Reduction Formulae

Reduction formulae are formulae that may be used repeatedly to express the integral of Complicated function in terms of simpler ones. The reduction formulae are generally obtained by the application of the rule of integration by parts.

Integration by parts:

If u and v be two differentiable functions of a single independent variable having continuous derivatives, we have,

$$d(uv) = u\,dv + v\,du$$

or transposing
$$u\,dv = d(uv) - v\,du$$

Integrating both sides, we obtain

$$\int u\,dv = uv - \int v\,du$$

In words, Integral of the product of two differentiable functions

= 1st. × Integral of 2nd
− Integral of [differential of 1st. × Integral of 2nd]

Reduction formulae for 1) $\int \sin^n x\,dx$ and 2) $\int_0^{\pi/2} \sin^n x\,dx$

n being positive integer, greater than 1.

1.) Here
$$I_n = \int \sin^n x\,dx$$
$$= \int \sin^{n-1}x \sin x\,dx$$
$$= \sin^{n-1}x(-\cos x) - \int (n-1)\sin^{n-2}x\cos x(-\cos x)dx$$
$$= -\sin^{n-1}x\cos x + (n-1)\int \sin^{n-2}x\cos^2 x\,dx$$
$$= -\sin^{n-1}x\cos x + (n-1)\int \sin^{n-2}x(1-\sin^2 x)dx$$
$$= -\sin^{n-1}x\cos x + (n-1)\int \sin^{n-2}x\,dx - (n-1)\int \sin^n x\,dx$$
$$= -\sin^{n-1}x\cos x + (n-1)I_{n-2} - (n-1)I_n$$

and transposing,

$$I_n = -\sin^{n-1}x \cos x + (n-1) I_{n-2} - n I_n + I_n$$

$$\text{or } I_n = -\frac{\sin^{n-1}x \cos x}{n} + \frac{n-1}{n} I_{n-2}$$

$$\therefore I_n = -\frac{\sin^{n-1}x \cos x}{n} + \frac{n-1}{n} I_{n-2} + c$$

(2) Let $J_n = \int_0^{\pi/2} \sin^n x \, dx$

then $J_n = -\left[\frac{\sin^{n-1}x \cos x}{n}\right]_0^{\pi/2} + \frac{n-1}{n} J_{n-2}$

$$= \frac{n-1}{n} J_{n-2}$$

Prob. Deduce $\int_0^{\pi/2} \sin^{10} x \, dx$

Solⁿ We know that $J_n = \frac{n-1}{n} J_{n-2}$

$$\therefore J_{10} = \frac{10-1}{10} J_8 = \frac{9}{10} J_8$$

$$= \frac{9}{10} \cdot \frac{7}{8} J_6$$

$$= \frac{9}{10} \cdot \frac{7}{8} \cdot \frac{5}{6} J_4$$

$$= \frac{9}{10} \cdot \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} J_2$$

$$= \frac{9}{10} \cdot \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} J_0$$

But $J_0 = \int_0^{\pi/2} dx = [x]_0^{\pi/2} = \pi/2$

Hence $J_{10} = \frac{9}{10} \cdot \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \pi/2$

(*) Reduction formulae for

1) $\int \cos^n x \, dx$ and 2) $\int_0^{\pi/2} \cos^n x \, dx$,

n being positive integer, greater than 1.

1) Here

$$I_n = \int \cos^n x \, dx$$

$$= \int \cos^{n-1} x \cos x \, dx$$

$= \cos^{n-1}x \sin x - \int (n-1)\cos^{n-2}x(-\sin x)\sin x\,dx$

$= \cos^{n-1}x \sin x + (n-1)\int \cos^{n-2}x \sin^2 x\,dx$

$= \cos^{n-1}x \sin x + (n-1)\int \cos^{n-2}x(1-\cos^2 x)dx$

$= \cos^{n-1}x \sin x + (n-1)\int \cos^{n-2}x\,dx - (n-1)\int \cos^n x\,dx$

$= \cos^{n-1}x \sin x + (n-1)I_{n-2} - (n-1)I_n$

$\therefore I_n + (n-1)I_n = \cos^{n-1}x \sin x + (n-1)I_{n-2}$

or $I_n[1+n-1] = \cos^{n-1}x \sin x + (n-1)I_{n-2}$

or $nI_n = \cos^{n-1}x \sin x + (n-1)I_{n-2}$

or $I_n = \dfrac{\cos^{n-1}x \sin x}{n} + \dfrac{(n-1)}{n} I_{n-2}$

$\therefore I_n = \dfrac{\cos^{n-1}x \sin x}{n} + \dfrac{n-1}{n} I_{n-2} + C.$

(2) Let $J_n = \int_0^{\pi/2} \cos^n x\,dx$

$= \left[\dfrac{\cos^{n-1}x \sin x}{n}\right]_0^{\pi/2} + \dfrac{n-1}{n} J_{n-2}$

