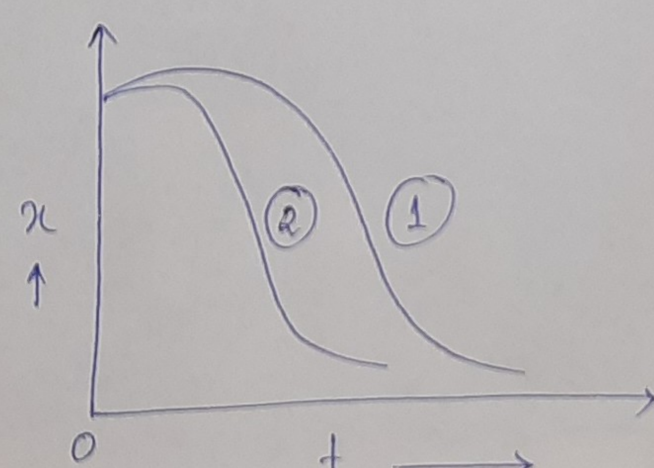
Study Material - Sem. 1 - C2T

- Oscillations - Dr. T. Kar - Class 2

$$x = e^{-bt}\left[A_1 e^{\sqrt{b^2-\omega_0^2}\,t} + A_2 e^{-\sqrt{b^2-\omega_0^2}\,t}\right]$$

$$= e^{-bt}\left(\frac{x_0}{2}\right)\left[\left\{1+\frac{\left(b+\frac{\dot{x}_0}{x_0}\right)}{\sqrt{b^2-\omega_0^2}}\right\} e^{\sqrt{b^2-\omega_0^2}\,t}\right.$$

$$\left. + \left\{1-\frac{\left(b+\dot{x}_0/x_0\right)}{\sqrt{b^2-\omega_0^2}}\right\} e^{-\sqrt{b^2-\omega_0^2}\,t}\right] \longrightarrow (9)$$



(1) $\Rightarrow b > \omega_0$

(2) $\Rightarrow b \simeq \omega_0$

Fig. 1.

Case II. Let $b = \omega_0$. To obtain the solution in this case, we let $b \to \omega_0$ in eq. (9). Thus, we get $\sqrt{b^2 - \omega_0^2} = \delta$ in eq. (9), where δ is a small quantity.

Expanding the R.H.S. of eq. (9) and neglecting the higher order term we get,

$$x = e^{-bt}\left(\frac{x_0}{2}\right)\left[\left\{1 + \frac{(b + \dot{x}_0/x_0)}{\delta}\right\} e^{\delta t} + \left\{1 - \frac{(b + \dot{x}_0/x_0)}{\delta}\right\} e^{-\delta t}\right]$$

$$= e^{-bt}\left(\frac{x_0}{2}\right)\left[\left\{1 + \frac{(b + \dot{x}_0/x_0)}{\delta}\right\}(1 + \delta t) + \left\{1 - \frac{(b + \dot{x}_0/x_0)}{\delta}\right\}(1 - \delta t)\right]$$

$$= e^{-bt}\left(\frac{x_0}{2}\right)\left[1 + \frac{(b + \dot{x}_0/x_0)}{\delta} + \delta t + (b + \dot{x}_0/x_0)t + 1 - \frac{(b + \dot{x}_0/x_0)}{\delta} - \delta t + (b + \dot{x}_0/x_0)t\right]$$

$$= e^{-bt}\left(\frac{x_0}{2}\right)\left[2 + 2\left(b + \frac{\dot{x}_0}{x_0}\right)t\right]$$

$$= x_0 e^{-bt}\left[1 + \left(b + \frac{\dot{x}_0}{x_0}\right)t\right] \longrightarrow (10)$$

Equation (10) shows that the motion is also aperiodic when $b = \omega_0$, but x approaches zero quicker than the overdamped case. The motion is now said to be critically damped and the variation of x with t is shown by curve 2 in fig. 1.

If the initial velocity $\dot{x}_0$ is imparted to the particle while at rest (i.e, $x_0 = 0$), then we get from eq. (10),

$$x = x_0 e^{-bt} \left[1 + \left(b + \frac{\dot{x}_0}{x_0}\right) t\right]$$

$$= x_0 e^{-bt} + t b x_0 e^{-bt} + \dot{x}_0 t e^{-bt}$$

$$= \dot{x}_0 t e^{-bt} \quad (\because x_0 = 0 \text{ at } t = 0)$$

$$\therefore x = \dot{x}_0 t e^{-bt} \longrightarrow (11)$$

$$\therefore \frac{dx}{dt} = \dot{x}_0 e^{-bt} + \dot{x}_0 t (-b) e^{-bt}$$

$$= \dot{x}_0 e^{-bt} [1 - bt] \longrightarrow (12)$$

The particle momentarily comes to rest when $\frac{dx}{dt} = 0$.

$$\therefore \dot{x}_0 e^{-bt} [1 - bt] = 0$$

or, $t = \frac{1}{b} = \frac{1}{\omega_0}$ [$\because b = \omega_0$, in case of critically damped motion]

At this stage, the particle attains the maximum displacement x_m.

$$\therefore x_m = \dot{x}_0 \frac{1}{b} e^{-1} = \frac{\dot{x}_0}{e \omega_0}$$

Therefore, we find that maximum displacement depends on initial velocity but the time required to attain the maximum displacement does not depend on initial velocity.

