(*) If $\varphi(n) = \int_0^{\pi/4} \tan^n x \, dx$, Show that $\varphi(n) + \varphi(n-2) = \frac{1}{n-1}$ and deduce the value of $\varphi(5)$.

Soln: We have $\varphi(n) = \int_0^{\pi/4} \tan^n x \, dx$

$$= \left[\frac{\tan^{n-1} x}{n-1}\right]_0^{\pi/4} - \int_0^{\pi/4} \tan^{n-2} x \, dx$$

$$= \frac{1}{n-1} - \varphi(n-2)$$

Hence $\varphi(n) + \varphi(n-2) = \frac{1}{n-1}$

Now $\varphi(5) = \frac{1}{4} - \varphi(3)$

$$= \frac{1}{4} - \left[\frac{1}{2} - \varphi(1)\right]$$

$$= -\frac{1}{4} + \varphi(1) = -\frac{1}{4} + \int_0^{\pi/4} \tan x \, dx$$

$$= -\frac{1}{4} + \left[\log \sec x\right]_0^{\pi/4} = -\frac{1}{4} + \log \sqrt{2}$$

$$= -\frac{1}{4} + \frac{1}{2}\log 2$$

(*) If $I_{m,n} = \int_0^{\pi/2} \cos^m x \cos nx \, dx$, prove that

$$I_{m,n} = \frac{m(m-1)}{m^2 - n^2} I_{m-2,n}$$

Soln: We have

$$I_{m,n} = \int_0^{\pi/2} \cos^m x \cos nx \, dx$$

$$= \left[\frac{\cos^m x \sin nx}{n}\right]_0^{\pi/2} - \int_0^{\pi/2} m \cos^{m-1} x (-\sin x) \frac{\sin nx}{n} dx$$

$$= 0 + \frac{m}{n} \int_0^{\pi/2} \cos^{m-1} x \sin x \sin nx \, dx$$

$$= \frac{m}{n} \int_0^{\pi/2} \cos^{m-1} x \sin x \sin nx \, dx$$

$$= \frac{m}{n}\left[\left\{(\cos^{m-1} x \sin x)\left(-\frac{\cos nx}{n}\right)\right\}_0^{\pi/2} - \int_0^{\pi/2} \left\{(m-1)\cos^{m-2} x (-\sin x) + \cos^m x\right\}\left(-\frac{\cos nx}{n}\right) dx\right]$$

$$= \frac{m}{n}\left[0 + \frac{1}{n}\int_0^{\pi/2}\left\{\cos^m x \cos nx - (m-1)\cos^{m-2}x \sin^2 x \cos nx\right\} dx\right]$$

$$= \frac{m}{n^2}\int_0^{\pi/2}\cos^m x \cos nx\, dx - \frac{m(m-1)}{n^2}\int_0^{\pi/2}\cos^{m-2}x(1-\cos^2 x)\cos nx\, dx$$

$$= \frac{m}{n^2} I_{m,n} - \frac{m(m-1)}{n^2}\int_0^{\pi/2}\cos^{m-2}x \cos nx\, dx + \frac{m(m-1)}{n^2}\int \cos^m x \cos nx\, dx$$

$$= \frac{m}{n^2} I_{m,n} - \frac{m(m-1)}{n^2} I_{m-2,n} + \frac{m(m-1)}{n^2} I_{m,n}$$

$$\therefore I_{m,n} - \frac{m}{n^2} I_{m,n} - \frac{m(m-1)}{n^2} I_{m,n} = -\frac{m(m-1)}{n^2} I_{m-2,n}$$

$$\therefore I_{m,n}\left[1 - \frac{m}{n^2} - \frac{m(m-1)}{n^2}\right] = -\frac{m(m-1)}{n^2} I_{m-2,n}$$

$$\therefore I_{m,n}\left[\frac{n^2 - m - m^2 + m}{n^2}\right] = -\frac{m(m-1)}{n^2} I_{m-2,n}$$

$$\therefore I_{m,n}\left(\frac{n^2 - m^2}{n^2}\right) = \frac{-m(m-1)}{n^2} I_{m-2,n}$$

$$\therefore -I_{m,n}\left(\frac{m^2 - n^2}{n^2}\right) = -\frac{m(m-1)}{n^2} I_{m-2,n}$$

$$\therefore I_{m,n} = \frac{m(m-1)}{m^2 - n^2} I_{m-2,n}$$

(*) Reduction formula for i) $\int \cos^m x \cos nx\, dx$
ii) $\int \cos^m x \sin nx\, dx$, $m, n$ being positive integers.

Sol$^n$: (i) Let $I_{m,n} = \int \cos^m x \cos nx\, dx$

$$= \frac{\cos^m x \cdot \sin nx}{n} + \frac{m}{n}\int \cos^{m-1}x \sin x \sin nx$$

Now $\cos(n-1)x = \cos nx \cos x + \sin nx \sin x$

$\therefore \sin nx \sin x = \cos(n-1)x - \cos nx \cos x$

$$= \frac{\cos^m x \cdot \sin nx}{n} + \frac{m}{n}\int \cos^{m-1}x\left\{\cos(n-1)x - \cos nx\right.$$

$$= \frac{\cos^m x \sin nx}{n} + \frac{m}{n}\int \cos^{m-1}x \cos(n-1)x\, dx - \frac{m}{n}\int \cos^m x \cos nx$$

$$\Rightarrow I_{m,n} = \frac{\cos^m x \sin nx}{n} + \frac{m}{n}\int \cos^{m-1} x \cos(n-1)x\,dx$$

$$\Rightarrow \left(1+\frac{m}{n}\right) I_{m,n} = \frac{\cos^m x \sin nx}{n} + \frac{m}{n}\int \cos^{m-1} x \cos(n-1)x\,dx - \frac{m}{n} I_{m,n}$$

$$\Rightarrow \left(\frac{m+n}{n}\right) I_{m,n} = \frac{\cos^m x \sin nx}{n} + \frac{m}{n}\int \cos^{m-1} x \cos(n-1)x\,dx$$

$$\Rightarrow I_{m,n} = \frac{\cos^m x \sin nx}{m+n} + \frac{m}{m+n}\int \cos^{m-1} x \cos(n-1)x\,dx$$

$$= \frac{\cos^m x \sin nx}{m+n} + \frac{m}{m+n} I_{m-1,n} + c$$

which is the required reduction formula.