$= \dfrac{n-1}{n} J_{n-2}$

Note: we have the same reduction formulae for $\int_0^{\pi/2} \sin^n x\,dx$ and $\int_0^{\pi/2} \cos^n x\,dx$ i.e. $J_n = \dfrac{n-1}{n} J_{n-2}$

i.e. $J_n = \dfrac{n-1}{n} J_{n-2}$

$\therefore J_{n-2} = \dfrac{n-3}{n-2} J_{n-4}$

$J_{n-4} = \dfrac{n-5}{n-4} J_{n-6}$

and so on

$\therefore\ J_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{1}{2} J_0$ when n is even,

and $J_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{2}{3} J_1$ when n is odd.

But $J_0 = \int_0^{\pi/2} 1 . dx = \pi/2$

and $J_1 = \int_0^{\pi/2} \sin x \, dx = \left[-\cos x\right]_0^{\pi/2}$

$= 1$

$\therefore$ we get

$$\int_0^{\pi/2} \sin^n x \, dx = \int_0^{\pi/2} \cos^n x \, dx$$

$= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{1}{2} \cdot \pi/2$ when n is a positive even integer

$= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{2}{3}$ when n is a positive odd integer.

(*) Reduction formulae for $\int \tan^n x \, dx$, n being a positive integer.

Soln: Let $I_n = \int \tan^n x \, dx$

$= \int \tan^{n-2} x \tan^2 x \, dx$

$= \int \tan^{n-2} x \, dx (\sec^2 x - 1) \, dx$

$= \int \tan^{n-2} x \sec^2 x \, dx - \int \tan^{n-2} x \, dx$

$= \int \tan^{n-2} x \sec^2 x \, dx - I_{n-2}$

$= \int z^{n-2} dz - I_{n-2}$ Substituting $\tan x = z$ so that $\sec^2 x \, dx = dz$

$= \frac{z^{n-1}}{n-1} - I_{n-2} + c$

$= \frac{\tan^{n-1} x}{n-1} - I_{n-2} + c$

Corollary: Reduction formulae for $\int_0^{\pi/4} \tan^n x \, dx$, n being a positive integer, greater than 1.

Soln: Let $J_n = \int_0^{\pi/4} \tan^n x \, dx$

$$= \left[\frac{\tan^{n-1} x}{n-1}\right]_0^{\pi/4} - J_{n-2}$$

$$= \frac{1}{n-1} - J_{n-2}$$

(*) Find $\int \tan^4 x \, dx$

Soln: Applying $I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2}$ successively, we find that

$$I_4 = \frac{\tan^3 x}{3} - I_2$$

$$= \frac{\tan^3 x}{3} - \left(\frac{\tan x}{1} - I_0\right)$$

$$= \frac{\tan^3 x}{3} - \tan x + I_0$$

$$= \frac{\tan^3 x}{3} - \tan x + \int \tan^0 x \, dx$$

$$= \frac{\tan^3 x}{3} - \tan x + \int dx.$$

$$= \frac{\tan^3 x}{3} - \tan x + x + c$$

(*) Evaluate $\int_0^{\pi/4} \tan^6 x \, dx$

Solution: Applying $J_n = \frac{1}{n-1} - J_{n-2}$ successively, we see that

$$J_6 = \frac{1}{5} - J_4$$

$$= \frac{1}{5} - \left(\frac{1}{3} - J_2\right)$$

$$= \frac{1}{5} - \frac{1}{3} + J_2$$

$$= \frac{1}{5} - \frac{1}{3} + 1 - J_0 \quad \text{where } J_0 = \int_0^{\pi/4} \tan^0 x \, dx = \int_0^{\pi/4} dx = [x]_0^{\pi/4} = \pi/4$$

$$= \frac{1}{5} - \frac{1}{3} + 1 - \pi/4 = \frac{13}{15} - \pi/4$$

Corollary: Reduction formulae for $\int_0^{\pi/4} \tan^n x\, dx$, n being a positive integer, greater than 1.

