cdots + b_n\frac{y^{n-1}}{x^{n-1}}) +$$
$$x^{n-2}(c_0 + c_1\frac{y}{x} + c_2\frac{y^2}{x^2} + \cdots + c_n\frac{y^{n-2}}{x^{n-2}}) + \cdots\cdots = 0$$

Or,

$$x^n\phi_n\left(\frac{y}{x}\right) + x^{n-1}\phi_{n-1}\left(\frac{y}{x}\right) + x^{n-2}\phi_{n-2}\left(\frac{y}{x}\right) + \cdots\cdots = 0$$

Oblique asymptotes are given by $y = mx + c$, where m is any of the real roots of the equation $\phi_n(m) = 0$ and for each value of m, c is given by

$c\phi_n'(m) + \phi_{n-1}(m) = 0$ if $\phi_n'(m) \neq 0$

If for any value of m, $\phi_n'(m) = 0$ then values of c are given by

$\frac{c^2}{2!}\phi_n''(m) + \frac{c}{1!}\phi_{n-1}'(m) + \phi_{n-2}(m) = 0$ and so on.

PROBLEMS ON RECTILINEAR ASYMPTOTES

EXAMPLE 1 : Find the asymptotes of the curve
$2x^3 - x^2y - 2xy^2 + y^3 - 4x^2 + 8xy - 4x + 1 = 0$.

SOLUTION : This is an algebraic curve of degree 3. So, it can have atmost 3 asymptotes. As the co-efficient of the highest power of x is $2(\neq 0)$, the given curve has no horizontal asymptote. Similarly, as the co-efficient of the highest power of y is $1(\neq 0)$, the given curve has no vertical asymptote. So, all the asymptotes (if exist) are oblique.
The equation of the given curve can be written as

$$x^3\left(2-\frac{y}{x}-2\frac{y^2}{x^2}+\frac{y^3}{x^3}\right)+x^2\left(-4+8\frac{y}{x}\right)+x(-4)+1=0.$$

Or, $x^3\phi_3\left(\frac{y}{x}\right)+x^2\phi_2\left(\frac{y}{x}\right)+x\phi_1\left(\frac{y}{x}\right)+1=0$

$\phi_3(m)=-2-m-2m^2+m^3$	$\phi_3'(m)=-1-4m+3m^2$	$\phi_3''(m)=-4m+6m$
$\phi_2(m)=-4+8m$	$\phi_2'(m)=8$	$\phi_2''(m)=0$
$\phi_1(m)=-4$	$\phi_1'(m)=0$	$\phi_1''(m)=0$

Oblique asymptotes are given by $y=mx+c$**, where** m **is any of the real roots of the equation** $\phi_3(m)=0$ **and for each value of** m**,** c **is given by** $c\phi_3'(m)+\phi_{3-1}(m)=0$

.

Therefore, $\phi_3(m)=-2-m-2m^2+m^3$**. Let** $\phi_3(m)=0$**, i.e.,** $-2-m-2m^2+m^3=0$ **or,** $m=1, -1, 2$.

For $m=1$, $c\phi_3'(m)+\phi_2(m)=0$ **or,** $c(-1-4m+3m^2)+(-4+8m)=0$

Or, $c(-1-4\times1+3\times1^2)+(-4+8\times1)=0$ **or,** $c=2$.

$$\boxed{m=1,\ c=2}$$

For $m=-1$, $c\phi_3'(m)+\phi_2(m)=0$ **or,** $c(-1-4m+3m^2)+(-4+8m)=0$

Or, $c(-1-4\times-1+3\times(-1)^2)+(-4+8\times-1)=0$ **or,** $c=2$.

$$\boxed{m=-1,\ c=2}$$

For $m=2$, $c\phi_3'(m)+\phi_2(m)=0$ **or,** $c(-1-4m+3m^2)+(-4+8m)=0$

Or, $c(-1-4\times2+3\times2^2)+(-4+8\times2)=0$ **or,** $c=-4$.

$$\boxed{m=2,\ c=-4}$$

Hence the required asymptotes are

$$\boxed{y=x+2,\quad y=-x+2,\quad y=2x-4}$$

EXAMPLE 2 : Find the asymptotes of the curve $x^3-2x^2y+xy^2+x^2-xy+2=0$.

SOLUTION : This is an algebraic curve of degree 3**. So, it can have atmost** 3 **asymptotes. As the co-efficient of the highest power of** x **is** $1(\neq 0)$**, the given curve has no horizontal asymptote. The co-efficient of the highest power of** y **is** 0**. The co-efficient of next highest available power of** y **is** x**. Hence the equation of the only vertical asymptote is**

$x=0$. So, the remaining two asymptotes are oblique. The equation of the given curve can be written as

$$x^3\left(1-2\frac{y}{x}+\frac{y^2}{x^2}\right)+x^2\left(1-\frac{y}{x}\right)+2=0.$$

Or, $x^3\phi_3\left(\frac{y}{x}\right)+x^2\phi_2\left(\frac{y}{x}\right)+2=0$

$\phi_3(m)=m^2-2m+1$	$\phi_3'(m)=2m-2$	$\phi_3''(m)=2$
$\phi_2(m)=1-m$	$\phi_2'(m)=-1$	$\phi_2''(m)=0$
$\phi_1(m)=0$	$\phi_1'(m)=0$	$\phi_1''(m)=0$

Oblique asymptotes are given by $y=mx+c$, where m is any of the real roots of the equation $\phi_3(m)=0$ and for each value of m, c is given by $c\phi_3'(m)+\phi_{3-1}(m)=0$.

Therefore, $\phi_3(m)=m^2-2m+1$. Let $\phi_3(m)=0$, i.e., $m^2-2m+1=0$ or, $m=1,1$.
For $m=1$, $\phi_3'(m)=2m-2=2\times1-2=0$
To obtain c let us consider the following relation.

$$\frac{c^2}{2!}\phi_3''(m)+\frac{c}{1!}\phi_{3-1}'(m)+\phi_{3-2}(m)=0$$

Or, $\frac{c^2}{2}(2)+\frac{c}{1}(-1)+0=0$, or $c^2-c=0$, or $c(c-1)=0$ or, $c=0$, $c=1$.

The two pair of values are

$$\boxed{m=1, c=0}$$

and

$$\boxed{m=1, c=1}$$

So, the oblique asymptotes are $y=x$, $y=x+1$.
Hence the required asymptotes are

$$\boxed{x=0,\ y=x,\ y=x+1,}$$

EXAMPLE 3 : Find the asymptotes of the curve $x^2y^2-x^2y-xy^2+x+y+1=0$.
SOLUTION : This is an algebraic curve of degree 4. So, it can have atmost 4 asymptotes.

The co-efficient of the highest power of x is 0. The co-efficient of next highest available power of x is $y^2 - y$. Let $y^2 - y = 0$. $y = 0$, $y = 1$. Hence equations of horizontal asymptotes are $y = 0$, $y = 1$.

The co-efficient of the highest power of y is 0. The co-efficient of next highest available power of y is $x^2 - x$. Let $x^2 - x = 0$. $x = 0$, $x = 1$. Hence equations of vertical asymptotes are $x = 0$, $x = 1$.

Hence the equations of required asymptotes are

$$\boxed{y = 0,\ y = 1,\ x = 0,\ x = 1}$$

ALTERNATIVE METHOD OF FINDING OBLIQUE ASYMPTOTES

(1) If the equation of the given curve is expressed as $(y - m_1 x)F_{n-1} + P_{n-1} = 0$, where

F_{n-1} contains the terms of degree $(n-1)$ and P_{n-1} contains the terms of degree not higher than $(n-1)$ then the equation of the asymptote parallel to $y - m_1 x = 0$ is given by $y - m_1 x + \underset{|x|\to\infty}{Lim} \dfrac{P_{n-1}}{F_{n-1}} = 0$ where $\underset{|x|\to\infty}{Lim} \dfrac{y}{x} = m_1$.

(2) If the equation of the given curve is expressed as ,

$(y - m_1 x)^2 F_{n-2} + (y - m_1 x)P_{n-2} + Q_{n-2} = 0$ where F_{n-2} & P_{n-2} contain the terms of Degree $(n-2)$ and Q_{n-2} contains the terms of degree not higher than $(n-2)$ then the equation of the asymptotes parallel to $y - m_1 x = 0$ are given by

$$(y - m_1 x)^2 + (y - m_1 x) \cdot \underset{|x|\to\infty}{Lim} \frac{P_{n-2}}{F_{n-2}} + \underset{|x|\to\infty}{Lim} \frac{Q_{n-2}}{F_{n-2}} = 0 \text{ where } \underset{|x|\to\infty}{Lim} \frac{y}{x} = m_1 .$$

Now let us try to find asymptotes of the curve given in example-(1) & example-(2) using the alternative method described above.