(ii) Let $I_{m,n} = \int \cos^m x \sin nx\,dx$

$$= -\frac{\cos^m x \cos nx}{n} - \frac{m}{n}\int \cos^{m-1} x(-\sin x)\left(-\frac{\cos nx}{n}\right)dx$$

$$= -\frac{\cos^m x \cos nx}{n} - \frac{m}{n}\int \cos^{m-1} x \sin x \cos nx\,dx$$

Now $\sin(n-1)x = \sin nx \cos x - \cos nx \sin x$

$$\Rightarrow \cos nx \sin x = \sin nx \cos x - \sin(n-1)x$$

$$= -\frac{\cos^m x \cos nx}{n} - \frac{m}{n}\int \cos^{m-1} x(\sin nx \cos x)\,dx + \frac{m}{n}\int \cos^{m-1} x \sin(n-1)x\,dx$$

$$= -\frac{\cos^m x \cos nx}{n} - \frac{m}{n}\int \cos^m x \sin nx\,dx + \frac{m}{n}\int \cos^{m-1} x \sin(n-1)x\,dx$$

$$= -\frac{\cos^m x \cos nx}{n} - \frac{m}{n} I_{m,n} + \frac{m}{n} I_{m-1,n-1}$$

$$\Rightarrow \left(1+\frac{m}{n}\right) I_{m,n} = -\frac{\cos^m x \cos nx}{n} + \frac{m}{n} I_{m-1,n-1}$$

$$\Rightarrow I_{m,n}\left(\frac{m+n}{n}\right) = -\frac{\cos^m x \cos nx}{n} + \frac{m}{n} I_{m-1,n-1}$$

$$\Rightarrow I_{m,n} = -\frac{\cos^m x \cos nx}{m+n} + \frac{m}{m+n} I_{m-1,n-1} + c$$

which is the required reduction formula.

(*) Prove that $\int_0^{\pi/2} \cos^n x \cos nx \, dx = \frac{\pi}{2^{n+1}}$, $n$ being a +ve integer

Solⁿ: Let $I_{m,n} = \int_0^{\pi/2} \cos^m x \cos nx \, dx$

By the reduction formulae

$$I_{m,n} = \frac{\cos^m x \sin nx}{m+n} + \frac{m}{m+n} I_{m-1,n-1}$$

we get

$$I_{n,n} = \left[\frac{\cos^n x \sin nx}{n+n}\right]_0^{\pi/2} + \frac{1}{2}\int_0^{\pi/2} \cos^{n-1} x \cos(n-1)x \, dx$$

$$= \frac{1}{2} I_{n-1,n-1}$$

$$= \frac{1}{2}\left[\frac{1}{2} I_{n-2,n-2}\right]$$

$$= \left[\frac{1}{2}\cdot\frac{1}{2}\cdots \text{ to } n \text{ factors}\right] I_{0,0}$$

$$= \frac{1}{2^n}\int_0^{\pi/2} \cos^0 x \cos(0\cdot x)\, dx$$

$$= \frac{1}{2^n}\int_0^{\pi/2} dx = \frac{1}{2^n}\cdot\frac{\pi}{2} = \frac{\pi}{2^{n+1}}$$

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(*) Find a reduction formula for

$$I_{m,n} = \int_0^{\pi/2} \cos^m x \sin nx \, dx,$$ $m, n$ being positive integer and hence deduce that

$$I_{m,m} = \frac{1}{2^{m+1}}\left[2 + \frac{2^2}{2} + \frac{2^3}{3} + \cdots + \frac{2^m}{m}\right]$$

Solⁿ:

# Length of an arc

The length of an arc can be found by one of the formulae's below for any differential curve defined by rectangular, polar or parametric equations.

**Formulas:** arc length $= \int_a^b ds$

where a and b represent $x, y, t$ or $\theta$ values as appropriate, and $ds$ can be found as follows

I. In rectangular form:

$$ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$

II. In parametric form:

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\, dt$$

III. In polar form:

$$ds = \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

(*) Find the length of an arc of the curve $y = \frac{1}{6}x^3 + \frac{1}{2}x^{-1}$ from $x=1$ to $x=2$

**Solⁿ:** Arc length $= \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$

$$= \int_1^2 \sqrt{1+\left(\frac{1}{2}x^2-\frac{1}{2}x^{-2}\right)^2}\, dx$$

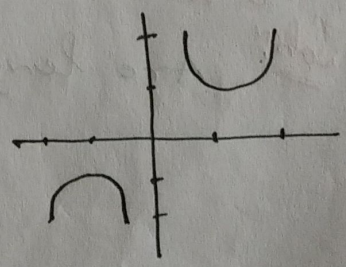
$y = \frac{1}{6}x^3 + \frac{1}{2}x^{-1}$

$\therefore \frac{dy}{dx} = \frac{1}{6}\cdot 3x^2 + \frac{1}{2}(-1)x^{-2}$

$= \frac{1}{2}x^2 - \frac{1}{2}x^{-2}$

$$= \int_1^2 \sqrt{1+\frac{1}{4}x^4-\frac{1}{2}+\frac{1}{4}x^{-4}}\, dx$$

$$= \int_1^2 \sqrt{\frac{1}{4}x^4+\frac{1}{2}+\frac{1}{4}x^{-4}}\, dx$$



$$= \int_1^2 \sqrt{\left(\frac{1}{2}x^2+\frac{1}{2}x^{-2}\right)^2}\, dx$$

$$= \int_1^2 \left(\frac{1}{2}x^2+\frac{1}{2}x^{-2}\right) dx$$

$$= \left[\frac{1}{2}\cdot\frac{x^3}{3} - \frac{1}{2}x^{-1}\right]_1^2 = \left[\frac{1}{6}x^3 - \frac{1}{2}x^{-1}\right]_1^2$$

$$= \left(\frac{8}{6}-\frac{1}{4}\right)-\left(\frac{1}{6}-\frac{1}{2}\right) = \left(\frac{16-3}{12}\right)-\left(\frac{1-3}{6}\right)$$