Solⁿ: Let $J_n = \int_0^{\pi/4} \tan^n x\, dx$

$$= \left[\frac{\tan^{n-1} x}{n-1}\right]_0^{\pi/4} - J_{n-2}$$

$$= \frac{1}{n-1} - J_{n-2}$$

(*) Find $\int \tan^4 x\, dx$

Solⁿ: Applying $I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2}$ successively, we find that

$$I_4 = \frac{\tan^3 x}{3} - I_2$$

$$= \frac{\tan^3 x}{3} - \left(\frac{\tan x}{1} - I_0\right)$$

$$= \frac{\tan^3 x}{3} - \tan x + I_0$$

$$= \frac{\tan^3 x}{3} - \tan x + \int \tan^0 x\, dx$$

$$= \frac{\tan^3 x}{3} - \tan x + \int dx.$$

$$= \frac{\tan^3 x}{3} - \tan x + x + C$$

(*) Evaluate $\int_0^{\pi/4} \tan^6 x\, dx$

Solution: Applying $J_n = \frac{1}{n-1} - J_{n-2}$ successively, we see that

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$$= \frac{1}{5} - \frac{1}{3} + J_2$$

$$= \frac{1}{5} - \frac{1}{3} + 1 - J_0 \quad \text{where } J_0 = \int_0^{\pi/4} \tan^0 x\, dx = \int_0^{\pi/4} dx = [x]_0^{\pi/4} = \pi/4$$

$$= \frac{1}{5} - \frac{1}{3} + 1 - \pi/4 = \frac{13}{15} - \pi/4$$

* Reduction formulae for $\int \sec^n x \, dx$, n being positive integer, greater than 1

Soln: Let $I_n = \int \sec^n x \, dx$

$$= \int \sec^{n-2} x \sec^2 x \, dx$$

$$= \sec^{n-2} x \tan x - \int (n-2) \sec^{n-3} x . \sec x \tan x \tan x \, dx$$

$$= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x \tan^2 x \, dx$$

$$= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x (\sec^2 x - 1) dx$$

$$= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x \sec^2 x \, dx + (n-2) \int \sec^{n-2} x \, dx$$

$$= \sec^{n-2} x \tan x - (n-2) \int \sec^n x \, dx + (n-2) \int \sec^{n-2} x \, dx$$

$$= \sec^{n-2} x \tan x - (n-2) I_n + (n-2) I_{n-2}$$

or $I_n + (n-2) I_n = \sec^{n-2} x \tan x + (n-2) I_{n-2}$

or $I_n [1 + n - 2] = \sec^{n-2} x \tan x + (n-2) I_{n-2}$

or $(n-1) I_n = \sec^{n-2} x \tan x + (n-2) I_{n-2}$

or $I_n = \dfrac{\sec^{n-2} x \tan x}{(n-1)} + \dfrac{n-2}{n-1} I_{n-2}$

$\therefore I_n = \dfrac{\sec^{n-2} x \tan x}{n-1} + \dfrac{n-2}{n-1} I_{n-2}$ which is the required reduction formulae.

* obtain $\int \sec^4 x \, dx$

Soln: Here $I_4 = \dfrac{\sec^2 x \tan x}{3} + \dfrac{2}{3} I_2$

$$= \frac{\sec^2 x \tan x}{3} + \frac{2}{3} \left[\frac{\tan x}{1} + 0 \right]$$

$$= \frac{\sec^2 x \tan x}{3} + \frac{2}{3} \tan x + c$$

(*) Obtain a reduction formula for

$I_n = \int \cot^n x \, dx$ and deduce a formula for $J_n = \int_{\pi/2}^{\pi/4} \cot^n x \, dx$, n being a positive integer, greater than one.

Solⁿ Let $I_n = \int \cot^n x \, dx$

$$= \int \cot^{n-2} x \cot^2 x \, dx$$

$$= \int \cot^{n-2} x (\operatorname{cosec}^2 x - 1) \, dx$$

$$= \int \cot^{n-2} x \operatorname{cosec}^2 x \, dx - \int \cot^{n-2} x \, dx$$

Substituting $\cot x = z$

$\therefore -\operatorname{cosec}^2 x \, dx = dz$

$$= -\int z^{n-2} \, dz - I_{n-2}$$

$$= -\frac{z^{n-1}}{n-1} - I_{n-2}$$

$$= -\frac{\cot^{n-1} x}{n-1} - I_{n-2} + c$$

Again $J_n = \left[-\frac{\cot^{n-1} x}{n-1} \right]_{\pi/2}^{\pi/4} - I_{n-2}$

$$= -\frac{1}{n-1} - J_{n-2}$$

(*) (i) Obtain a reduction formula for

$I_{m,n} = \int \sin^m x \cos^n x \, dx$, where m and n are positive integers. Hence, find a reduction formula for $J_{m,n} = \int_0^{\pi/2} \sin^m x \cos^n x \, dx$