EXAMPLE 4 : **Find the asymptotes of the curve**
$2x^3 - x^2y - 2xy^2 + y^3 - 4x^2 + 8xy - 4x + 1 = 0$.

SOLUTION : This is an algebraic curve of degree 3**. So, it can have atmost** 3 **asymptotes. As the co-efficient of the highest power of** x **is** $2(\neq 0)$**, the given curve has no horizontal asymptote. Similarly, as the co-efficient of the highest power of** y **is** $1(\neq 0)$**, the given curve has no vertical asymptote. So, all the asymptotes (if exist) are oblique.**

ALTERNATIVE METHOD (for oblique asymptotes): The given equation can be written as
$(x^2 - y^2)(2x - y) - 4x^2 + 8xy - 4x + 1 = 0$

Or, $(y + x)(y - x)(y - 2x) + (8xy - 4x^2) + (1 - 4x) = 0$

Therefore, asymptote parallel to $y + x = 0$ **is given by**

$y + x + \underset{|x|\to\infty}{Lim} \dfrac{P_{n-2}}{F_{n-2}} = 0$ **where** $P_{n-2} = (8xy - 4x^2) + (1 - 4x)$ **and**

$F_{n-2} = (y - x)(y - 2x)$. **Also** $\underset{|x|\to\infty}{Lim} \dfrac{y}{x} = -1$**. Hence asymptote**

parallel to $y + x = 0$ **is** $y + x + \underset{|x|\to\infty}{Lim} \dfrac{(8xy - 4x^2) + (1 - 4x)}{(y - x)(y - 2x)} = 0$

or, $$y + x + \underset{|x|\to\infty}{Lim} \frac{\left(8\dfrac{y}{x} - 4\right) + \left(\dfrac{1}{x^2} - \dfrac{4}{x}\right)}{\left(\dfrac{y}{x} - 1\right)\left(\dfrac{y}{x} - 2\right)} = 0$$

or, $$y + x + \underset{|x|\to\infty}{Lim} \frac{(8\times(-1) - 4) + (0 - 0)}{(-1 - 1)(-1 - 2)} = 0 \quad \left(\because \underset{|x|\to\infty}{Lim} \frac{y}{x} = -1\right)$$

or, $y + x - 2 = 0$.

Again, asymptote parallel to $y - x = 0$ **is given by**

$y + x + \underset{|x|\to\infty}{Lim} \dfrac{P_{n-2}}{F_{n-2}} = 0$ **where** $P_{n-2} = (8xy - 4x^2) + (1 - 4x)$ **and**

$F_{n-2} = (y - x)(y - 2x)$. **Also** $\underset{|x|\to\infty}{Lim} \dfrac{y}{x} = 1$**. Hence asymptote**

parallel to $y - x = 0$ **is** $y - x + \underset{|x|\to\infty}{Lim} \dfrac{(8xy - 4x^2) + (1 - 4x)}{(y + x)(y - 2x)} = 0$

or, $$y-x+\underset{|x|\to\infty}{Lim}\frac{\left(8\frac{y}{x}-4\right)+\left(\frac{1}{x^2}-\frac{4}{x}\right)}{\left(\frac{y}{x}+1\right)\left(\frac{y}{x}-2\right)}=0$$

or, $$y-x+\underset{|x|\to\infty}{Lim}\frac{(8\times(1)-4)+(0-0)}{(1+1)(1-2)}=0 \quad \left(\because \underset{|x|\to\infty}{Lim}\frac{y}{x}=1\right)$$

or, $y-x-2=0$.

Similarly, asymptote parallel to $y-2x=0$ **is given by**

$$y-2x+\underset{|x|\to\infty}{Lim}\frac{P_{n-2}}{F_{n-2}}=0$$ **where** $P_{n-2}=(8xy-4x^2)+(1-4x)$ **and**

$F_{n-2}=(y+x)(y-x)$. **Also** $\underset{|x|\to\infty}{Lim}\frac{y}{x}=2$. **Hence asymptote**

parallel to $y-2x=0$ **is** $$y-2x+\underset{|x|\to\infty}{Lim}\frac{(8xy-4x^2)+(1-4x)}{(y+x)(y-x)}=0$$

or, $$y-2x+\underset{|x|\to\infty}{Lim}\frac{\left(8\frac{y}{x}-4\right)+\left(\frac{1}{x^2}-\frac{4}{x}\right)}{\left(\frac{y}{x}+1\right)\left(\frac{y}{x}-1\right)}=0$$

or, $$y-2x+\underset{|x|\to\infty}{Lim}\frac{(8\times(2)-4)+(0-0)}{(2+1)(2-1)}=0 \quad \left(\because \underset{|x|\to\infty}{Lim}\frac{y}{x}=2\right)$$

or, $y-2x+4=0$.

Therefore, oblique asymptotes are $y+x-2=0$, $y-x-2=0$ **and** $y-2x+4=0$
Hence the required asymptotes are

$$\boxed{y=x+2,\quad y=-x+2,\quad y=2x-4}$$

EXAMPLE 5 : Find the asymptotes of the curve $x^3-2x^2y+xy^2+x^2-xy+2=0$.

SOLUTION : This is an algebraic curve of degree 3. **So, it can have atmost** 3 **asymptotes. As the co-efficient of the highest power of** x **is** $1(\neq 0)$, **the given curve has no horizontal asymptote. The co-efficient of the highest power of** y **is** 0. **The co-efficient of next highest available power of** y **is** x. **Hence the equation of the only vertical asymptote is** $x=0$. **So, the remaining two asymptotes are oblique.**

ALTERNATIVE METHOD (for oblique asymptotes): The given equation can be written as $x(y-x)^2 - x(y-x) + 2 = 0$

Equation of the given curve is expressed as ,

$(y-x)^2 F_{3-2} + (y-x)P_{3-2} + Q_{3-2} = 0$ **where** $F_{3-2} = x$ **&** $P_{3-2} = -x$ **contain the terms of Degree** $(3-2)$ **and** Q_{3-2} **contains the terms of degree not higher than** $(3-2)$ **, then the equation of the asymptotes parallel to** $y - x = 0$ **are given by**

$(y-x)^2 + (y-x)\cdot \underset{|x|\to\infty}{Lim} \dfrac{P_{3-2}}{F_{3-2}} + \underset{|x|\to\infty}{Lim} \dfrac{Q_{3-2}}{F_{3-2}} = 0$ **where** $F_{3-2} = x$, $P_{3-2} = -x$ **and** $Q_{3-2} = 2$. **Also,** $\underset{|x|\to\infty}{Lim} \dfrac{y}{x} = 1$. **That is, the equation of the asymptotes parallel to** $y - x = 0$ **are** $(y-x)^2 + (y-x)\cdot \underset{|x|\to\infty}{Lim} \dfrac{-x}{x} + \underset{|x|\to\infty}{Lim} \dfrac{2}{x} = 0$.

Or, $(y-x)^2 - (y-x) + 0 = 0$

Or, $(y-x)\{(y-x)-1\} = 0$

Or, $y - x = 0$, $(y-x) - 1 = 0$

Therefore, oblique asymptotes are $y - x = 0$ **and** $y - x = 1$.

Hence the required asymptotes are

$$\boxed{x = 0,\ \ y - x = 0,\ \ y - x = 1}$$

ASYMPTOTES BY INSPECTION

If the equation of a curve be of the form $F_n + F_{n-2} = 0$ **where** F_n **is a polynomial of degree** n **and** F_{n-2} **is a polynomial of degree not higher than** $(n-2)$ **and if** F_n **can be broken up into** n **distinct linear factors then all the asymptotes of the curve are given by** $F_n = 0$.

EXAMPLE 6 : Find the asymptotes of the curve $(x-y+2)(2x-3y+4)(4x-5y+6) + 5x - 6y + 7 = 0$.