Case III. Let $b < \omega_0$, i.e, the damping is small.

Now, $\sqrt{b^2-\omega_0^2} = j\sqrt{\omega_0^2-b^2}$, where $j=\sqrt{-1}$

we have, $x = e^{-bt}\left[A_1 e^{\sqrt{b^2-\omega_0^2}\,t} + A_2 e^{-\sqrt{b^2-\omega_0^2}\,t}\right]$

which reduces to —

$$x = e^{-bt}\left[A_1 e^{j\sqrt{\omega_0^2-b^2}\,t} + A_2 e^{-j\sqrt{\omega_0^2-b^2}\,t}\right]$$

$$= e^{-bt}\left[A_1\left\{\cos\sqrt{\omega_0^2-b^2}\,t + j\sin\sqrt{\omega_0^2-b^2}\,t\right\} + A_2\left\{\cos\sqrt{\omega_0^2-b^2}\,t - j\sin\sqrt{\omega_0^2-b^2}\,t\right\}\right]$$

$$= e^{-bt}\left[(A_1+A_2)\cos\sqrt{\omega_0^2-b^2}\,t + j(A_1-A_2)\sin\sqrt{\omega_0^2-b^2}\,t\right] \rightarrow (13)$$

The constants A_1 and A_2 are, in general, complex numbers and either the real or the imaginary part of eq. (13) is the physical solution. Both parts can be written as —

$$x = e^{-bt}\left(A\cos\sqrt{\omega_0^2-b^2}\,t + B\sin\sqrt{\omega_0^2-b^2}\,t\right) \longrightarrow (14)$$

where A and B are arbitrary real constants to be determined from the initial conditions.

Let the initial displacement of the particle be x_0 and the initial velocity $\dot{x}_0$. Putting $x = x_0$ at $t = 0$ in eq. (14) we get,

$$x_0 = A \longrightarrow (15)$$

$$\frac{dx}{dt} = -b e^{-bt}\left(A\cos\sqrt{\omega_0^2 - b^2}\,t + B\sin\sqrt{\omega_0^2 - b^2}\,t\right)$$
$$+ e^{-bt}\left[\left(-A\sin\sqrt{\omega_0^2 - b^2}\,t\right)\sqrt{\omega_0^2 - b^2} + B\sqrt{\omega_0^2 - b^2}\cos\sqrt{\omega_0^2 - b^2}\,t\right]$$

$$= -b e^{-bt}\left(A\cos\sqrt{\omega_0^2 - b^2}\,t + B\sin\sqrt{\omega_0^2 - b^2}\,t\right)$$
$$+ \sqrt{\omega_0^2 - b^2}\, e^{-bt}\left[-A\sin\sqrt{\omega_0^2 - b^2}\,t + B\cos\sqrt{\omega_0^2 - b^2}\,t\right]$$

At $t = 0$,

$$\dot{x}_0 = -bA + \sqrt{\omega_0^2 - b^2}\,B$$
$$= -b x_0 + \sqrt{\omega_0^2 - b^2}\,B$$

or, $\left(\sqrt{\omega_0^2 - b^2}\right)B = \dot{x}_0 + b x_0$

or, $$B = \frac{\dot{x}_0 + b x_0}{\sqrt{\omega_0^2 - b^2}} \longrightarrow (16)$$

$$\therefore x = e^{-bt}\left[x_0\cos\sqrt{\omega_0^2 - b^2}\,t + \frac{\dot{x}_0 + b x_0}{\sqrt{\omega_0^2 - b^2}}\sin\sqrt{\omega_0^2 - b^2}\,t\right]$$

$$= e^{-bt}\left[C\cos\theta\cos\sqrt{\omega_0^2 - b^2}\,t + C\sin\theta\sin\sqrt{\omega_0^2 - b^2}\,t\right]$$

$$= C e^{-bt}\cos\left(\sqrt{\omega_0^2 - b^2}\,t - \theta\right) \longrightarrow (17)$$

where, $C\cos\theta = x_0$

$$C\sin\theta = \frac{\dot{x}_0 + b x_0}{\sqrt{\omega_0^2 - b^2}}$$

$$\therefore \tan\theta = b + (\dot{x}_0/x_0) \Big/ \sqrt{\omega_0^2 - b^2}$$

$$c^2 = x_0^2 + \frac{(bx_0+\dot{x}_0)^2}{\omega_0^2 - b^2} = \frac{x_0^2\omega_0^2 - b^2x_0^2 + b^2x_0^2 + 2bx_0\dot{x}_0 + \dot{x}_0^2}{\omega_0^2 - b^2}$$

$$\therefore c = \left[\frac{x_0^2\omega_0^2 + \dot{x}_0^2 + 2b\,x_0\dot{x}_0}{\omega_0^2 - b^2}\right]^{1/2}$$