(ii) Use the above formula to evaluate

$$\int_0^{\pi/2} \sin^8 x \cos^6 x \, dx.$$

Solⁿ Let $I_{m,n} = \int \sin^m x \cos^n x \, dx$

Then $I_{m,n} = \int \cos^{n-1} x \sin^m x . \cos x \, dx$

$$= \cos^{n-1} x \int \sin^m x \cos x \, dx - \int \left\{ (n-1) \cos^{n-2} x (-\sin x) \int \sin^m x \cos x \, dx \right\} dx$$

$$= \cos^{n-1}x \cdot \frac{1}{m+1}\sin^{m+1}x + (n-1)\int \cos^{n-2}x \cdot \sin x \cdot \frac{1}{m+1}\sin^{m+1}x$$

$$= \frac{1}{m+1}\cos^{n-1}x\,\sin^{m+1}x + \frac{n-1}{m+1}\int \cos^{n-2}x\,\sin^{m}x\,\sin^{2}x\,dx$$

$$= \frac{1}{m+1}\cos^{n-1}x\,\sin^{m+1}x + \frac{n-1}{m+1}\int \cos^{n-2}x\,\sin^{m}x\,(1-\cos^{2}x)\,dx$$

$$= \frac{1}{m+1}\cos^{n-1}x\,\sin^{m+1}x + \frac{n-1}{m+1}\int \sin^{m}x\,\cos^{n-2}x\,dx$$

$$- \frac{n-1}{m+1}\int \sin^{m}x\,\cos^{n}x\,dx$$

$$= \frac{1}{m+1}\cos^{n-1}x\,\sin^{m+1}x + \frac{n-1}{m+1}I_{m,n-2} - \frac{n-1}{m+1}I_{m,n}$$

or $\left(1+\frac{n-1}{m+1}\right)I_{m,n} = \frac{1}{m+1}\cos^{n-1}x\,\sin^{m+1}x + \frac{n-1}{m+1}I_{m,n-2}$

or $\left(\frac{m+n}{m+1}\right)I_{m,n} = \frac{1}{m+1}\cos^{n-1}x\,\sin^{m+1}x + \frac{n-1}{m+1}I_{m,n-2} + C$

which is the required reduction formulae.

finally, we have

$$\left(\frac{m+n}{m+1}\right)I_{m,n} = \frac{1}{m+1}\cos^{n-1}x\,\sin^{m+1}x + \frac{n-1}{m+1}I_{m,n-2} \quad \text{---(1)}$$

Let $J_{m,n} = \int_0^{\pi/2} \sin^{m}x\,\cos^{n}x\,dx = \left[I_{m,n}\right]_0^{\pi/2}$ ---

Then from eqn. (1), we have

$$J_{m,n} = \frac{1}{m+n}\left[\cos^{n-1}x\,\sin^{m+1}x\right]_0^{\pi/2} + \frac{n-1}{m+n}J_{m,n-2}$$

$$\therefore\; J_{m,n} = \frac{n-1}{m+n}J_{m,n-2} \quad \text{---- (2)}$$

This is the reduction formulae.

(ii) $\int_0^{\pi/2} \sin^8 x \cos^6 x \, dx = J_{8,6}$

$= \frac{5}{8+6} J_{8,4}$ Here $m = 8, n = 6$

$= \frac{5}{14} \cdot \frac{3}{12} J_{8,2}$

$= \frac{5}{14} \cdot \frac{3}{12} \cdot \frac{1}{10} J_{8,0}$ ——— (3)

Now $J_{8,0} = \int_0^{\pi/2} \sin^8 x (\cos x)^0 \, dx$

$= \int_0^{\pi/2} \sin^8 x \, dx$ Again we know that $J_n = \frac{n-1}{n} J_{n-2}$

$= \frac{7}{8} J_6 = \frac{7}{8} \cdot \frac{5}{6} J_4 = \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} J_2 = \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} J_0$

$= \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$ where $J_0 = \pi/2$

$= \frac{35}{256} \pi$

Again from eqn. (3), we get

$\int_0^{\pi/2} \sin^8 x \cos^6 x \, dx = \frac{5}{14} \cdot \frac{3}{12} \cdot \frac{1}{10} \cdot \frac{35}{256} \pi$

$= \frac{5\pi}{4096}$

(*) If $I_n = \int_0^{\pi/3} \tan^n x \, dx$, show that $(n-1)(I_n + I_{n-2}) = (\sqrt{3})^{n-1}$

Soln: We have $I_n = \int_0^{\pi/3} \tan^n x \, dx$

$= \int_0^{\pi/3} \tan^{n-2} x \tan^2 x \, dx$

$= \int_0^{\pi/3} \tan^{n-2} x (\sec^2 x - 1) \, dx$

$= \int_0^{\pi/3} \tan^{n-2} x \sec^2 x \, dx - \int_0^{\pi/3} \tan^{n-2} x \, dx$

$= \left[\frac{\tan^{n-1} x}{n-1} \right]_0^{\pi/3} - I_{n-2}$

or $I_n + I_{n-2} = \left[\frac{\tan^{n-1} x}{n-1} \right]_0^{\pi/3} = \frac{(\sqrt{3})^{n-1}}{n-1}$

Hence $(n-1)[I_n + I_{n-2}] = (\sqrt{3})^{n-1}$