Study Material - Sem. 1 - C2T

- Oscillations - Dr. T. Kar - Class 3

Forced Vibration

A vibrating system gradually loses its amplitude since energy is dissipated due to frictional forces. To maintain the system in vibration, energy must be supplied from outside. If an external periodic force is applied to the vibrating system, the system tends to vibrate with its own natural frequency. But the applied driving force tries to impress its own frequency on the vibrating system. Initially, the system vibrates with both the frequencies. The natural vibrations die out in course of time due to the prevailing resisting forces and the system finally in steady state, vibrates with the frequency of the driving force with a constant amplitude. Such vibration where the system oscillates with a frequency the same as that of an externally impressed periodic force is known as the forced vibration of the system.

Usually, a feeble sound is heard from a vibrating tuning fork. But if the stem of the vibrating fork is pressed on a wooden table, the sound becomes considerably louder. The vibrating fork induces a forced vibration of the table which, in turn, excites the vibrations of a large volume of the surrounding air at a frequency equal to that of the tuning fork. Thus, the sound becomes louder. This, however, does not violate the principle of conservation of energy. The fork vibrates for a longer time, if it is not pressed on the table. When it is placed in contact with the table, its energy is drained very quickly to set the table and the large volume of air into vibration. Thus, though the sound becomes louder, it lasts for a shorter time, keeping the total energy constant.

Analytical Treatment of Forced Vibration

Let a particle of mass m, capable of executing a damped simple harmonic motion, be subjected to an external simple harmonic force of constant amplitude and frequency. Suppose, x be the displacement of the particle from its mean position at time t. The forces acting on the particle are the following:

a) The force of restitution or the restoring force $-sx$ tending to bring the particle back to its mean position, 's' being the stiffness factor.

b) The retarding or the resisting force $-k\frac{dx}{dt}$, k being resisting force per unit velocity.

c) The driving periodic force $F\cos\omega t$, where F is the amplitude and ω is the angular frequency of the force.

The net force in the positive x-direction is $F\cos\omega t - sx - k\frac{dx}{dt}$. Hence, the equation of motion of the particle is —

$$m\frac{d^2x}{dt^2} = F\cos\omega t - sx - k\frac{dx}{dt}$$

or, $$\frac{d^2x}{dt^2} + \left(\frac{k}{m}\right)\frac{dx}{dt} + \left(\frac{s}{m}\right)x = \frac{F}{m}\cos\omega t$$

or, $$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega_0^2 x = f\cos\omega t \longrightarrow (1)$$

where $2b = \frac{k}{m}$, $\omega_0^2 = \frac{s}{m}$, $\frac{F}{m} = f$. The quantity ω_0 is the undamped natural angular frequency of the particle and b is the decay constant.

Let $x = x_1 = C e^{-bt} \cos(\sqrt{\omega_0^2 - b^2}\, t - \theta)$ be the solution of the equation [where, C & θ are constants]

$$\frac{d^2x_1}{dt^2} + 2b\frac{dx_1}{dt} + \omega_0^2 x_1 = 0 \longrightarrow (2)$$

Let another value of $x = x_2$ be the particular solution of equation

$$\frac{d^2x_2}{dt^2} + 2b\frac{dx_2}{dt} + \omega_0^2 x_2 = f\cos\omega t \longrightarrow (3)$$

Therefore, the general solution of eq. (1) is $x = x_1 + x_2$

To find x_2, let us take $x_2 = A\cos(\omega t - \alpha)$. We can suppose this on the ground that the system will ultimately vibrate with the same frequency as that of the impressed periodic force.

$x_2 = A\cos(\omega t - \alpha)$

$$\therefore \frac{dx_2}{dt} = -A\omega\sin(\omega t - \alpha)$$

$$\therefore \frac{d^2x_2}{dt^2} = -A\omega^2\cos(\omega t - \alpha)$$

From (3), we get,

$$-A\omega^2\cos(\omega t-\alpha) - 2bA\omega\sin(\omega t-\alpha) + \omega_0^2 A\cos(\omega t-\alpha)$$
$$= f\cos[(\omega t-\alpha)+\alpha]$$
$$= f\cos(\omega t-\alpha)\cos\alpha - f\sin(\omega t-\alpha)\sin\alpha \longrightarrow (4)$$