SOLUTION : This is an algebraic curve of degree 3**. So, it can have atmost** 3 **asymptotes.** **As the co-efficient of the highest power of** x **is not equal to** 0**, the given curve has no horizontal asymptote. Similarly, as the co-efficient of the highest power of** y **is not**

equal to 0 **, the given curve has no vertical asymptote. So, all the asymptotes (if exist) are oblique.**

The equation of the given curve can be written as $F_3 + F_{3-2} = 0$**. Hence by the method of inspection asymptotes are given by** $F_3 = 0$**.**

That is, $(x - y + 2)(2x - 3y + 4)(4x - 5y + 6) = 0$**. Therefore required asymptotes are**

$$\boxed{(x - y + 2) = 0,\ (2x - 3y + 4) = 0,\ (4x - 5y + 6) = 0}$$

THEOREM : Any asymptote of an algebraic curve of nth degree intersects the curve at $(n-2)$ **points.**

COROLLARY : The n **asymptotes of an algebraic curve of nth degree intersects the curve at** $n(n-2)$ **points.**

IMPORTANT NOTE

We know that if the equation of a curve be of the form $F_n + F_{n-2} = 0$ **where** F_n **is a polynomial of degree** n **and** F_{n-2} **is a polynomial of degree not higher than** $(n-2)$ **and if** F_n **can be broken up into** n **distinct linear factors then all the asymptotes of the curve are given by** $F_n = 0$**.**

Now equation of the given curve is $F_n + F_{n-2} = 0 \ldots\ldots\ldots\ldots(1)$ **and equation of all the asymptotes of the curve are** $F_n = 0 \ldots\ldots\ldots\ldots(2)$**. So points of intersection of the curve and the asymptotes will satisfy both the equations (1) & (2). Again as** $F_n = 0$**, points of intersection of the curve and the asymptotes will satisfy the equation** $F_{n-2} = 0$**. Hence all the points of intersection of the given curve and the asymptotes will lie on the curve** $F_{n-2} = 0$**.**

EXAMPLE 7 : Show that the four asymptotes of the curve $(x^2 - y^2)(y^2 - 4x^2) + 6x^3 - 5x^2y - 3xy^2 + 2y^3 - x^2 + 3xy - 1 = 0$ **intersect the curve in eight points which lie on the circle** $x^2 + y^2 = 1$

SOLUTION : This is an algebraic curve of degree 4 **. So, it can have atmost** 4 **asymptotes. As the co-efficient of the highest power of** x **is not equal to** 0**, the given curve has no horizontal asymptote. Similarly, as the co-efficient of the highest power of** y **is not**

equal to 0 **, the given curve has no vertical asymptote. So, all the asymptotes (if exist) are oblique.**
Now the equation of given curve can be written as

$$(x-y)(x+y)(y-2x)(y+2x)+6x^3-5x^2y-3xy^2+2y^3-x^2+3xy-1=0$$

So, the equation of the asymptote parallel to $(x-y)=0$ **is**

$$x-y+\underset{|x|\to\infty}{Lim}\frac{P_{4-1}}{F_{4-1}}=0$$

Or, $x-y+\underset{|x|\to\infty}{Lim}\dfrac{6x^3-5x^2y-3xy^2+2y^3-x^2+3xy-1}{(x+y)(y-2x)(y+2x)}=0$ **where**

$F_{4-1}=(x+y)(y-2x)(y+2x)$ **,** $P_{4-1}=6x^3-5x^2y-3xy^2+2y^3-x^2+3xy-1$**.**

Also, $\underset{|x|\to\infty}{Lim}\dfrac{y}{x}=1$.

Or, $x-y+\underset{|x|\to\infty}{Lim}\dfrac{6-5\left(\frac{y}{x}\right)-3\left(\frac{y}{x}\right)^2+2\left(\frac{y}{x}\right)^3-\frac{1}{x}+3\left(\frac{y}{x}\right)\cdot\frac{1}{x}-\frac{1}{x^3}}{(1+\frac{y}{x})(\frac{y}{x}-2)(\frac{y}{x}+2)}=0$

Or, $x-y+0=0$
Or,

$$\boxed{x-y=0}$$

The equation of the asymptote parallel to $(x+y)=0$ **is**

$$x+y+\underset{|x|\to\infty}{Lim}\frac{P_{4-1}}{F_{4-1}}=0$$

Or, $x-y+\underset{|x|\to\infty}{Lim}\dfrac{6x^3-5x^2y-3xy^2+2y^3-x^2+3xy-1}{(x-y)(y-2x)(y+2x)}=0$ **where**

$F_{4-1}=(x-y)(y-2x)(y+2x)$**,** $P_{4-1}=6x^3-5x^2y-3xy^2+2y^3-x^2+3xy-1$**.**

Also, $\underset{|x|\to\infty}{Lim}\dfrac{y}{x}=-1$.

Or, $$x+y+\underset{|x|\to\infty}{Lim}\frac{6-5\left(\frac{y}{x}\right)-3\left(\frac{y}{x}\right)^2+2\left(\frac{y}{x}\right)^3-\frac{1}{x}+3\left(\frac{y}{x}\right)\cdot\frac{1}{x}-\frac{1}{x^3}}{(1-\frac{y}{x})(\frac{y}{x}-2)(\frac{y}{x}+2)}=0$$

Or, $x+y-1=0 \quad \left(\because \underset{|x|\to\infty}{Lim}\frac{y}{x}=-1\right)$

Or,

$$\boxed{x+y-1=0}$$

The equation of the asymptote parallel to $(y-2x)=0$ **is**

$$y-2x+\underset{|x|\to\infty}{Lim}\frac{P_{4-1}}{F_{4-1}}=0$$

Or, $y-2x+\underset{|x|\to\infty}{Lim}\dfrac{6x^3-5x^2y-3xy^2+2y^3-x^2+3xy-1}{(x-y)(x+y)(y+2x)}=0$ **where** $F_{4-1}=(x-y)(x+y)(y+2x)$, $P_{4-1}=6x^3-5x^2y-3xy^2+2y^3-x^2+3xy-1$**. Also,** $\underset{|x|\to\infty}{Lim}\frac{y}{x}=2$.

Or, $$x+y+\underset{|x|\to\infty}{Lim}\frac{6-5\left(\frac{y}{x}\right)-3\left(\frac{y}{x}\right)^2+2\left(\frac{y}{x}\right)^3-\frac{1}{x}+3\left(\frac{y}{x}\right)\cdot\frac{1}{x}-\frac{1}{x^3}}{(1-\frac{y}{x})(1+\frac{y}{x})(\frac{y}{x}+2)}=0$$

Or, $y-2x+0=0 \quad \left(\because \underset{|x|\to\infty}{Lim}\frac{y}{x}=2\right)$

Or,

$$\boxed{y-2x=0}$$

The equation of the asymptote parallel to $(y+2x)=0$ **is**

$$y+2x+\underset{|x|\to\infty}{Lim}\frac{P_{4-1}}{F_{4-1}}=0$$

Or, $y+2x+\underset{|x|\to\infty}{Lim}\dfrac{6x^3-5x^2y-3xy^2+2y^3-x^2+3xy-1}{(x-y)(x+y)(y-2x)}=0$ where

$F_{4-1}=(x-y)(x+y)(y-2x)$, $P_{4-1}=6x^3-5x^2y-3xy^2+2y^3-x^2+3xy-1$. Also,

$\underset{|x|\to\infty}{Lim}\dfrac{y}{x}=-2$.

Or, $$y+2x+\underset{|x|\to\infty}{Lim}\frac{6-5\left(\frac{y}{x}\right)-3\left(\frac{y}{x}\right)^2+2\left(\frac{y}{x}\right)^3-\frac{1}{x}+3\left(\frac{y}{x}\right)\cdot\frac{1}{x}-\frac{1}{x^3}}{(1-\frac{y}{x})(1+\frac{y}{x})(\frac{y}{x}-2)}=0$$

Or, $y+2x-1=0$ $\quad\left(\because \underset{|x|\to\infty}{Lim}\dfrac{y}{x}=-2\right)$

Or,

$$\boxed{y+2x-1=0}$$

Therefore, equations of all the four asymptotes(oblique) are

$x-y=0$	$x+y-1=0$
$y-2x=0$	$y-2x-1=0$

The joint equation of the asymptotes of the given curve is

$(x-y)(x+y-1)(y-2x)(y+2x-1)=0$

Or, $(x^2-y^2)(y^2-4x^2)+6x^3-5x^2y-3xy^2+2y^3-2x^2+3xy-y^2=0$

$\equiv F_4$ **(say)**

Now the equation of the given curve can be written

$\{(x^2-y^2)(y^2-4x^2)+6x^3-5x^2y-3xy^2+2y^3-2x^2+3xy-y^2\}+(x^2+y^2-1)=0$

or, $F_4+F_2=0$ $\quad[\equiv F_n+F_{n-2}=0]$

Where F_4 **represents the joint equation of four asymptotes of the given curve. Also four asymptotes cut the given curve in** $4(4-2)=8$ **(eight) points. Hence the eight points of intersection of the given curve and the asymptotes must lie on** $F_2=0$**, i.e., on** $x^2+y^2=1$.