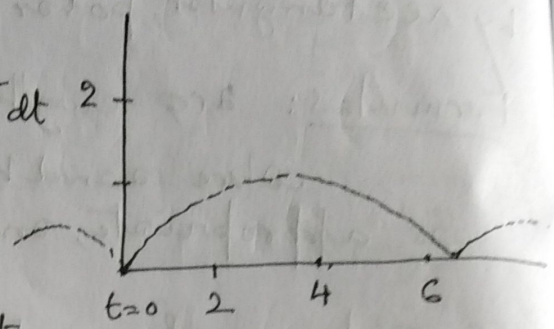
$$= \frac{13}{12}+\frac{1}{3} = \frac{13+4}{12} = \frac{17}{12} \text{ unit}$$

(*) Find the length of the arc in one period of the cycloid

$$x = t - \sin t, \quad y = 1 - \cos t.$$

The values of $t$ run from $0$ to $2\pi$

Soln:




$$\text{Arc length} = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{(1-\cos t)^2 + (\sin t)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t}\, dt$$

$$= \int_0^{2\pi} \sqrt{1 - 2\cos t + 1}\, dt$$

$$= \int_0^{2\pi} \sqrt{2 - 2\cos t}\, dt$$

$$= \int_0^{2\pi} \sqrt{2(1-\cos t)}\, dt$$

$$= \int_0^{2\pi} \sqrt{2 \cdot 2\sin^2 t/2}\, dt$$

$$= \int_0^{2\pi} 2\,|\sin t/2|\, dt$$

$$= \int_0^{2\pi} 2 \sin t/2\, dt = \left[-4\cos t/2\right]_0^{2\pi}$$

$$= (-4\cos\pi) - (-4\cos 0)$$

$$= -4(-1) - (-4 \cdot 1)$$

$$= 4 + 4 = 8 \text{ unit}$$

Here $x = t - \sin t$

$\therefore \frac{dx}{dt} = 1 - \cos t$

$y = 1 - \cos t$

$\frac{dy}{dt} = -(-\sin t) = \sin t$

As $\cos 2\theta = 1 - 2\sin^2\theta$

$\Rightarrow 2\sin^2\theta = 1 - \cos 2\theta$

$\Rightarrow 2\sin^2 \theta/2 = 1 - \cos \frac{2\theta}{2}$

$= 1 - \cos\theta$

(*) Find the length of the arc of the first rotation of the logarithmic spiral $r = e^\theta$. The values of $\theta$ run from $0$ to $2\pi$.

Soln:

$$\text{Arc length} = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$r = e^\theta$

$\Rightarrow \frac{dr}{d\theta} = e^\theta$

$$= \int_0^{2\pi} \sqrt{(e^\theta)^2 + (e^\theta)^2}\, d\theta$$

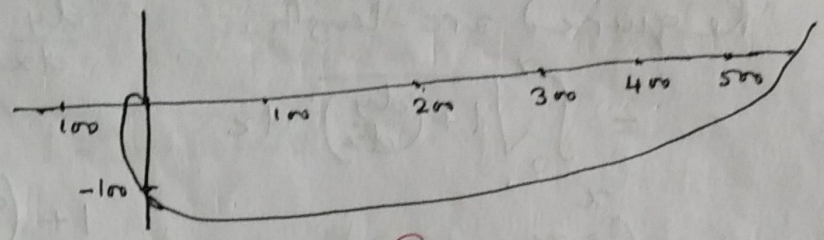
$$= \int_0^{2\pi} \sqrt{e^{2\theta} + e^{2\theta}}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{2e^{2\theta}}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{2}\, e^\theta\, d\theta$$

$$= \left[\sqrt{2}\, e^{\theta}\right]_0^{2\pi}$$

$$= \sqrt{2}\left(e^{2\pi} - 1\right)$$




(*) Find the length of arc of the following curves between the indicated points.

(i) $x = e^{\theta}\sin\theta,\ y = e^{\theta}\cos\theta;\quad \theta = 0$ and $\theta = \pi/2$

(ii) $y = \frac{1}{2}\left(e^{x/2} + e^{-x/2}\right);\quad x = 0$ and $x = x$

(iii) $y = \log\sec x;\quad x = 0$ and $x = \pi/3$

Solⁿ: (i) $x = e^{\theta}\sin\theta \quad \therefore \frac{dx}{d\theta} = e^{\theta}\sin\theta + e^{\theta}\cos\theta$

$$= e^{\theta}(\sin\theta + \cos\theta)$$

$y = e^{\theta}\cos\theta \quad \therefore \frac{dy}{d\theta} = e^{\theta}\cos\theta - e^{\theta}\sin\theta$

$$= e^{\theta}(\cos\theta - \sin\theta)$$

$$\therefore \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2$$

$$= \left\{e^{\theta}(\sin\theta + \cos\theta)\right\}^2 + \left\{e^{\theta}(\cos\theta - \sin\theta)\right\}^2$$

$$= e^{2\theta}\left(\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta\right) + e^{2\theta}\left(\cos^2\theta - 2\sin\theta\cos\theta + \sin^2\theta\right)$$

$$= 2e^{2\theta}$$

Length of the arc of the curve from $\theta = 0$ to $\theta = \pi/2$

$$= \int_0^{\pi/2} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{\pi/2} \sqrt{2e^{2\theta}}\, d\theta$$

$$= \int_0^{\pi/2} \sqrt{2}\, e^{\theta}\, d\theta$$

$$= \left[\sqrt{2}\, e^{\theta}\right]_0^{\pi/2} = \sqrt{2}\left(e^{\pi/2} - 1\right) \text{ units}$$