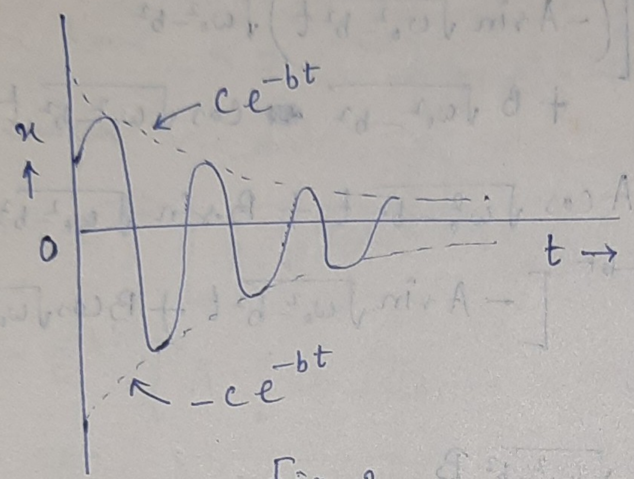


Fig. 2.

Equation (17) shows that the motion is oscillatory with an angular frequency $\omega = \sqrt{\omega_0^2 - b^2}$ and an amplitude ce^{-bt}. Since the amplitude decays exponentially with time, the motion is damped oscillatory. The variation of x with time t in case of underdamped motion is shown in Fig. 2.

Differential equation of damped motion from consideration of the energy of the system

Let x be the displacement of a particle of mass m executing a damped motion at time t. The potential energy E_p of the particle at this instant is the work done against the force of restitution. Thus,

$$E_p = \int_0^x sx\,dx = \frac{1}{2}sx^2 \longrightarrow (1)$$

The instantaneous kinetic energy is

$$E_K = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 \longrightarrow (2)$$

The sum of kinetic and potential energies is $E = E_K + E_p = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}sx^2 \longrightarrow (3)$

The instantaneous rate of dissipation of the energy E due to the resisting force $k\left(\frac{dx}{dt}\right)$ is the product of this force and the velocity. Thus,

$$-\frac{dE}{dt} = k\left(\frac{dx}{dt}\right)^2 \longrightarrow (4)$$

From eq. (3), we get,

$$\frac{dE}{dt} = m\left(\frac{dx}{dt}\right)\left(\frac{d^2x}{dt^2}\right) + sx\left(\frac{dx}{dt}\right)$$

or, $$-k\left(\frac{dx}{dt}\right)^2 = m\left(\frac{dx}{dt}\right)\left(\frac{d^2x}{dt^2}\right) + sx\left(\frac{dx}{dt}\right)$$

or, $$m\left(\frac{d^2x}{dt^2}\right) + k\left(\frac{dx}{dt}\right) + sx = 0,$$ since $\frac{dx}{dt} \neq 0$ for all values of t.

$\longrightarrow (5)$

Therefore, eq. 5 is the differential equation of the damped motion, obtained from the energy principle.

The restoring force acting on a harmonic oscillator is conservative in nature. This force is derivable from the potential energy of the oscillator: $F = -sx = -\frac{dE_p}{dx}$. The total energy of the oscillator remains constant and the undamped oscillator is said to be a conservative system.

For a damped oscillator, the viscous friction comes into play when the system is in motion and always acts opposite to the velocity. Always the work has to be done against the resisting force so that the work done over a full cycle is not zero. Energy must be drained off the oscillator to perform this work which is dissipated as heat. The energy of a damped oscilla-tor therefore does not remain constant, and it decays with time. The damped oscillator is thus a nonconservative system.

Q-factor of a damped Oscillator

Displacement of a damped oscillator at any instant t is given by —

$$x = c e^{-bt} \cos\left(\sqrt{\omega_0^2 - b^2}\, t - \theta\right) \longrightarrow (1)$$

Let x_0, x_1, x_2, x_3 etc. be the maximum displacement of the system in both directions at times given by —

$\sqrt{\omega_0^2 - b^2}\, t - \theta = 0, \pi, 2\pi, 3\pi$ etc. respectively.