EXAMPLE 8 : Show that the eight points of intersection of the curve $x^4-5x^2y^2+4y^4+x^2-y^2+x+y+1=0$ and its asymptotes lie on a rectangular hyperbola.

SOLUTION : This is an algebraic curve of degree 4 . So, it can have atmost 4 asymptotes. As the co-efficient of the highest power of x is not equal to 0, the given curve has no horizontal asymptote. Similarly, as the co-efficient of the highest power of y is not equal to 0, the given curve has no vertical asymptote. So, all the asymptotes (if exist) are oblique.

Now the equation of given curve can be written as

$x^4-4x^2y^2-x^2y^2+4y^4+x^2-y^2+x+y+1=0$

Or, $x^2(x^2-4y^2)-y^2(x^2-4y^2)+x^2-y^2+x+y+1=0$

Or, $(x^2-4y^2)(x^2-y^2)+x^2-y^2+x+y+1=0$

Or, $(x-y)(x+y)(x+2y)(x-2y)+x^2-y^2+x+y+1=0$

Therefore, the equation of the given curve is expressed as $F_4+F_{4-2}=0$. Hence by the method of inspection joint equation of the asymptotes of the given curve is $F_4=0$. i.e.,

$(x-y)(x+y)(x+2y)(x-2y)=0$.

That is, $x^4-4x^2y^2-x^2y^2+4y^4=0$

$\equiv F_4$

The equation of given is $x^4-5x^2y^2+4y^4+x^2-y^2+x+y+1=0$

Or, $(x^4-4x^2y^2-x^2y^2+4y^4)+(x^2-y^2+x+y+1)=0$

Or, $F_4+F_{4-2}=0$**. Where F_4 represents the joint equation of four asymptotes of the given curve. Also four asymptotes cut the given curve in $4(4-2)=8$ (eight) points. Hence the eight points of intersection of the given curve and the asymptotes must lie on $F_2=0$, i.e., on $(x^2-y^2+x+y+1)=0$.**

i.e., on $\left(y-\frac{1}{2}\right)^2-\left(x-\frac{1}{2}\right)^2=1$ **which is a rectangular hyperbola.**

ASYMPTOTES IN POLAR CO-ORDINATE IN SYSTEM

Let the equation of the given curve be $r = f(\theta)$**. Let us change** $\frac{1}{r}$ **to** u **. Then the given curve becomes** $u = F(\theta)$**. Let us find** $\frac{du}{d\theta} = F'(\theta)$**. If** $r \to \infty$ **then** $u \to 0$ **&** $\theta \to \alpha$ **(say). Let us find** $F'(\alpha)$**, that is** $\frac{du}{d\theta}$ **at** $\theta = \alpha$**. Then the required asymptote is** $r\sin(\theta - \alpha) = \frac{1}{F'(\alpha)}$.

EXAMPLE 9 : Find the asymptotes of the polar curve $r = a\tan\theta$.

SOLUTION : Let $u = \frac{1}{r}$**. Then the given curve becomes** $u = \frac{1}{a\tan\theta} = \frac{\cot\theta}{a} = F(\theta)$.

Then $\frac{du}{d\theta} = F'(\theta) = \frac{-\operatorname{cosec}^2\theta}{a}$**. If** $r \to \infty$ **then** $u \to 0$ **&** $\cot\theta \to 0$**. Therefore,**

$\theta \to (2n+1)\frac{\pi}{2}$ **{** $= \alpha$ **say },** $n = 1,2,3,\cdots\cdots$**. So,** $F'(\alpha) = \dfrac{-\operatorname{cosec}^2(2n+1)\frac{\pi}{2}}{a}$.

Therefore, the required asymptotes are $r\sin(\theta - \alpha) = \frac{1}{F'(\alpha)}$

Or, $r\sin(\theta - (2n+1)\frac{\pi}{2}) = -a\sin^2(2n+1)\frac{\pi}{2} = -a$

Or, $-r\sin(2n+1)\frac{\pi}{2} - \theta) = -a$

Or, $\pm r\cos\theta = a$

Or, $r\cos\theta = \pm a$.

EXAMPLE 10 : Find the asymptotes of the polar curve $r^n\sin(n\theta) = a^n$.

SOLUTION : Let $u=\frac{1}{r}$. Then the given curve becomes $u^n=\frac{\sin(n\theta)}{a^n}=F(\theta)$ (say).

Then $n\log u=\log\sin(n\theta)-n\log a$. Differentiating, we get $\frac{n}{u}\cdot\frac{du}{d\theta}=\frac{n\cos(n\theta)}{\sin(n\theta)}$.

Or, $\frac{du}{d\theta}=u\cot(n\theta)=\frac{\sin^{\frac{1}{n}}(n\theta)}{a}\cdot\cot(n\theta)\left(\because u^n=\frac{\sin(n\theta)}{a^n}\right)$

Or, $\frac{du}{d\theta}=F'(\theta)=\frac{\sin^{\frac{1}{n}-1}(n\theta)}{a}\cdot\cos(n\theta)$. If $r\to\infty$ then $u\to 0$ & $\sin(n\theta)\to 0$.

Therefore, $n\theta\to k\pi$, i.e., $\theta\to\frac{k\pi}{n}$ { $=\alpha$ say }, $k=1,2,3,\cdots\cdots$.

So, $F'(\alpha)=\frac{(\sin k\pi)^{\frac{1}{n}-1}}{a}\cdot\cos(n\theta)$. Therefore, the required asymptotes are

$r\sin(\theta-\alpha)=\frac{1}{F'(\alpha)}$, or, $r\sin(\theta-\frac{k\pi}{n})=\frac{1}{F'(\alpha)}$. If $n>1$ then $\frac{1}{n}<1$, i.e.,

$\frac{1}{n}-1<0$ then $F'(\alpha)=\frac{\cos(n\theta)}{a(\sin(k\pi))^{1-\frac{1}{n}}}=\infty$. In that case asymptotes are

$r\sin(\theta-\frac{k\pi}{n})=\frac{1}{\infty}=0$ or, $\sin(\theta-\frac{k\pi}{n})=0$ or, $\theta=\frac{k\pi}{n}$ or,

$$\boxed{n\theta=k\pi}$$

. Again If $n<1$ then $\frac{1}{n}>1$, i.e., $\frac{1}{n}-1>0$ then $F'(\alpha)=0$. That is, $\frac{1}{F'(\alpha)}=\infty$ and in that case no asymptote will exist.

EXAMPLE 11 : **Find the asymptotes of the polar curve** $r=2a\sin\theta\tan\theta$.

SOLUTION : Let $u=\frac{1}{r}$. Then the given curve becomes $u=\frac{\cot\theta}{2a\sin\theta}=F(\theta)$ (say).

Then $\frac{du}{d\theta}=F'(\theta)=\frac{-1}{2a\sin\theta}$. If $r\to\infty$ then $u\to 0$ & $\cot\theta\to 0$. Therefore,

$\theta\to(2n+1)\frac{\pi}{2}$ { $=\alpha$ say }, Then $F'(\alpha)=\frac{-1}{2a\sin(2n+1)\frac{\pi}{2}}=\frac{-1}{2a\cdot(\pm 1)}=\frac{-1}{\pm 2a}$.

Then the required asymptotes are $r\sin(\theta-\alpha)=\dfrac{1}{F'(\alpha)}$ or,

$$r\sin\left(\theta-(2n+1)\frac{\pi}{2}\right)=\frac{1}{\frac{-1}{\pm 2a}}=\frac{\pm 2a}{-1}$$

or, $-r\sin\left((2n+1)\dfrac{\pi}{2}-\theta\right)=-(\pm 2a)$ or, $\pm r\cos\theta=\pm 2a$ or, $r\cos\theta=2a$.

The required asymptotes are

$$\boxed{r\cos\theta=2a}$$

EXAMPLE 12 : Find the asymptotes of the polar curve $(2r-3)\sin\theta=5$.