(ii)

$$y = \frac{a}{2}\left(e^{x/a} + e^{-x/a}\right)$$

$$\frac{dy}{dx} = \frac{a}{2}\cdot\frac{1}{a}\left(e^{x/a} - e^{-x/a}\right)$$

$$= \frac{1}{2}\left(e^{x/a} - e^{-x/a}\right)$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{1}{4}\left(e^{x/a} - e^{-x/a}\right)^2$$

$$= 1 + \frac{1}{4}\left(e^{2x/a} + e^{-2x/a} - 2\cdot e^{x/a}\cdot e^{-x/a}\right)$$

$$= 1 + \frac{1}{4}\left(e^{2x/a} + e^{-2x/a} - 2\right)$$

$$= 1 + \frac{1}{4}\left(e^{2x/a} + e^{-2x/a}\right) - \frac{1}{2}$$

$$= \frac{1}{2} + \frac{1}{4}\left(e^{2x/a} + e^{-2x/a}\right)$$

$$= \frac{1}{4}\left(e^{x/a} + e^{-x/a}\right)^2$$

Required arc-length

$$= \int_0^x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \frac{1}{2}\int_0^x \left(e^{x/a} + e^{-x/a}\right) dx$$

$$= \frac{1}{2}\left[a e^{x/a} - a e^{-x/a}\right]_0^x$$

$$= \frac{a}{2}\left(e^{x/a} - e^{-x/a}\right) \text{ units}$$

(iii) $y = \log \sec x$

$$\therefore \frac{dy}{dx} = \frac{1}{\sec x}\cdot \sec x \tan x$$

$$= \tan x$$

Required arc-length

$$= \int_0^{\pi/3} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_0^{\pi/3} \sqrt{1 + \tan^2 x}\, dx$$

$$= \int_0^{\pi/3} \sqrt{\sec^2 x}\, dx$$

$$= \int_0^{\pi/3} \sec x\, dx = \left[\log|\sec x + \tan x|\right]_0^{\pi/3}$$

$$= \log\left(2 + \sqrt{3}\right) \text{ units}$$

* Find the total length of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$

Solⁿ: Parametric equations of the curve are

$$x = a\cos^3\theta, \quad y = a\sin^3\theta$$

So $\frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta$, $\frac{dy}{d\theta} = 3a\sin^2\theta\cdot\cos\theta$

$$\therefore \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = 9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta$$

$$= 9a^2\sin^2\theta\cos^2\theta\left(\cos^2\theta + \sin^2\theta\right)$$

$$= 9a^2\sin^2\theta\cos^2\theta$$

Length of a quadrant of the astroid

$$= \int_0^{\pi/2} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{\pi/2} \sqrt{9a^2 \sin^2\theta \cos^2\theta}\, d\theta$$

$$= \int_0^{\pi/2} 3a \sin\theta \cos\theta \, d\theta$$

$$= \int_0^{\pi/2} \frac{3a}{2} \cdot 2\sin\theta\cos\theta \, d\theta = \frac{3a}{2}\int_0^{\pi/2} \sin 2\theta \, d\theta$$

$$= -\frac{3a}{2} \cdot \left[\frac{\cos 2\theta}{2}\right]_0^{\pi/2} = \frac{3a}{2} \text{ units}$$

$\therefore$ the length of the entire perimeter $= 4 \times \frac{3a}{2}$ unit

$= 6a$ unit,

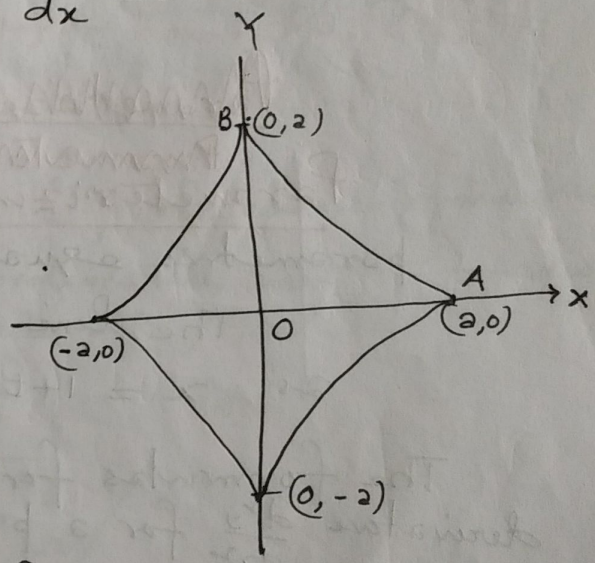
Since each of the four parts is of equal length.

(*) Find the length of the perimeter of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$

Solⁿ: from the equⁿ. $x^{2/3} + y^{2/3} = a^{2/3}$, we get by differentiating both sides w.r.t. $x$,

$$\frac{2}{3} x^{-\frac{1}{3}} + \frac{2}{3} y^{-1/3} \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{y^{1/3}}{x^{1/3}}$$




Now length of the arc BA

$$= \int_0^a \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_0^a \sqrt{1 + \left(-\frac{y^{1/3}}{x^{1/3}}\right)^2}\, dx$$

$$= \int_0^a \sqrt{1 + \frac{y^{2/3}}{x^{2/3}}}\, dx = \int_0^a \sqrt{\frac{x^{2/3} + y^{2/3}}{x^{2/3}}}\, dx$$

$$= \int_0^a \sqrt{\frac{a^{2/3}}{x^{2/3}}}\, dx = a^{1/3} \int_0^a x^{-\frac{1}{3}}\, dx = a^{1/3} \cdot \left[\frac{x^{2/3}}{2/3}\right]_0^a$$

$$= a^{1/3} \cdot \frac{3}{2} a^{2/3} = \frac{3}{2} \cdot a$$

$\therefore$ the length of the ~~exact~~ entire perimeter $= 4 \times \frac{3a}{2} = 6a$ unit,

Since each of the four parts is of equal length.

# Surface area of a Surface of revolution

The formulas below give the surface area of a surface of revolution. The axis of rotation must be either the x-axis or the y-axis. The curve being rotating can be defined using rectangular, polar or parametric equations.

