Semester-III

Core Course 5T

Chapter-5

Partial Differential Equations

Class Note 5 (1 hour)

(#Wave equation and its solution for vibrational modes of a stretched string, rectangular and circular membranes)

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Class Note-4

7. Wave equation and its solution for vibrational modes of a stretched string, rectangular and circular membranes.

Differential equation of transverse wave in a string:

Let a string is held under uniform tension T with its equilibrium position along the X axis of a Cartesian coordinate system. Let m is the mass per unite length of the string and $u(x,t)$ is the instantaneous transverse displacement of the string at position x and time t. Let for any value of x and t, $u(x,t)$ is very small compared to the length of the string, so that the length of the string and the tension T can be assumed to remain constant always.

Fig. 3 shows the equilibrium and displaced position (AB and CD respectively) of a small portion of the string. The net force on the small section in a direction perpendicular to the equilibrium position of the wire is:

$$\Delta F = T\sin\theta_2 - T\sin\theta_1$$

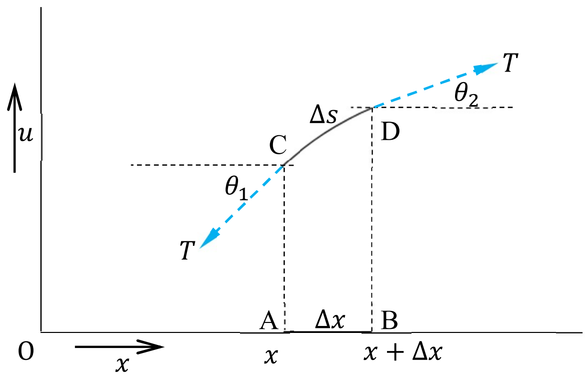

Fig 3

Since $u(x,t)$ is small, θ_1 and θ_2 can also be assumed to be small, and we can write:

$$\sin\theta_1 \approx \tan\theta_1 = \frac{\partial u(x,t)}{\partial x}$$

$$\sin\theta_2 \approx \tan\theta_2 = \frac{\partial u(x+\Delta x,t)}{\partial x} = \left.\frac{\partial u(x,t)}{\partial x}\right|_{x+\Delta x} = \left.\frac{\partial u(x,t)}{\partial x}\right|_{x} + \left.\frac{\partial^2 u(x,t)}{\partial x^2}\right|_{x}\Delta x + \cdots$$

$$= \frac{\partial u(x,t)}{\partial x} + \frac{\partial^2 u(x,t)}{\partial x^2}\Delta x + \cdots$$

Then:

$$\Delta F = T\left(\frac{\partial u(x+\Delta x,t)}{\partial x} - \frac{\partial u(x,t)}{\partial x}\right) = T\left[\frac{\partial u(x,t)}{\partial x} + \frac{\partial^2 u(x,t)}{\partial x^2}\Delta x + \cdots - \frac{\partial u(x,t)}{\partial x}\right] \approx T\frac{\partial^2 u(x,t)}{\partial x^2}\Delta x$$

Mass of the small section is $m\Delta x$. Then from Newton's law:

$$m\Delta x\frac{\partial^2 u(x,t)}{\partial t^2} = T\frac{\partial^2 u(x,t)}{\partial x^2}\Delta x \qquad \Rightarrow \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{m}{T}\frac{\partial^2 u(x,t)}{\partial t^2}$$

$$\Rightarrow \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u(x,t)}{\partial t^2}, \quad where\ c = \sqrt{T/m} \ldots\ldots\ldots (7.1)$$

Equation (7.1) is the standard wave equation for transverse waves in a stretched string. You can verify that any function of the form $f(x - ct)$ is a solution for $u(x,t)$ and can represent a propagating wave in the string where c is the velocity of the wave. However to obtain the solutions for the vibrations of stretched string, fastened between fixed supports, we shall solve equation (7.1) by the method of separation of variables. Let:

$$u(x,t) = X(x)T(t) \dots\dots\dots (7.2)$$

Where $X(x)$ and $T(t)$ are respectively function of x only and function of t only. Substituting (7.2) in (7.1) and rearranging we get:

$$\frac{c^2}{X}\frac{d^2X(x)}{dx^2} = \frac{1}{T}\frac{d^2T(t)}{dt^2} = constant = -\omega^2\ (say) \dots\dots\dots (7.3)$$

We shall discuss about the negative sign of the constant $(-\omega^2)$ later.