SOLUTION : Let $u=\dfrac{1}{r}$. **Then the given curve becomes** $u=\dfrac{2\sin\theta}{5+3\sin\theta}=F(\theta)$ **(say).**

Then $\dfrac{du}{d\theta}=F'(\theta)=\dfrac{(5+3\sin\theta)2\cos\theta-2\sin\theta\times 3\cos\theta}{(5+3\sin\theta)^2}=\dfrac{10\cos\theta}{(5+3\sin\theta)^2}$. **If** $r\to\infty$ **then** $u\to 0$ **&** $\sin\theta\to 0$. **Therefore,** $\theta\to n\pi$ **{** $=\alpha$ **say },** **Then**

$$F'(\alpha)=\frac{10\cos n\pi}{(5+3\sin n\pi)^2}=\frac{10\cos n\pi}{25}=\pm\frac{2}{5}.$$

Then the required asymptotes are $r\sin(\theta-\alpha)=\dfrac{1}{F'(\alpha)}$ **or,**

$$r\sin(\theta-n\pi)=\frac{1}{\pm\frac{2}{5}}=\pm\frac{5}{2}$$

or, $-r\sin(n\pi-\theta)=\pm\dfrac{5}{2}$ **or,** $\pm r\sin\theta=\pm\dfrac{5}{2}$ **or,** $r\sin\theta=\dfrac{5}{2}$. **The required asymptotes are**

$$\boxed{r\sin\theta=\frac{5}{2}}$$

EXAMPLE 13 : Show that there is an infinite series of parallel asymptotes to the curve $r=\dfrac{a}{\theta\sin\theta}+b$.

SOLUTION : Let $u = \dfrac{1}{r}$ **. Then the given curve becomes** $u = \dfrac{\theta \sin\theta}{a + b\theta\sin\theta} = F(\theta)$ **(say).**

Then $\dfrac{du}{d\theta} = F'(\theta) = \dfrac{(a+b\theta\sin\theta)(\sin\theta+\theta\cos\theta) - \theta\sin\theta(b\sin\theta + b\theta\cos\theta)}{(a+b\theta\sin\theta)^2}$

Or, $\dfrac{du}{d\theta} = F'(\theta) = \dfrac{a(\sin\theta+\theta\cos\theta)}{(a+b\theta\sin\theta)^2}$ **.**

If $r \to \infty$ **then** $u \to 0$ **&** $\sin\theta \to 0$**. Therefore,** $\theta \to n\pi$ **{** $= \alpha$ **say }. Then** $F'(\alpha) = \dfrac{a(\sin n\pi + n\pi\cos n\pi)}{(a+bn\pi\sin n\pi)^2} = \pm\dfrac{n\pi}{a}$ **. Then the required asymptotes are**

$r\sin(\theta-\alpha) = \dfrac{1}{F'(\alpha)}$ **or,** $r\sin(\theta - n\pi) = \dfrac{1}{\pm\dfrac{n\pi}{a}} = \dfrac{a}{\pm n\pi}$ **or,**

$-r\sin(n\pi - \theta) = \dfrac{a}{\pm n\pi}$ **or,** $(-1)\cdot \pm r\sin\theta = \dfrac{a}{\pm n\pi}$ **or,** $r\sin\theta = \dfrac{a}{n\pi}$ **. Giving different values of** n **, we get an infinite series of parallel asymptotes.**

INDETERMINATE FORMS

We know that $\underset{x\to a}{Lim}\dfrac{f(x)}{g(x)}=\dfrac{\underset{x\to a}{Lim}\,f(x)}{\underset{x\to a}{Lim}\,g(x)}$. That is, limiting value of the quotient of two functions $f(x)$ and $g(x)$ is , in general, quotient of their individual limits $\underset{x\to a}{Lim}\,f(x)$ and $\underset{x\to a}{Lim}\,g(x)$. But if both the limits $\underset{x\to a}{Lim}\,f(x)$ and $\underset{x\to a}{Lim}\,g(x)$ are equal to zero then this rule is no longer applicable because in that case the limit will be of the form $\dfrac{0}{0}$ $\left(\because \underset{x\to a}{Lim}\dfrac{f(x)}{g(x)}=\dfrac{\underset{x\to a}{Lim}\,f(x)}{\underset{x\to a}{Lim}\,g(x)}=\dfrac{0}{0}\right)$ which is clearly, meaningless. For example, if we consider the limit $\underset{x\to 0}{Lim}\dfrac{\sin x}{x}=\dfrac{\underset{x\to 0}{Lim}\sin x}{\underset{x\to 0}{Lim}\,x}=\dfrac{0}{0}$. So, this limit $\underset{x\to 0}{Lim}\dfrac{\sin x}{x}$ takes the indeterminate form $\dfrac{0}{0}$. But we know that $\underset{x\to 0}{Lim}\dfrac{\sin x}{x}=1$. In this article we shall consider the cases where given limit takes the indeterminate forms like $\dfrac{0}{0}$, $\dfrac{\infty}{\infty}$, $0\times\infty$, $\infty-\infty$. We w shall consider also the cases where given limit takes the indeterminate forms like 0^0, ∞^0, 1^∞, $1^{-\infty}$.

L' HOSPITAL'S RULE

L' Hospital's Theorem : If two functions $f(x)$ and $g(x)$ as also their derivatives $f'(x)$ and $g'(x)$ are continuous at $x=a$ and if $\underset{x\to a}{Lim}\,f(x)=0=\underset{x\to a}{Lim}\,g(x)$ then $\underset{x\to a}{Lim}\dfrac{f(x)}{g(x)}=\underset{x\to a}{Lim}\dfrac{f'(x)}{g'(x)}$ provided $\underset{x\to a}{Lim}\,g'(x)\neq 0$.

Generalization : If $\underset{x\to a}{Lim}\,f'(x)=\underset{x\to a}{Lim}\,g'(x)=0$ then

$\underset{x\to a}{Lim}\dfrac{f(x)}{g(x)}=\underset{x\to a}{Lim}\dfrac{f'(x)}{g'(x)}=\underset{x\to a}{Lim}\dfrac{f''(x)}{g''(x)}$ provided $\underset{x\to a}{Lim}\,g''(x)\neq 0$ and so on.

LIMITS WHICH TAKE $\frac{0}{0}$ form

EX-1: **Evaluate the limit** $\lim\limits_{x\to 0}\frac{\sin x}{x}$.

SOLUTION : Given limit is $\lim\limits_{x\to 0}\frac{\sin x}{x}=\frac{\lim\limits_{x\to 0}\sin x}{\lim\limits_{x\to 0}x}=\frac{0}{0}\ \left(\frac{0}{0}\ form\right)$.

Therefore, given limit $=\lim\limits_{x\to 0}\frac{\sin x}{x}$

$=\lim\limits_{x\to 0}\frac{\cos x}{1}$ **[Applying L' Hospital's Rule]**

$=1$

EX-2: **Evaluate the limit** $\lim\limits_{x\to a}\frac{x^n-a^n}{x-a}$.

SOLUTION : Given limit is $\lim\limits_{x\to a}\frac{x^n-a^n}{x-a}=\frac{\lim\limits_{x\to a}(x^n-a^n)}{\lim\limits_{x\to a}(x-a)}=\frac{0}{0}\ \left(\frac{0}{0}\ form\right)$

Therefore, given limit $=\lim\limits_{x\to a}\frac{x^n-a^n}{x-a}$

$=\lim\limits_{x\to a}\frac{nx^{n-1}}{1}$ **[Applying L' Hospital's Rule]**

$=na^{n-1}$.

EX-3: **Evaluate the limit** $\lim\limits_{x\to 0}\frac{x-\sin x\cos x}{x^3}$.

SOLUTION : Given limit is $\lim\limits_{x\to 0}\frac{x-\sin x\cos x}{x^3}=\frac{\lim\limits_{x\to 0}(x-\sin x\cos x)}{\lim\limits_{x\to 0}x^3}=\frac{0}{0}$.

So, the given limit is of $\frac{0}{0}$ **form.**

Therefore, given limit $=\lim\limits_{x\to 0}\frac{x-\sin x\cos x}{x^3}$

$$= \lim_{x\to 0} \frac{1+\sin^2 x - \cos^2 x}{3x^2}$$ **[Applying L' Hospital's Rule we see again it is of $\frac{0}{0}$ form]**

$$= \lim_{x\to 0} \frac{2\sin x\cos x + 2\cos x\sin x}{6x}$$

$$= \lim_{x\to 0} \frac{4\sin x\cos x}{6x}$$ **[Applying L' Hospital's Rule we see again it is of $\frac{0}{0}$ form]**

$$= \lim_{x\to 0} \frac{4(\cos^2 x - \sin^2 x)}{6}$$

$$= \lim_{x\to 0} \frac{4}{6}$$

$$= \frac{2}{3}$$

TASK : **(i)** $\lim_{x\to 0} \frac{e^x + e^{\sin x}}{x - \sin x}$ **(ii)** $\lim_{x\to 0} \frac{\sin\log(1+x)}{\log(1+\sin x)}$

LIMITS WHICH TAKE $\frac{\infty}{\infty}$ form

EX-4: Evaluate the limit $\lim_{x\to \frac{\pi}{2}} \frac{\tan 5x}{\tan x}$.