Rotation about the x-axis:

$$\text{Surface area} = \int_A^B 2\pi y \, ds$$

Rotation about the y-axis:

$$\text{Surface area} = \int_A^B 2\pi x \, ds$$

In either formulae, ds can be found by any of the following:

I. In rectangular form, use

$$ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$\text{or } ds = \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$

II. In parametric form, use

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\, dt$$

III. In polar form, use

$$ds = \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

~~Parametrized~~

~~Parameterization~~

**Parameterizing a curve**: To write in terms of parametric equations

The line $x+y=2$ can be parametrized as $x = 1+t$, $y = 1-t$

The formulas for first derivative $\frac{dy}{dx}$ and second derivative $\frac{d^2y}{dx^2}$ for a parametrically defined curve are given below:

$$\text{Slope of the tangent line} = \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

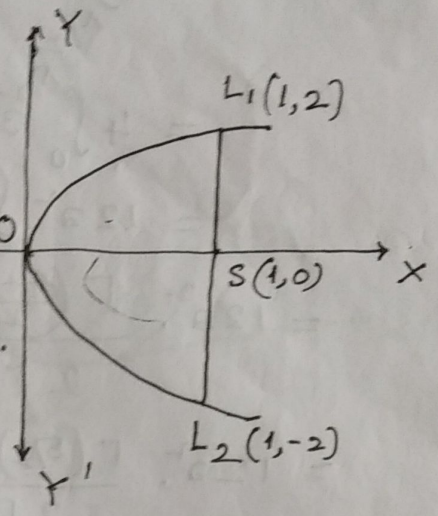
$$\text{Second derivative} = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dt}\left(\frac{dy}{dx}\right)\cdot\frac{dt}{dx} = \frac{\frac{dx}{dt}\cdot\frac{d^2y}{dt^2} - \frac{dy}{dt}\cdot\frac{d^2x}{dt^2}}{(dx/dt)^2}$$

$$= \frac{\frac{d}{dt}(dy/dx)}{}$$

(*) Find the area of the region bounded by the parabola $y^2 = 4x$ and its latus rectum.

Solⁿ: The parabola $y^2 = 4x$, has its vertex at $O(0,0)$ and focus at $S(1,0)$,




The equation to the latus rectum $L_1SL_2$ is $x = 1$ and latus rectum intersects the parabola at $L_1(1,2)$ and $L_2(1,-2)$.

Area bounded by the parabola and its rectum is

$OL_1SL_2O$

$= 2 \times$ area $OL_1SO$ (Since the curve is symmetrical about the x-axis)

$$= 2\int_0^1 y\,dx = 2\int_0^1 2x^{1/2}\,dx \quad [\text{As } y = 2\sqrt{x}]$$

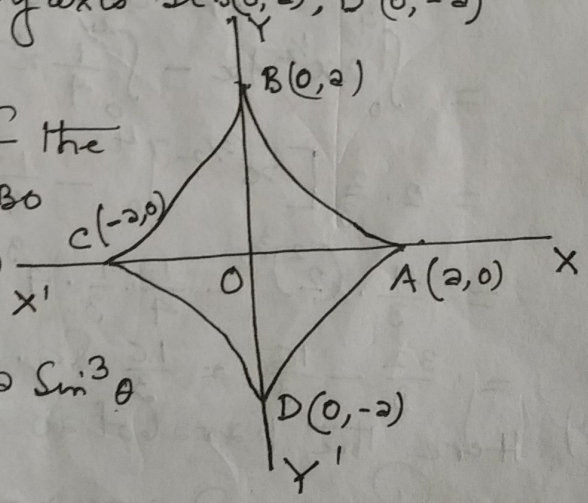
$$= 4\int_0^1 x^{1/2}\,dx$$

$$= 4\cdot\left[\frac{x^{3/2}}{3/2}\right]_0^1 = 4\cdot\frac{2}{3}\left[x^{3/2}\right]_0^1$$

$$= 8/3 \text{ sq. unit}$$

(*) Find the area of the region bounded by the astroid $x^{2/3} + y^{2/3} = a^{2/3}$

Solution: The curve $x^{2/3} + y^{2/3} = a^{2/3}$, which is symmetrical about both the axes of co-ordinates, meet the x-axis at $A(a,0)$, $C(-a,0)$, and the y axis at $B(0,a)$, $D(0,-a)$




Required area $= 4 \times$ area of the region $OABO$

$$\therefore \quad A = 4\int_0^a y\,dx \quad \text{—— (1)}$$

We put $x = a\cos^3\theta$, $y = a\sin^3\theta$

$\therefore dx = -3a\cos^2\theta\sin\theta\,d\theta$

when $x = 0$, $\theta = \pi/2$

and $x = a$, $\theta = 0$

From equ$^n$ ①, we have

$$A = 4\int_{\pi/2}^{0} a\sin^3\theta\cdot(-3a\cos^2\theta\sin\theta)\,d\theta$$

$$= 4\int_0^{\pi/2} 3a^2\sin^4\theta\cos^2\theta\,d\theta$$

$$= 12a^2\int_0^{\pi/2}\sin^4\theta\cos^2\theta\,d\theta$$

$$= 12a^2\cdot\frac{\Gamma\left(\frac{4+1}{2}\right)\Gamma\left(\frac{2+1}{2}\right)}{2\Gamma\left(\frac{4+2+2}{2}\right)}$$

$$= 12a^2\cdot\frac{\Gamma(5/2)\Gamma(3/2)}{2\Gamma(4)}$$

$$= 12a^2\cdot\frac{\frac{3}{2}\cdot\frac{1}{2}\Gamma(1/2)\cdot\frac{1}{2}\Gamma(1/2)}{2\cdot 3!}$$

$$= 12a^2\cdot\frac{\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}\cdot\frac{1}{2}\sqrt{\pi}}{2\times3\times2} = 12a^2\times\frac{3}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\times\frac{1}{2\times6}\pi$$

$$= \frac{3}{8}\pi a^2 \text{ sq. unit}$$

**Formula**

$$\int_0^{\pi/2}\sin^p\theta\cos^q\theta\,d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$$