Equation (7.3) gives:

$$\frac{d^2X(x)}{dx^2} + \frac{\omega^2}{c^2}X(x) = 0 \dots\dots\dots (7.4)$$

With solution:

$$X(x) = A\cos kx + B\sin kx\,, \quad where\ k = \omega/c\ \dots\dots\dots (7.4A)$$

A, B are constants to be determined from boundary conditions. And:

$$\frac{d^2T(t)}{dt^2} + \omega^2T(t) = 0 \dots\dots\dots (7.5)$$

With solution:

$$T(t) = C\cos\omega t + D\sin\omega t\ \dots\dots\dots (7.5A)$$

C, D are constants to be determined from boundary conditions. Then from (7.2), (7.4A), (7.5A) we get:

$$u(x,t) = X(x)T(t) = (A\cos kx + B\sin kx)(C\cos\omega t + D\sin\omega t) \dots\dots\dots (7.6)$$

Note that the solution is oscillatory in x and t, which is consistent with the vibration of a sting or transverse wave in a string. If the constant in equation (7.3) were positive, say ω^2, then we should have obtain:

$$u(x,t) = (a\exp(kx) + b\exp(-kx))(c\exp(\omega t) + d\exp(-\omega t)), \qquad a, b, c, d\ being\ constants.$$

And $u(x,t)$ would not be oscillatory but continuously increasing or decreasing with x or t, which would not represent vibrations or waves. Therefore we have taken the constant in equation (7.3) to be negative.

Vibration of a stretched string fastened with two fixed supports:

Now if the string is fastened at its two ends with two fixed supports separated by a distance l, then:

$$u(0,t) = u(l,t) = 0 \dots \dots \dots (7.7)$$

Applying the boundary condition (7.7) to equation (7.6), we can write:

$$u(0,t) = 0 = A(C\cos\omega t + D\sin\omega t)$$

Since $(C\cos\omega t + D\sin\omega t)$ cannot be zero for all t, therefore we must have:

$$A = 0$$

Then (7.6) converts to:

$$u(x,t) = B\sin(kx)\,(C\cos\omega t + D\sin\omega t)$$

$$= \sin(kx)\,(C'\cos\omega t + D'\sin\omega t), \qquad \text{with } C', D' = C, D \text{ multiplied by}$$

We also have:

$$u(l,t) = 0 = \sin(kl)\,(C'\cos\omega t + D'\sin\omega t)$$

$$\Rightarrow \sin(kl) = 0 = \sin n\pi\,, where\ n = 1,2 \dots \dots$$

$$\Rightarrow k = \frac{n\pi}{l}, where\ n = 1,2 \dots \dots$$

We did not take $n = 0$, since this gives $k = 0$ and $u(x,t)$ becomes zero for all values of x and t, which is not an acceptable solution.

Therefore:

$$\omega = kc = \frac{n\pi c}{l},\ \ n = 1,2 \dots \dots$$

Also the constants C' and D' now can depend on n and we should write them as C_n and D_n.

Then we get a solution for each value of n:

$$u_n(x,t) = C_n \sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi ct}{l}\right) + D_n \sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{n\pi ct}{l}\right), \qquad n = 1,2,3 \dots\dots\ . \ \dots \dots \dots (7.8)$$

And the general solution will be:

$$u(x,t) = \sum_{n=1}^{\infty}\left[C_n \sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi ct}{l}\right) + D_n \sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{n\pi ct}{l}\right)\right] \quad \dots \dots \dots (7.9)$$

The constants C_n and D_n remains undetermined till now. They can be evaluated with the help of more boundary / initial conditions.

Transverse vibration of rectangular membrane:

Consider a rectangular elastic membrane with its boundary fixed in a rigid frame. In rectangular Cartesian coordinates, the differential equation of the transverse vibration of the membrane can be written, in analogy of that of a vibrating string as:

$$\frac{\partial^2 u(x,y,t)}{\partial x^2}+\frac{\partial^2 u(x,y,t)}{\partial y^2}=\frac{1}{c^2}\frac{\partial^2 u(x,y,t)}{\partial t^2} \dots\dots\dots (7.10)$$

Let $u(x,y,t)$ can be expressed as:

$$u(x,y,t)=X(x)Y(y)T(t) \dots\dots\dots (7.11)$$

Where $X(x), Y(y)$ and $T(t)$ are respectively the functions of x only, y only and t only.

Application of the method of separation of variables gives:

$$\frac{c^2}{X}\frac{d^2X(x)}{dx^2}+\frac{c^2}{Y}\frac{d^2Y(y)}{dy^2}=\frac{1}{T}\frac{d^2T(t)}{dt^2}=constant=-\omega^2\ (say)$$

$$\Rightarrow \frac{d^2T(t)}{dt^2}+\omega^2T(t)=0 \dots\dots\dots (7.12)$$

$$\frac{1}{X}\frac{d^2X(x)}{dx^2}=-\frac{1}{Y}\frac{d^2Y(y)}{dy^2}-\frac{\omega^2}{c^2}=constant=-\alpha^2\ (say)$$

$$\Rightarrow \frac{d^2X(x)}{dx^2}+\alpha^2X(x)=0 \dots\dots\dots (7.13)$$

$$\frac{d^2Y(y)}{dy^2}+\left(\frac{\omega^2}{c^2}-\alpha^2\right)Y(y)=0$$

$$\Rightarrow \frac{d^2Y(y)}{dy^2}+\beta^2Y(y)=0 \dots\dots\dots (7.14), where\ \beta^2=\frac{\omega^2}{c^2}-\alpha^2$$