Then, $x_0 = c e^{-bt_0}$

here, $\sqrt{\omega_0^2 - b^2}\, t_0 - \theta = 0$ or, $t_0 = \dfrac{\theta}{\sqrt{\omega_0^2 - b^2}}$

$$\therefore x_0 = c e^{-b\theta/\sqrt{\omega_0^2 - b^2}} \longrightarrow (2)$$

In case of x_1, $\sqrt{\omega_0^2 - b^2}\, t_1 - \theta = \pi$

or, $t_1 = (\theta + \pi)/\sqrt{\omega_0^2 - b^2}$

$$\therefore x_1 = -c e^{-b(\theta+\pi)/\sqrt{\omega_0^2 - b^2}} \longrightarrow (3)$$

Similarly, $x_2 = c e^{-b(\theta+2\pi)/\sqrt{\omega_0^2 - b^2}} \longrightarrow (4)$

$$x_3 = -c e^{-b(\theta+3\pi)/\sqrt{\omega_0^2 - b^2}} \longrightarrow (5)$$

Neglecting the signs of displacements, we get

$$\frac{x_0}{x_1} = \frac{c e^{-b\theta/\sqrt{\omega_0^2 - b^2}}}{c e^{-b(\theta+\pi)/\sqrt{\omega_0^2 - b^2}}} = e^{b\pi/\sqrt{\omega_0^2 - b^2}} = e^{bT/2}$$

$$= \frac{x_1}{x_2} = \frac{x_2}{x_3} \text{ etc.}$$

$\because$ Time period of oscillation of a damped oscillator $T = \dfrac{2\pi}{\sqrt{\omega_0^2 - b^2}}$

$\therefore \pi/\sqrt{\omega_0^2 - b^2} = T/2$

Let $e^{+bT/2} = p$

The quantity p is termed as the decrement. The logarithmic decrement is defined by $\lambda = \ln p = \frac{bT}{2} \longrightarrow (6)$

Starting from the maximum displacement x_1' on one side of the swing, the particle after time period T has the maximum displacement x_3' on the same side. Let x_2' be the maximum displacement on the other side.

$$\therefore \frac{x_1'}{x_2'} = \frac{x_2'}{x_3'} = e^{\lambda} \longrightarrow (7)$$

At x_1' and x_3', the particle is momentarily at rest and its energy is entirely potential. The loss of energy in one time period is thus —

$$\Delta E = \frac{1}{2} s \left(x_1'^2 - x_3'^2\right) = \frac{1}{2} s x_1'^2 \left[1 - \frac{x_3'^2}{x_1'^2}\right]$$

Now,

$$E = \frac{1}{2} s x_1'^2$$

$$\frac{x_1'}{x_3'} = \frac{x_1'}{x_2'} \times \frac{x_2'}{x_3'} = e^{\lambda} \times e^{\lambda} = e^{2\lambda}$$

$$\frac{x_3'}{x_1'} = e^{-2\lambda}$$

$$= E\left[1 - \left(e^{-2\lambda}\right)^2\right]$$

$$= E\left[1 - e^{-4\lambda}\right] \longrightarrow (8)$$

The average fractional loss of energy per cycle is $\frac{\Delta E}{E} = 1 - e^{-4\lambda} \simeq 1 - (1 - 4\lambda)$

$= 4\lambda$ if λ is small

$$\therefore \frac{\Delta E}{E} = 4\lambda \longrightarrow (9)$$

The efficiency of a damped oscillator to store energy is measured by a figure of merit or a quality factor, Q. It is defined as —

$$Q = 2\pi \times \frac{\text{maximum energy stored per cycle}}{\text{energy lost or dissipated per cycle}}$$

$$= 2\pi \times \frac{E}{\Delta E} = \frac{2\pi}{4\lambda} = \frac{\pi}{2\lambda} \longrightarrow (10)$$

Now, $\lambda = \frac{bT}{2}$ and $T = \frac{2\pi}{\omega_0}$ for small damping

$$\therefore \lambda = \frac{b(2\pi)}{2\omega_0} = \frac{\pi b}{\omega_0}$$

$$\therefore Q = \frac{\pi\omega_0}{2(\pi b)} = \frac{\omega_0}{2b} = \frac{\omega_0 m}{k} \quad \left[\text{as } 2b = \frac{k}{m}\right] \longrightarrow (11)$$

The time in which the amplitude decreases to $\frac{1}{e}$ of its initial value is called time constant or modulus of decay or the relaxation time of the damped motion (τ).

$$\therefore b\tau = 1 \quad \text{or,} \quad \tau = \frac{1}{b} = \frac{2m}{k} \longrightarrow (12)$$

$$\therefore Q = \frac{\omega_0 \tau}{2} \longrightarrow (13) \quad [\text{using } (12)]$$

The higher the Q, the less the damping, and the higher the efficiency of the oscillator in storing energy.