& $\Gamma(n+1) = n\Gamma(n)$ if $n>0$

where $n$ is a +ve integer $\Gamma(n+1) = n!$

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

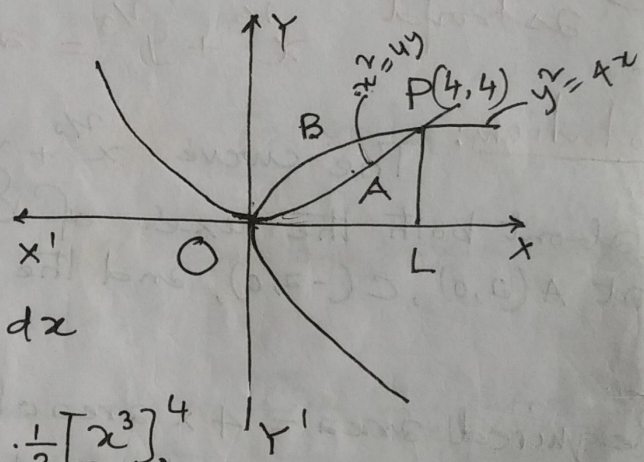
$$\Gamma(m)\Gamma(1-m) = \frac{\pi}{\sin m\pi}$$

$\Gamma(\frac{1}{2}) = \sqrt{\pi}$ & $\Gamma(1) = 1$

(*) (i) Find the area bounded by the parabola $x^2 = 4y$ and $y^2 = 4x$

(ii) Show that the area between $y = x^2$ and $x = y^2$ is $\frac{1}{3}$ sq. unit

Solⁿ: (i) The parabola $x^2 = 4y$ and $y^2 = 4x$ intersect at $O(0,0)$ and $P(4,4)$




Required area

= area OBPLO − area OAPLO

$$= \int_0^4 2x^{1/2}\,dx - \int_0^4 \frac{1}{4}x^2\,dx$$

$$= 2\cdot\frac{2}{3}\left[x^{3/2}\right]_0^4 - \frac{1}{4}\cdot\frac{1}{3}\left[x^3\right]_0^4$$

$$= \frac{4}{3}\cdot(2^2)^{3/2} - \frac{1}{4}\cdot\frac{1}{3}\cdot4^3$$

$$= \frac{32}{3} - \frac{16}{3} = \frac{16}{3} \text{ sq. unit}$$

(ii) Here the parabolas intersects at $(0,0)$ and $(1,1)$.

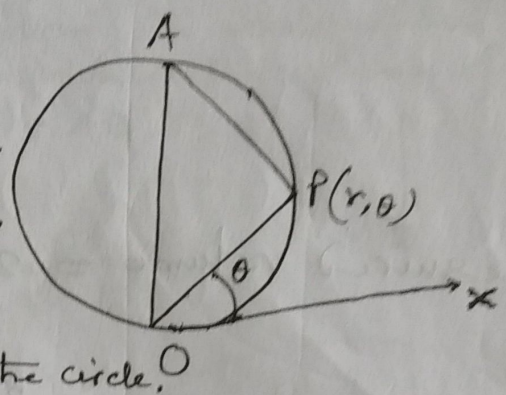
$$\text{Required area} = \int_0^1 x^{1/2}\,dx - \int_0^1 x^2\,dx$$

$$= \frac{2}{3}\left[x^{3/2}\right]_0^1 - \frac{1}{3}\left[x^3\right]_0^1$$

$= \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$ sq. unit.

(*) Find the area of the circle $r = 2a\sin\theta$

Solution:




With O as pole and OX initial line, the equation of the circle is $r = 2a\sin\theta$, where $OA = 2a$ = diameter of the circle.

$P(r,\theta)$ is any point on the circle.

Area of the circle

$$= \frac{1}{2}\int_0^{\pi} r^2\,d\theta = \frac{1}{2}\int_0^{\pi} 4a^2\sin^2\theta\,d\theta$$

$$= a^2\int_0^{\pi} 2\sin^2\theta\,d\theta = a^2\int_0^{\pi}(1-\cos 2\theta)\,d\theta$$

$$= a^2\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi} = \pi a^2 \text{ sq. unit.}$$

(*) Find the volume generated by revolving about OX, the area bounded by $y = x^3$ between $x=0$ and $x=2$.

Solution:

Required volume of revolution

$$= \pi\int_0^2 y^2\,dx$$

$$= \pi\int_0^2 x^6\,dx = \pi\left[\frac{x^7}{7}\right]_0^2 = \frac{128\pi}{7} \text{ cubic units}$$

(*) The circle $x^2+y^2=a^2$ revolves round the x-axis. Find the surface area and the volume of the whole surface generated.

Solution: $x^2+y^2=a^2$ $\quad\therefore y = \sqrt{a^2-x^2}$

$$\therefore \frac{dy}{dx} = -\frac{x}{\sqrt{a^2-x^2}}$$

Now $\left(\frac{ds}{dx}\right)^2 = 1+\left(\frac{dy}{dx}\right)^2 = 1+\frac{x^2}{a^2-x^2} = \frac{a^2}{a^2-x^2}$

Required Surface area $(S) = 4\pi \int_0^a y \left(\frac{ds}{dx}\right) dx$

$$= 4\pi \int_0^a \sqrt{a^2-x^2} \cdot \frac{a}{\sqrt{a^2-x^2}}\, dx$$

$$= 4\pi a \int_0^a dx = 4\pi a . [x]_0^a$$

$$= 4\pi a^2 \text{ sq. unit}$$

Required volume $= 2\pi \int_0^a y^2 dx$

$$= 2\pi \int_0^a (a^2-x^2)\, dx$$

$$= 2\pi \left[a^2 x - \frac{x^3}{3}\right]_0^a = 2\pi \left[a^3 - \frac{1}{3}a^3\right]$$

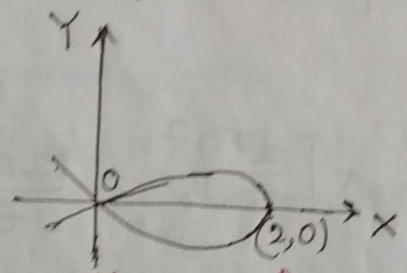
$$= 2\pi . \frac{2}{3} a^3 = \frac{4}{3}\pi a^3 \text{ cubic units}$$

(*) Find the volume generated by the revolution about x-axis of the area bounded by the loop of the curve $y^2 = x^2(2-x)$

Solution: Equation to the curve is $y^2 = x^2(2-x)$

Since $y = 0$, when $x = 0$ and $x = 2$, for the loop of the curve, $x$ varies from 0 to 2.