Note that ω^2, α^2 and β^2 should be positive in order to make $T(t)$, $X(x)$ and $Y(y)$ oscillatory, which the required condition of a vibrating membrane with fixed boundary. Equation (7.12), (7.13) and (7.14) have solutions:

$$T(t)=A\cos\omega t+B\sin\omega t$$

$$X(x)=C\cos\alpha x+D\sin\alpha x$$

$$Y(y)=E\cos\beta y+F\sin\beta y$$

Where A, B, C, D, E and F are constants to be determined from boundary conditions. The solution fro $u(x,y,t)$ will be then given by:

$$u(x,y,t)=(C\cos\alpha x+D\sin\alpha x)(E\cos\beta y+F\sin\beta y)(A\cos\omega t+B\sin\omega t) \dots\dots\dots (7.15)$$

If a and b are the sides of the rectangular membrane, then the boundary conditions in the present problem are:

$$u(x,y,t) = 0 \text{ for } x = 0 \text{ } and \text{ } a \text{ ……… } (7.16A)$$

$$u(x,y,t) = 0 \text{ for } y = 0 \text{ } and \text{ } b \text{ ……… } (7.16B)$$

$u(x,y,t) = 0$ for $x = 0$ gives:

$$0 = C(E\cos\beta y + F\sin\beta y)(A\cos\omega t + B\sin\omega t)$$

$$\Rightarrow C = 0$$

$u(x,y,t) = 0$ for $x = a$ gives:

$$0 = D\sin\alpha a\,(E\cos\beta y + F\sin\beta y)(A\cos\omega t + B\sin\omega t)$$

$$\Rightarrow \sin\alpha a = 0 = \sin n\pi$$

$$\Rightarrow \alpha = \frac{n\pi}{a}, n = 1,2,3 \text{ ………}$$

Similarly $(7.16B)$ gives:

$$E = 0 \text{ and } \beta = \frac{m\pi}{b}, m = 1,2,3 \text{ ………}$$

Now:

$$\beta^2 = \frac{\omega^2}{c^2} - \alpha^2$$

$$\Rightarrow \omega^2 = c^2(\beta^2 + \alpha^2) = \pi^2 c^2\left(\frac{n^2x^2}{a^2} + \frac{m^2y^2}{b^2}\right)$$

Therefore we should write ω_{mn} in place of ω. i.e.

$$\omega_{mn}{}^2 = \pi^2 c^2\left(\frac{n^2x^2}{a^2} + \frac{m^2y^2}{b^2}\right) \text{ ……… } (7.17)$$

Also writing A_{mn} in place of FDA and B_{mn} in place of FDB, (7.15) becomes:

$$u_{mn}(x,y,t) = \sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}(A_{mn}\cos\omega_{mn}t + B_{mn}\sin\omega_{mn}t) \text{ ……… } (7.18)$$

Then the general solution becomes:

$$\left.\begin{aligned} u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}(A_{mn}\cos\omega_{mn}t + B_{mn}\sin\omega_{mn}t) \\ \text{with } \omega_{mn} = \pi^2 c^2\left(\frac{n^2x^2}{a^2} + \frac{m^2y^2}{b^2}\right) \end{aligned}\right\} \text{ …… } (7.19)$$

Transverse vibration of circular membrane (drum skin):

For the transverse vibration of the circular membrane, using polar coordinates (r, θ), we can write the wave equation as:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u(r,\theta,t)}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u(r,\theta,t)}{\partial \theta^2}=\frac{1}{c^2}\frac{\partial^2 u(r,\theta,t)}{\partial t^2}$$

$$\Rightarrow \frac{\partial^2 u(r,\theta,t)}{\partial r^2}+\frac{1}{r}\frac{\partial u(r,\theta,t)}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u(r,\theta,t)}{\partial \theta^2}=\frac{1}{c^2}\frac{\partial^2 u(r,\theta,t)}{\partial t^2} \dots\dots\dots (7.20)$$

Let $u(r, \theta, t)$ can be expressed as:

$$u(r,\theta,t) = R\,(r)\Theta(\theta)T(t) \dots\dots\dots (7.21)$$

Where $R\,(r)$, $\Theta(\theta)$ and $T(t)$ are respectively the functions of r only, θ only and t only.