Since eq. (4) is true for all values of t, we can equate co-efficients of $\cos(\omega t-\alpha)$ and $\sin(\omega t-\alpha)$ from both sides.

$$\therefore f\cos\alpha = A(\omega_0^2-\omega^2)$$

and $f\sin\alpha = 2bA\omega$

$$\therefore \tan\alpha = \frac{2bA\omega}{A(\omega_0^2-\omega^2)} = \frac{2b\omega}{\omega_0^2-\omega^2} \longrightarrow (5)$$

$$\therefore f^2 = A^2(\omega_0^2-\omega^2)^2 + 4b^2A^2\omega^2$$

$$= A^2\left[(\omega_0^2-\omega^2)^2 + 4b^2\omega^2\right]$$

$$\therefore A = \frac{f}{\left[(\omega_0^2-\omega^2)^2+4b^2\omega^2\right]^{1/2}} \longrightarrow (6)$$

$$\therefore x_2 = \frac{f\cos(\omega t-\alpha)}{\left[(\omega_0^2-\omega^2)^2+4b^2\omega^2\right]^{1/2}} \longrightarrow (7)$$

Therefore, the complete solution of eq. (1) is —

$$x = x_1 + x_2$$

$$= Ce^{-bt}\cos\left(\sqrt{\omega_0^2-b^2}\,t-\theta\right) + \frac{f\cos(\omega t-\alpha)}{\left[(\omega_0^2-\omega^2)^2+4b^2\omega^2\right]^{1/2}}$$

$$\longrightarrow (8)$$

The first part of the solution for x, i.e, x_1 represents natural vibration set up in the damped system by the harmonic force at the start. These vibrations, however, become negligible very soon as the amplitude diminishes exponentially with time. If the damping is very small, the natural vibrations will persist for a longer time. The resultant vibration x at any instant is the sum of the natural vibrations represented by x_1 and the forced sustained vibration represented by x_2. After a lapse of time, when x_1 becomes negligible, we can write $x = A\cos(\omega t - \alpha)$ which represents sustained forced vibration (steady state motion).

If $\sqrt{\omega_0^2 - b^2}$ and ω are nearly equal, the natural vibration will interfere with the forced vibration at the initial stage, and produce beats. These beats are transient, as natural vibrations become imperceptible after a short interval of time.

In the steady state, the particle oscillates simple harmonically with a constant amplitude $f[(\omega_0^2 - \omega^2)^2 + 4b^2\omega^2]^{-1/2}$ with the period of the impressed force, but lags the impressed force by the angle α.

Resonance

Resonance is a particular case of forced vibration when the frequency of the driving periodic force equals the natural frequency of the system. Here the system oscillates with a considerably large amplitude. In the state of steady oscillations, the energy supplied by the external source fully replenishes the energy drained away from the system in overcoming the frictional forces. Resonant vibrations are also called sympathetic vibrations, and the system is said to resonate with the impressed periodic force.

In forced vibration, the steady-state displacement and velocity amplitudes of the driven system depend on the frequency of the driving force. When the displacement amplitude is a maximum for some frequency of the driver, we have the phenomenon of amplitude (or displacement) resonance. On the other hand, when the velocity amplitude attains a maximum for a certain frequency of the driving force, we have velocity resonance.

A. Amplitude (or displacement) resonance:

The steady state displacement amplitude of forced vibration is

$$A = \frac{f}{\left[(\omega_0^2-\omega^2)^2+4b^2\omega^2\right]^{1/2}} = \frac{F/m}{\left[(\omega_0^2-\omega^2)^2+4b^2\omega^2\right]^{1/2}}$$

$$= \frac{F}{\left[m^2(\omega_0^2-\omega^2)^2+4b^2m^2\omega^2\right]^{1/2}}$$

$$= \frac{F}{\left[(m\omega_0^2-m\omega^2)^2+(2b)^2m^2\omega^2\right]^{1/2}}$$

$$= \frac{F.}{\left[\left(m\times\frac{s}{m}-m\omega^2\right)^2+\frac{k^2}{m^2}m^2\omega^2\right]^{1/2}}$$

$$= \frac{F}{\left[\omega^2k^2+(m\omega^2-s)^2\right]^{1/2}} \longrightarrow (1)$$