Required volume $= \pi \int_0^a y^2 dx$




$$= \pi \int_0^2 x^2 (2-x)\, dx$$

$$= \pi \left[\frac{2}{3}x^3 - \frac{1}{4}x^4\right]_0^2$$

$$= \pi \left[\frac{16}{3} - 4\right] = \frac{4}{3}\pi \text{ cubic Unit}$$

(*) Find the volume of the solid generated by revolving the part of the parabola $x^2 = 4ay$, $(a>0)$ between the ordinates $y=0$ and $y=a$ about its axis.

Solution: Since here $x^2 = 4ay$

Required volume $= \pi \int_0^a x^2 dy$

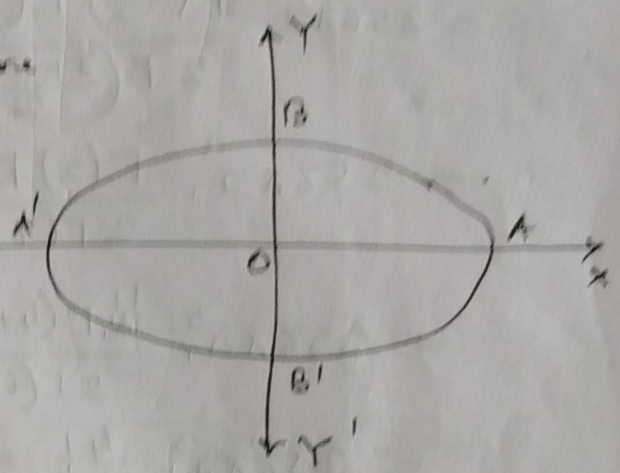
$$= \pi \int_0^a 4ay\, dy = 4\pi a \int_0^a y\, dy$$

$$= 4\pi a . \left[\frac{y^2}{2}\right]_0^a = 4\pi a . \frac{a^2}{2}$$

$$= 2\pi a^3 \text{ cubic unit}$$

(*) Find the volume of the solid obtained by revolving the ellipse $x^2/a^2 + y^2/b^2 = 1$ about the axis of $x$.

Solution: It is easy to see that the solid obtained by revolving the arc $ABA'$ about $x$-axis is the same as the solid obtained by revolving the whole ellipse. Also the volume of the solid is double the volume of the solid obtained by revolving the arc $AB$.




The required volume, therefore

$$= 2\pi \int_0^a y^2\,dx$$

$$= 2\pi \cdot \frac{b^2}{a^2} \int_0^a (a^2 - x^2)\,dx$$

$$= 2\pi \cdot \frac{b^2}{a^2} \left[a^2 x - \frac{x^3}{3}\right]_0^a$$

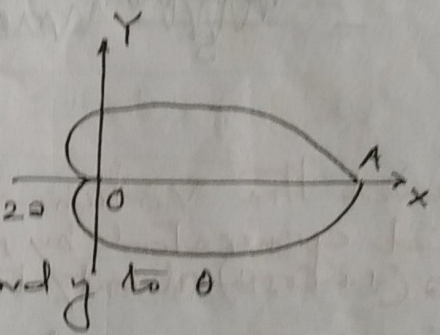
$$= 2\pi \cdot \frac{b^2}{a^2} \cdot \left[a^3 - \frac{a^3}{3}\right]$$

$$= 2\pi \cdot \frac{b^2}{a^2} \cdot \frac{2}{3} a^3$$

$$= \frac{4}{3}\pi a b^2 \text{ cubic unit}$$

(*) Find the volume of the solid obtained by revolving the cardioide $r = a(1+\cos\theta)$ about the initial line.

Solution:




The required volume

$$= \pi \int_0^{2a} y^2\,dx, \text{ for } OA = 2a$$

We change the variables $x$ and $y$ to $\theta$

We have $x = r\cos\theta = a\cos\theta(1+\cos\theta)$

$y = r\sin\theta = a\sin\theta(1+\cos\theta)$

$\therefore\ dx = -a(\sin\theta + 2\sin\theta\cos\theta)\,d\theta$

Also $\theta = \pi$ when $x = 0$ and $\theta = 0$, when $x = 2a$

$$\therefore\ V = -\pi \int_\pi^0 a^2 \sin^2\theta (1+\cos\theta)^2 \cdot a(\sin\theta + 2\sin\theta\cos\theta)\,d\theta$$

$$= \pi a^3 \int_0^\pi \sin^3\theta (1+\cos\theta)^2 (1+2\cos\theta)\,d\theta$$

$$= \pi a^3 \int_0^\pi \left(2\sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2}\right)^3 \cdot \left(2\cos^2\tfrac{\theta}{2}\right)^2 \left\{1 + 2\left(2\cos^2\tfrac{\theta}{2} - 1\right)\right\} d\theta$$

$$= \pi a^3 \int_0^\pi 8\sin^3\tfrac{\theta}{2}\cos^3\tfrac{\theta}{2} \cdot 4\cos^4\tfrac{\theta}{2}\left(4\cos^2\tfrac{\theta}{2} - 1\right) d\theta$$

$$= 128\pi a^3 \int_0^{\pi} \sin^3\theta/2 \cos^9\theta/2 \, d\theta \; - 32\pi a^3 \int_0^{\pi} \sin^3\theta/2 \cos^7\theta/2 \, d\theta$$

$$= 128\pi a^3 \cdot 2 \int_0^{\pi/2} \sin^3\varphi \cos^9\varphi \, d\varphi - 32\pi a^3 \cdot 2 \int_0^{\pi/2} \sin^3\varphi \cos^7\varphi \, d\varphi$$

$$= 256\pi a^3 \int_0^{\pi/2} \sin^3\varphi \cos^9\varphi \, d\varphi - 64\pi a^3 \int_0^{\pi/2} \sin^3\varphi \cos^7\varphi \, d\varphi \quad \text{where } \varphi = \theta/2$$

$$= 256\pi a^3 \cdot \frac{\Gamma\left(\frac{3+1}{2}\right)\Gamma\left(\frac{9+1}{2}\right)}{2\Gamma\left(\frac{3+9+2}{2}\right)} - 64\pi a^3 \cdot \frac{\Gamma\left(\frac{3+1}{2}\right)\Gamma\left(\frac{7+1}{2}\right)}{2\Gamma\left(\frac{3+7+2}{2}\right)}$$