SOLUTION : Given limit is $\lim_{x\to \frac{\pi}{2}} \frac{\tan 5x}{\tan x} = \frac{\lim_{x\to \frac{\pi}{2}} \tan 5x}{\lim_{x\to \frac{\pi}{2}} \tan x} = \frac{\infty}{\infty} \left(\frac{\infty}{\infty}\ form\right)$.

Therefore, given limit $= \lim_{x\to \frac{\pi}{2}} \frac{\tan 5x}{\tan x}$

$$= \lim_{x\to \frac{\pi}{2}} \frac{\cot x}{\cot 5x}$$ **[Now it is of $\frac{0}{0}$ form]**

$$= \lim_{x\to \frac{\pi}{2}} \frac{-\operatorname{cosec}^2 x}{-5\operatorname{cosec}^2 5x}$$ **[Applying L' Hospital's Rule]**

$$= \frac{-1}{-5} = \frac{1}{5}$$

EX-5: **Evaluate the limit** $\underset{x \to 0}{Lim} \frac{\log\left(1+\frac{1}{x}\right)}{\frac{1}{x}}$.

SOLUTION : **Given limit is** $\underset{x \to 0}{Lim} \frac{\log\left(1+\frac{1}{x}\right)}{\frac{1}{x}} = \frac{\underset{x \to 0}{Lim} \log\left(1+\frac{1}{x}\right)}{\underset{x \to 0}{Lim} \frac{1}{x}} = \frac{\infty}{\infty} \quad \left(\frac{\infty}{\infty}\ form\right)$.

Therefore, given limit $= \underset{x \to 0}{Lim} \frac{\log\left(1+\frac{1}{x}\right)}{\frac{1}{x}}$

$$= \underset{x \to 0}{Lim} \frac{\left(\frac{1}{1+\frac{1}{x}} \times \left(-\frac{1}{x^2}\right)\right)}{\left(-\frac{1}{x^2}\right)}$$

$$= \underset{x \to 0}{Lim} \frac{x}{1+x}$$

$$= 0$$

TASK : **(i)** $\underset{x \to 0}{Lim} \frac{\log x^2}{\log \cot^2 x}$ **(ii)** $\underset{x \to \infty}{Lim} \frac{x^n}{e^x}$ **[** n **being positive]**

LIMITS WHICH TAKE $0 \times \infty$ form

EX-6: **Evaluate the limit** $\underset{x \to 0}{Lim}\, x^2 \log x^2$.

SOLUTION : **Given limit is** $\underset{x \to 0}{Lim}\, x^2 \log x^2 = \underset{x \to 0}{Lim}\, x^2 \times \underset{x \to 0}{Lim} \log x^2 = 0 \times \infty$.

$$(0 \times \infty \quad form)$$

Now the given limit can be written as $= \underset{x \to 0}{Lim}\, x^2 \log x^2$

$$= \underset{x \to 0}{Lim} \frac{\log x^2}{\left(\frac{1}{x^2}\right)} = \frac{\underset{x \to 0}{Lim} \log x^2}{\underset{x \to 0}{Lim}\left(\frac{1}{x^2}\right)} \left(\frac{\infty}{\infty}\ form\right)$$

Therefore, given limit $= \underset{x \to 0}{Lim}\, x^2 \log x^2$

$$= \underset{x \to 0}{Lim} \frac{\log x^2}{\left(\frac{1}{x^2}\right)} \quad \left(\frac{\infty}{\infty}\ form\right)$$

$$= \underset{x \to 0}{Lim} \frac{\frac{1}{x^2} \times 2x}{\left(\frac{-2}{x^3}\right)}$$

$$= \underset{x \to 0}{Lim}(-x^2)$$

$$= 0$$

EX-7: Evaluate the limit $\underset{x \to 0}{Lim}\, x \log\left(\sin^2 x\right)$.

SOLUTION : Given limit is $\underset{x \to 0}{Lim}\, x \log\left(\sin^2 x\right) = \underset{x \to 0}{Lim}\, x \times \underset{x \to 0}{Lim} \log\left(\sin^2 x\right) = 0 \times \infty$.

$$(0 \times \infty \quad form)$$

Now the given limit can be written as $\underset{x \to 0}{Lim}\, x \log\left(\sin^2 x\right)$

$$= \underset{x \to 0}{Lim} \frac{\log\left(\sin^2 x\right)}{\left(\frac{1}{x}\right)} = \frac{\underset{x \to 0}{Lim} \log\left(\sin^2 x\right)}{\underset{x \to 0}{Lim}\left(\frac{1}{x}\right)} \left(\frac{\infty}{\infty}\ form\right)$$

Therefore, given limit $= \underset{x \to 0}{Lim}\, x \log\left(\sin^2 x\right)$

$$= \underset{x \to 0}{Lim} \frac{\log\left(\sin^2 x\right)}{\left(\frac{1}{x}\right)} \quad \left(\frac{\infty}{\infty}\ form\right)$$

$$= \lim_{x \to 0} \frac{\frac{1}{\sin^2 x} \times 2 \sin x \cos x}{\left(-\frac{1}{x^2}\right)}$$

$$= -2 \lim_{x \to 0} x^2 \cot x \quad (0 \times \infty \quad form)$$

$$= -2 \lim_{x \to 0} \frac{x^2}{\tan x}$$ **[Now it is of $\frac{0}{0}$ form]**

$$= -2 \lim_{x \to 0} \frac{2x}{\sec^2 x}$$ **[Applying L' Hospital's Rule]**

$$= 0$$

TASK : **(i)** $\lim_{x \to 0} \sin x \log x^2$ **(ii)** $\lim_{x \to \frac{\pi}{2}} \sec 5x \cos 7x$

LIMITS WHICH TAKE $\infty - \infty$ form

EX-8: **Evaluate the limit** $\lim_{x \to \frac{\pi}{2}} (\sec x - \tan x)$

SOLUTION : Given limit is $= \lim_{x \to \frac{\pi}{2}} (\sec x - \tan x) = \lim_{x \to \frac{\pi}{2}} \sec x - \lim_{x \to \frac{\pi}{2}} \tan x$

$$(\infty - \infty \quad form)$$

Therefore, given limit $= \lim_{x \to \frac{\pi}{2}} (\sec x - \tan x) \quad (\infty - \infty \quad form)$

$$= \lim_{x \to \frac{\pi}{2}} \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x} \right)$$

$$= \lim_{x \to \frac{\pi}{2}} \left(\frac{1 - \sin x}{\cos x} \right)$$ **[Now it is of $\frac{0}{0}$ form]**

$$= \lim_{x \to \frac{\pi}{2}} \left(\frac{-\cos x}{-\sin x} \right)$$ **[Applying L' Hospital's Rule]**

$$= \frac{0}{1} = 0$$

TASK : (i) $\lim\limits_{x\to 0}\left(\frac{1}{x^2}-\frac{1}{\sin^2 x}\right)$ (ii) $\lim\limits_{x\to 0}\left(\frac{4}{x^2-4}-\frac{1}{x-2}\right)$

LIMITS WHICH TAKE 0^0, ∞^0, 1^∞, $1^{-\infty}$ forms

EX-9: Evaluate the limit $\lim\limits_{x\to 0} x^{2x}$

SOLUTION : Given limit is $=\lim\limits_{x\to 0} x^{2x}$ $\left(0^0 \ form\right)$

Let $f(x)=x^{2x}$. **Then** $\log f(x)=2x\log x$

Or, $\lim\limits_{x\to 0}(\log f(x))=\lim\limits_{x\to 0}(2x\log x)$

Or, $\log\left(\lim\limits_{x\to 0} f(x)\right)=\lim\limits_{x\to 0} 2x\log x$

Or, $\lim\limits_{x\to 0} f(x)=e^{\lim\limits_{x\to 0}2x\log x}$.