The amplitude A is a maximum for that value of ω for which the denominator on the R.H.S of eq. (1) is a minimum. Therefore, for this value of angular frequency ω of the forcing function, we have

$$\frac{d}{d\omega}\left[\omega^2k^2+(m\omega^2-s)^2\right]=0$$

or, $2\omega k^2+2(m\omega^2-s)(2m\omega)=0$

or, $\omega k^2+2(m\omega^2-s)m\omega=0$

or, $k^2+2m(m\omega^2-s)=0$ or, $k^2+2m^2\omega^2-2ms=0$

or, $2m^2\omega^2 = 2ms - k^2$ or, $\omega^2 = \frac{2ms}{2m^2} - \frac{k^2}{2m^2}$

or, $\omega^2 = \frac{s}{m} - \frac{k^2}{2m^2} = \omega_0^2 - \frac{(2b)^2}{2}$

$\therefore \omega^2 = \omega_0^2 - 2b^2$ $\left[\because \omega_0^2 = \frac{s}{m} \text{ \& } 2b = \frac{k}{m}\right]$ $\longrightarrow$ (2)

Putting the above value of ω^2 in eq. (1), we get the maximum displacement amplitude

$$A_m = \frac{F}{\left[(\omega_0^2 - 2b^2)k^2 + \{m(\omega_0^2 - 2b^2) - s\}^2\right]^{1/2}}$$

$$= \frac{F}{\left[\omega_0^2 k^2 - 2b^2k^2 + (m\omega_0^2 - 2b^2m - s)^2\right]^{1/2}}$$

$$= \frac{F}{\left[\frac{s}{m}k^2 - 2b^2k^2 + (m\frac{s}{m} - 2b^2m - s)^2\right]^{1/2}}$$

$$= \frac{F}{\left[\frac{s}{m}k^2 - 2b^2k^2 + 4b^4m^2\right]^{1/2}}$$

$$= \frac{F}{\left[\frac{s}{m}k^2 - 2b^2k^2 + 4b^4\left(\frac{k^2}{4b^2}\right)\right]^{1/2}}$$

$$= \frac{F}{\left[\frac{s}{m}k^2 - 2b^2k^2 + b^2k^2\right]^{1/2}}$$

$$= \frac{F}{k\left[s/m - b^2\right]^{1/2}} = \frac{F}{2bm\left[\omega_0^2 - b^2\right]^{1/2}}$$

$$\therefore A_m = \frac{F/m}{2b\left[\omega_0^2 - b^2\right]^{1/2}} \longrightarrow (3)$$

The angular frequency for amplitude resonance $\sqrt{\omega_0^2 - 2b^2}$ is less than both undamped natural frequency ω_0 and the angular frequency for the damped vibration of the forced system $\sqrt{\omega_0^2 - b^2}$. When the amplitude is maximum, the potential energy of the driven system $\frac{1}{2}sx^2$ is a maximum. Thus, the amplitude resonance occurs simultaneously with the maximum potential energy of the driven system w.r.t the variation of frequency of the driver.

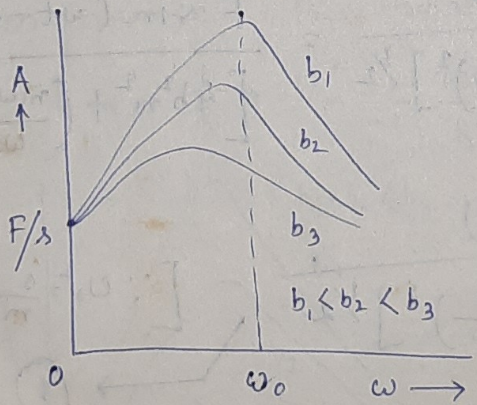

fig. 1.

The variation of displacement amplitude with the angular frequency ω of the forcing system for three different values of decay constant b_1, b_2, b_3 is shown in fig. 1. At $\omega = 0$, we have $A = \frac{F}{s}$ (eq. (1)) for all values of damping. As damping decreases, the angular frequency at which A is maximum, moves towards ω_0. Also A_m increases with diminishing damping. If $b = 0$, A becomes infinite at $\omega = \omega_0$ (eq. 3).