$$= 256\pi a^3 \cdot \frac{\Gamma(2)\Gamma(5)}{2\Gamma(7)} - 64\pi a^3 \cdot \frac{\Gamma(2)\Gamma(4)}{2\Gamma(6)}$$

$$= 256\pi a^3 \cdot \frac{1!\,\Gamma(4+1)}{2\Gamma(6+1)} - 64\pi a^3 \cdot \frac{1!\,\Gamma(3+1)}{2\Gamma(5+1)}$$

$$= 256\pi a^3 \cdot \frac{4!}{2\times 6!} - 64\pi a^3 \cdot \frac{3!}{2\times 5!}$$

$$= 256\pi a^3 \cdot \frac{4\times 3\times 2}{2\times 6\times 5\times 4\times 3\times 2} - 64\pi a^3 \frac{3\times 2}{2\times 5\times 4\times 3\times 2}$$

$$= \frac{128\pi a^3}{6\times 5} - \frac{8\pi a^3}{5} = \frac{64\pi a^3}{15} - \frac{8\pi a^3}{5}$$

$$= \frac{64\pi a^3 - 24\pi a^3}{15}$$

$$= \frac{40\pi a^3}{15} = \frac{8}{3}\pi a^3 \text{ cubic units}$$

(*) Find the volume and the surface area of the solid generated by revolving the cycloid $x = a(\theta + \sin\theta)$, $y = a(1+\cos\theta)$ about its base.

<u>Solution</u>: The equation show that the cycloid has the x-axis as its base, the extreme values of $x$ are given by $\theta = \pm\pi$, i.e. $x = \pm a\pi$

The required volume $= \pi \int_{-a\pi}^{a\pi} y^2 \, dx$

$$= \pi \int_{-\pi}^{\pi} a^2 (1+\cos\theta)^2 \cdot a(1+\cos\theta)\, d\theta$$

Since $x = a(\theta + \sin\theta)$

$\therefore dx = a(1+\cos\theta)\, d\theta$

& $y = a(1+\cos\theta)$

$$= \pi a^3 \int_{-\pi}^{\pi} (1+\cos\theta)^3 \, d\theta$$

$$= 2\pi a^3 \int_0^{\pi} (1+\cos\theta)^3 \, d\theta$$

$$= 2\pi a^3 \cdot \int_0^{\pi} (2\cos^2\theta/2)^3 \, d\theta$$

$$= 16\pi a^3 \int_0^{\pi} \cos^6 \theta/2 \, d\theta \qquad \text{Put } \theta/2 = \varphi$$

$$= 32\pi a^3 \int_0^{\pi/2} \cos^6 \varphi \, d\varphi$$

$$= 32\pi a^3 \cdot \frac{5}{6} J_4$$

$$= 32\pi a^3 \cdot \frac{5}{6} \cdot \frac{3}{4} J_2$$

$$= 32\pi a^3 \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} J_0$$

$$= \cancel{32}\pi a^3 \cdot \frac{5}{\cancel{6}_{\cancel{2}}} \cdot \frac{\cancel{3}}{\cancel{4}} \cdot \frac{1}{\cancel{2}} \cdot \cancel{\pi/2}$$

$$= 5\pi^2 a^3 \text{ cubic unit}$$

using the reduction formula

$$J_n = \int_0^{\pi/2} \cos^n x \, dx = \frac{n-1}{n} J_{n-2}$$

Now $J_0 = \int_0^{\pi/2} \cos^0 \varphi \, d\varphi$

$$= \int_0^{\pi/2} d\varphi = [\varphi]_0^{\pi/2}$$

$$= \pi/2$$

Now the Required Surface area

$$= 2\pi \int y \, ds = 2\pi \int y \sqrt{dx^2 + dy^2}$$

$$= 2\pi \int_{-\pi}^{\pi} a(1+\cos\theta) \sqrt{\{a(1+\cos\theta)\,d\theta\}^2 + (-a\sin\theta\, d\theta)^2}$$

$$= 2\pi \int_{-\pi}^{\pi} a(1+\cos\theta) \sqrt{a^2(1+\cos\theta)^2 d\theta^2 + a^2 \sin^2\theta \, d\theta^2}$$

$$= 2\pi a \int_{-\pi}^{\pi} (1+\cos\theta) \cdot a \sqrt{(1+\cos\theta)^2 + \sin^2\theta} \cdot d\theta$$

$$= 2\pi a^2 \int_{-\pi}^{\pi} (1+\cos\theta) \sqrt{1+\cos^2\theta + 2\cos\theta + \sin^2\theta} \, d\theta.$$

$$= 2\pi a^2 \int_{-\pi}^{\pi} (1+\cos\theta) \sqrt{2+2\cos\theta} \, d\theta$$

$$= 2\pi a^2 \int_{-\pi}^{\pi} (1+\cos\theta) \sqrt{2(1+\cos\theta)} \, d\theta.$$

$$= 2\pi a^2 \int_{-\pi}^{\pi} (1+\cos\theta) \cdot \sqrt{2 \cdot 2\cos^2 \theta/2} \, d\theta.$$

$$= 2\pi a^2 \int_{-\pi}^{\pi} (1+\cos\theta) \cdot 2\cos\theta/2 \, d\theta$$

$$= 4\pi a^2 \int_{-\pi}^{\pi} 2\cos^2\theta/2 \cdot \cos\theta/2 \, d\theta$$

$$= 8\pi a^2 \int_{-\pi}^{\pi} \cos^3 \theta/2 \, d\theta$$

$$= 16\pi a^2 \int_0^{\pi} \cos^3 \theta/2 \, d\theta \qquad \text{Put } \varphi = \theta/2$$

$$= 32\pi a^2 \int_0^{\pi/2} \cos^3 \varphi \, d\varphi$$

$$= 32\pi a^2 \cdot \frac{2}{3} \cdot J_1 = 32\pi a^2 \cdot \frac{2}{3} \cdot 1 \quad \text{as } J_1 = 1$$

$$= \frac{64}{3}\pi a^2 \text{ sq. unit.}$$