That is, given limit $=\lim\limits_{x\to 0} x^{2x}=e^{\lim\limits_{x\to 0}2x\log x}$ **……………………(1)**

Now let us consider the limit $\lim\limits_{x\to 0} 2x\log x$ **which clearly takes the form** $0\times\infty$.

Therefore, $\lim\limits_{x\to 0} 2x\log x=2\lim\limits_{x\to 0}\frac{\log x}{\left(\frac{1}{x}\right)}$ $\left(\frac{\infty}{\infty}\ form\right)$

$$=2\lim_{x\to 0}\frac{\frac{1}{x}}{\left(-\frac{1}{x^2}\right)}$$

$$=2\lim_{x\to 0}(-x)$$

$$=0$$

Hence from (1), the given limit $=\lim\limits_{x\to 0} x^{2x}=e^{\lim\limits_{x\to 0}2x\log x}=e^0=1$

EX-10: **Evaluate the limit** $\lim\limits_{x\to 0}(\cos x)^{\cot^2 x}$

SOLUTION : **Given limit is** $= \lim\limits_{x\to 0}(\cos x)^{\cot^2 x} \quad (1^\infty \text{ form})$

Let $f(x) = (\cos x)^{\cot^2 x}$. **Then** $\log f(x) = \cot^2 x \log\cos x$

Or, $\lim\limits_{x\to 0}(\log f(x)) = \lim\limits_{x\to 0}(\cot^2 x \log\cos x)$

Or, $\log\left(\lim\limits_{x\to 0} f(x)\right) = \lim\limits_{x\to 0}(\cot^2 x \log\cos x)$

Or, $\lim\limits_{x\to 0} f(x) = e^{\lim\limits_{x\to 0}(\cot^2 x \log\cos x)}$.

That is, given limit $= \lim\limits_{x\to 0}(\cos x)^{\cot^2 x} = e^{\lim\limits_{x\to 0}(\cot^2 x \log\cos x)}$ **............(1)**

Now let us consider the limit $\lim\limits_{x\to 0}(\cot^2 x \log\cos x)$ **which clearly takes the form** $0\times\infty$.

Therefore, $\lim\limits_{x\to 0}(\cot^2 x \log\cos x) = \lim\limits_{x\to 0}\dfrac{\log\cos x}{(\tan^2 x)} \quad \left(\dfrac{0}{0}\text{ form}\right)$

$$= \lim_{x\to 0}\frac{\dfrac{-\sin x}{\cos x}}{(2\tan x \times \sec^2 x)}$$ **[Applying L' Hospital's Rule]**

$$= -\frac{1}{2}\lim_{x\to 0}\cos^2 x$$

$$= -\frac{1}{2}$$

Hence from (1), the given limit $= \lim\limits_{x\to 0}(\cos x)^{\cot^2 x} = e^{\lim\limits_{x\to 0}(\cot^2 x \log\cos x)}$

$$= e^{-\frac{1}{2}}$$

EX-11: **Evaluate the limit** $\lim\limits_{x \to 1} x^{\frac{1}{1-x}}$

SOLUTION : Given limit is $\lim\limits_{x \to 1} x^{\frac{1}{1-x}} \quad (1^{\infty}\ form)$

Let $f(x) = x^{\frac{1}{1-x}}$. **Then** $\log f(x) = \frac{1}{1-x} \log x$

$$\text{Or, } \lim_{x \to 1} (\log f(x)) = \lim_{x \to 1} \left(\frac{1}{1-x} \log x \right)$$

$$\text{Or, } \log \left(\lim_{x \to 1} f(x) \right) = \lim_{x \to 1} \left(\frac{1}{1-x} \log x \right)$$

$$\text{Or, } \lim_{x \to 1} f(x) = e^{\lim\limits_{x \to 1} \left(\frac{1}{1-x} \log x \right)}.$$

That is, given limit $= \lim\limits_{x \to 1} x^{\frac{1}{1-x}} = e^{\lim\limits_{x \to 1} \left(\frac{1}{1-x} \log x \right)}$ **......................(1)**

Now let us consider the limit $\lim\limits_{x \to 1} \left(\frac{1}{1-x} \log x \right)$ **which clearly takes the form** $0 \times \infty$.

Therefore, $\lim\limits_{x \to 1} \left(\frac{1}{1-x} \log x \right) = \lim\limits_{x \to 1} \left(\frac{\log x}{1-x} \right) \quad \left(\frac{0}{0}\ form \right)$

$= \lim\limits_{x \to 1} \frac{\frac{1}{x}}{(-1)}$ **[Applying L' Hospital's Rule]**

$= -1$

Hence from (1), the given limit $= \lim\limits_{x \to 1} x^{\frac{1}{1-x}} = e^{\lim\limits_{x \to 1} \left(\frac{1}{1-x} \log x \right)} = e^{-1} = \frac{1}{e}$

EX-12: **Evaluate the limit** $\lim\limits_{x \to \infty} \left(1 + \frac{1}{x^2} \right)^x$

SOLUTION : Given limit is $\lim\limits_{x \to \infty} \left(1 + \frac{1}{x^2} \right)^x \quad (1^{\infty}\ form)$

Let $f(x)=\left(1+\frac{1}{x^2}\right)^x$ **. Then** $\log f(x)=x\log\left(1+\frac{1}{x^2}\right)$

Or, $\lim_{x\to\infty}(\log f(x))=\lim_{x\to\infty}\left(x\log\left(1+\frac{1}{x^2}\right)\right)$

Or, $\log\left(\lim_{x\to\infty} f(x)\right)=\lim_{x\to\infty}\left(x\log\left(1+\frac{1}{x^2}\right)\right)$

Or, $\lim_{x\to\infty} f(x)=e^{\lim_{x\to\infty}\left(x\log\left(1+\frac{1}{x^2}\right)\right)}$.

That is, given limit $=\lim_{x\to\infty}\left(1+\frac{1}{x^2}\right)^x=e^{\lim_{x\to\infty}\left(x\log\left(1+\frac{1}{x^2}\right)\right)}$ **……………………(1)**

Now let us consider the limit $\lim_{x\to\infty}\left(x\log\left(1+\frac{1}{x^2}\right)\right)$ **which clearly takes the form** $0\times\infty$.

Therefore, $\lim_{x\to\infty}\left(x\log\left(1+\frac{1}{x^2}\right)\right)=\lim_{x\to\infty}\left(\frac{\log\left(1+\frac{1}{x^2}\right)}{\frac{1}{x}}\right)$ $\left(\frac{0}{0}\ form\right)$

$$=\lim_{x\to\infty}\frac{\frac{1}{\left(1+\frac{1}{x^2}\right)}\times\left(\frac{-2}{x^3}\right)}{\left(\frac{-1}{x^2}\right)}$$ **[Applying L' Hospital's Rule]**

$$=\lim_{x\to\infty}\frac{2x}{x^2+1}\qquad\left(\frac{\infty}{\infty}\ form\right)$$

$$=\lim_{x\to\infty}\frac{2}{2x}$$

$$=0$$

Hence from (1), the given limit $=\lim_{x\to\infty}\left(1+\frac{1}{x^2}\right)^x=e^0=0$

EX-13: Evaluate the limit $\lim\limits_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x}}$

SOLUTION : Given limit is $\lim\limits_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x}}$ $\quad (1^{\infty}\ form)$

Let $f(x)=\left(\frac{\sin x}{x}\right)^{\frac{1}{x}}$. **Then** $\log f(x)=\frac{1}{x}\log\left(\frac{\sin x}{x}\right)$

Or, $\lim\limits_{x\to 0}(\log f(x))=\lim\limits_{x\to 0}\frac{1}{x}\log\left(\frac{\sin x}{x}\right)$

Or, $\log\left(\lim\limits_{x\to 0} f(x)\right)=\lim\limits_{x\to 0}\frac{\log\left(\frac{\sin x}{x}\right)}{x}$

Or, $\lim\limits_{x\to 0} f(x)=e^{\lim\limits_{x\to 0}\frac{\log\left(\frac{\sin x}{x}\right)}{x}}$.

That is, given limit $=\lim\limits_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x}}=e^{\lim\limits_{x\to 0}\frac{\log\left(\frac{\sin x}{x}\right)}{x}}$ **......................(1)**

Now let us consider the limit $\lim\limits_{x\to 0}\frac{\log\left(\frac{\sin x}{x}\right)}{x}$ **which clearly takes the form** $\frac{0}{0}$.

Therefore, $\lim\limits_{x\to 0}\frac{\log\left(\frac{\sin x}{x}\right)}{x}$ $\quad\left(\frac{0}{0}\ form\right)$

$$=\lim_{x\to 0}\frac{\frac{1}{\left(\frac{\sin x}{x}\right)}\times\left(\frac{x\cos x-\sin x}{x^2}\right)}{1}$$ **[Applying L' Hospital's Rule]**

$$=\lim_{x\to 0}\frac{x\cos x-\sin x}{x\sin x}\qquad\left(\frac{0}{0}\ form\right)$$

$$= \lim_{x\to 0} \frac{\cos x - x\sin x - \cos x}{\sin x + x\cos x}$$ **[Applying L' Hospital's Rule]**

$$= \lim_{x\to 0} \frac{-x\sin x}{\sin x + x\cos x} \qquad \left(\frac{0}{0}\ form\right)$$

$$= \lim_{x\to 0} \frac{-\sin x - x\cos x}{\cos x + \cos x - x\sin x}$$ **[Applying L' Hospital's Rule]**

$$= \frac{0}{2}$$

$$= 0$$

Hence from (1), the given limit $= \lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x}} = e^0 = 1$

TASK : **(i)** $\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}$ **(ii)** $\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}}$

EX-14 : **If** $\lim_{x\to 0} \frac{a\sin x - \sin 2x}{\tan^3 x}$ **is finite, find the value of** a **and hence find the limit.**

SOLUTION : Given limit is $\lim_{x\to 0} \frac{a\sin x - \sin 2x}{\tan^3 x} \left(\frac{0}{0}\ form\right)$.

Therefore, given limit $= \lim_{x\to 0} \frac{a\sin x - \sin 2x}{\tan^3 x} \left(\frac{0}{0}\ form\right)$

$$= \lim_{x\to 0} \frac{a\cos x - 2\cos 2x}{3\tan^2 x\sec^2 x}$$ **[Applying L' Hospital's Rule]**

As the given limit is finite and the denominator $3\tan^2 x\sec^2 x \to 0$ **as** $x \to 0$, **the numerator** $a\cos x - 2\cos 2x$ **must tend to** 0 **as** $x \to 0$.

That is, $a\cos 0 - 2\cos 2\cdot 0 = 0$.

Or, $a = 2$ **[First part solved].**

Putting $a=2$, **the above limit becomes** $\underset{x\to 0}{Lim}\dfrac{2\cos x-2\cos 2x}{3\tan^2 x\sec^2 x}\left(\dfrac{0}{0}\ form\right)$ [$\because a\cos 0-2\cos 2\cdot 0=0$]

$$=\underset{x\to 0}{Lim}\frac{(2\cos x-2\cos 2x)\cos^4 x}{3\sin^2 x}\left(\frac{0}{0}\ form\right)$$

$$=\underset{x\to 0}{Lim}\frac{(-2\cos^4 x\sin x+4\sin 2x\cos^4 x+8\cos 2x\cos^3 x\sin x)}{6\sin x\cos x}$$

[Applying L' Hospital's Rule]

$$=\underset{x\to 0}{Lim}\frac{(-5\cos^3 x\sin 2x+4\sin 2x\cos^4 x+2\cos^2 x\sin 4x)}{6\sin x\cos x}$$

$$=\underset{x\to 0}{Lim}\frac{(15\cos^2 x\sin x\sin 2x-10\cos^3 x\cos 2x+2\cos 2x\cos^4 x-16\cos^3\sin x\sin 2x-4\cos x\sin x\sin 4x+8\cos^2 x\cos 4x)}{6\cos 2x}$$

[Applying L' Hospital's Rule]

$$=\frac{-10+8+8}{6}$$

$=1$ **[Second part solved].**

EX-15 : If $\underset{x\to 0}{Lim}\dfrac{x(1+a\cos x)-b\sin x}{x^3}=1$ **then find the values of** a **&** b **.**

SOLUTION : Given limit is $\underset{x\to 0}{Lim}\dfrac{x(1+a\cos x)-b\sin x}{x^3}\left(\dfrac{0}{0}\ form\right)$

$$=\underset{x\to 0}{Lim}\frac{(1+a\cos x-ax\sin x)-b\cos x}{3x^2}$$

[Applying L' Hospital's Rule]

As the value of the given limit is 1 **(that is, finite) and the denominator** $3x^2\to 0$ **as** $x\to 0$, **the numerator** $(1+a\cos x-ax\sin x)-b\cos x$ **must tend to** 0 **as** $x\to 0$.

That is, $(1+a\cos 0-a\times 0\times\sin 0)-b\cos 0=0$.

Or, $a-b=-1\cdots\cdots\cdots\cdots(1)$

Therefore, the given limit $=\underset{x\to 0}{Lim}\dfrac{(1+a\cos x-ax\sin x)-b\cos x}{3x^2}\left(\dfrac{0}{0}\ form\right)$

$$= \underset{x \to 0}{Lim} \frac{(-a\sin x - a\sin x - ax\cos x) + b\sin x}{6x} \left(\frac{0}{0}\ form\right)$$

[Applying L' Hospital's Rule]

$$= \underset{x \to 0}{Lim} \frac{(-2a\cos x - a\cos x + ax\sin x) + b\cos x}{6}$$

[Applying L' Hospital's Rule]

As per given condition the value of the given limit is 1. **Therefore,**

$$\frac{(-2a\cos 0 - a\cos 0 + a \times 0 \times \sin 0) + b\cos 0}{6} = 1$$

Or, $-3a + b = 6 \cdots\cdots\cdots\cdots (2)$

Solving (1) **&** (2)**, we get** $a = \frac{-5}{2}$ **&** $b = \frac{-3}{2}$.

EX-15 : Determine the values of a, b **&** c **so that** $\frac{ae^x - b\cos x + ce^{-x}}{x\sin x} \to 2$ **as** $x \to 0$.

SOLUTION : Given that $\frac{ae^x - b\cos x + ce^{-x}}{x\sin x} \to 2$ **as** $x \to 0$.

That is, $\underset{x \to 0}{Lim} \frac{ae^x - b\cos x + ce^{-x}}{x\sin x} = 2$.

Now $\underset{x \to 0}{Lim} \frac{ae^x - b\cos x + ce^{-x}}{x\sin x}$ **is finite and the denominator** $x\sin x \to 0$ **as** $x \to 0$ **the numerator** $ae^x - b\cos x + ce^{-x}$ **must tend to** 0 **as** $x \to 0$. **That is,** $ae^0 - b\cos 0 + ce^{-0} = 0$ **or,** $a - b + c = 0 \cdots\cdots\cdots\cdots (1)$. **So, in that case, the above limit** $= \underset{x \to 0}{Lim} \frac{ae^x - b\cos x + ce^{-x}}{x\sin x} \left(\frac{0}{0}\ form\right)$ **[**$\because ae^x - b\cos x + ce^{-x} = 0$ **]**

[Applying L' Hospital's Rule]

$$= \underset{x \to 0}{Lim} \frac{ae^x + b\sin x - ce^{-x}}{\sin x + x\cos x}$$

Since the above limit is finite and the denominator $\sin x + x\cos x \to 0$ **as** $x \to 0$ **the numerator** $ae^x + b\sin x - ce^{-x}$ **must tend to** 0 **as** $x \to 0$. **That is,**

$ae^0 + b\sin 0 - ce^{-0} = 0$ **or,** $a - c = 0 \cdots\cdots\cdots\cdots(2)$. **So, in that case, the above limit** $= \lim\limits_{x\to 0} \dfrac{ae^x + b\sin x - ce^{-x}}{\sin x + x\cos x}\left(\dfrac{0}{0}\, form\right)$ **[**$\because ae^x + b\sin x - ce^{-x} = 0$**]**

$$= \lim_{x\to 0} \frac{ae^x + b\cos x + ce^{-x}}{\cos x + \cos x - x\sin x}$$

[Applying L' Hospital's Rule]

$$= \frac{ae^0 + b\cos 0 + ce^{-0}}{\cos 0 + \cos 0 - 0\sin 0}$$

$$= \frac{a+b+c}{1+1-0}$$

$$= \frac{a+b+c}{2}.$$

According to given condition $\dfrac{a+b+c}{2} = 2$

Or, $a + b + c = 4 \cdots\cdots\cdots\cdots(3)$.

Solving (1)**,** (2) **&** (3)**, we get** $a = 1,\ b = 2,\ c = 1$

$$\boxed{a = 1,\ b = 2,\ c = 1}$$