Application of the method of separation of variables gives:

$$\frac{c^2}{R}\frac{d^2R(r)}{dr^2}+\frac{c^2}{rR}\frac{dR(r)}{dr}+\frac{c^2}{r^2\Theta}\frac{d^2\Theta(\theta)}{d\theta^2}=\frac{1}{T}\frac{d^2T(t)}{dt^2}=constant=-\omega^2\ (say)$$

$$\Rightarrow \frac{d^2T(t)}{dt^2}+\omega^2T(t)=0 \dots\dots\dots (7.22)$$

$$\frac{r^2}{R}\frac{d^2R(r)}{dr^2}+\frac{r}{R}\frac{dR(r)}{dr}+\frac{\omega^2r^2}{c^2}=-\frac{1}{\Theta}\frac{d^2\Theta(\theta)}{d\theta^2}=constant=m^2\ (say)$$

$$\Rightarrow \frac{d^2\Theta(\theta)}{d\theta^2}+m^2\Theta(\theta)=0 \dots\dots\dots (7.23)$$

$$r^2\frac{d^2R(r)}{dr^2}+r\frac{dR(r)}{dr}+\left(\frac{\omega^2}{c^2}r^2-m^2\right)R(r)=0$$

$$\Rightarrow r^2\frac{d^2R(r)}{dr^2}+r\frac{dR(r)}{dr}+(k^2r^2-m^2)R(r)=0, where\ k=\omega/c$$

Let $kr = \xi$. Then:

$$r\frac{dR(r)}{dr}=\xi\frac{dR(\xi/k)}{d\xi}=\xi\frac{dP(\xi)}{d\xi}; \qquad r^2\frac{d^2R(r)}{dr^2}=\xi^2\frac{d^2R(\xi/k)}{d\xi^2}=\xi^2\frac{d^2P(\xi)}{d\xi^2}$$

Where $P(\xi) = R(\xi/k)$. Therefore the radial equation becomes:

$$\xi^2\frac{d^2P(\xi)}{d\xi^2}+\xi\frac{dP(\xi)}{d\xi}+(\xi^2-m^2)P(\xi)=0 \dots\dots\dots (7.24)$$

Equation (7.22) has solution:

$$T(t) = A\cos\omega t + B\sin\omega t \dots\dots\dots (7.25)$$

Equation (7.23) has solution:

$$\Theta(\theta) = C\cos m\theta + D\sin m\theta \dots\dots\dots (7.26)$$

Note that $\omega^2$and m^2 must be positive for $T(t)$ and $\Theta(\theta)$ to be oscillatory, a condition required by the physical situation. Moreover since increasing θ by 2π the same point is reached, therefore for the sake of single valued ness of the solution we must have $\Theta(\theta) = \Theta(\theta + 2\pi)$. This requires m to be positive or negative integers or zero. i.e. $m = 0, \pm 1, \pm 2, \dots \dots$.

Eqn. **(7.24)** is the well-known **Bessel differential equation** with solution. When m an integer, its solution is given by:

$$P(\xi) = EJ_m(\xi) + FN_m(\xi) \quad \dots \dots \dots (7.27)$$

Where $J_m(\xi)$ is the Bessel function of first kind of order m. For integer m, $J_m(\xi)$ can be given by:

$$J_m(\xi) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,(s+m)!} \left(\frac{\xi}{2}\right)^{2s+m} \quad \dots \dots \dots (7.28)$$

And $N_m(\xi)$ is called the Bessel function of second kind or Neumann function and is given by:

$$N_m(\xi) = \frac{\cos m\pi \, J_m(\xi) - J_{-m}(\xi)}{\sin m\pi} \quad \dots \dots \dots (7.29)$$

$N_m(\xi)$ contains the function $J_{-m}(\xi) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(s-m)!} \left(\frac{\xi}{2}\right)^{2s-m}$. Due to it, $N_m(\xi)$ becomes indeterminate for $\xi = kr = 0$ or $r = 0$. Thus at the centre of the circular membrane, the solution becomes indeterminate for $N_m(\xi)$, which is unacceptable. Therefore we must have for $F = 0$. Then equation (7.27) have the form:

$$R(r) = EJ_m(kr) \quad \dots \dots \dots (7.27)$$

Then the solution can be given by:

$$u(r,\theta,t) = (A\cos\omega t + B\sin\omega t)(C\cos m\theta + D\sin m\theta)J_m(kr) \quad \dots \dots \dots (7.28)$$

The solution can be written as:

$$u(r,\theta,t) = ACJ_m(kr)\cos m\theta\cos\omega t + BCJ_m(kr)\cos m\theta\sin\omega t$$
$$+ADJ_m(kr)\sin m\theta\cos\omega t + BDJ_m(kr)\sin m\theta\sin\omega t \quad \dots \dots \dots (7.29)$$

Each term in the R.H.S of the above equation is a possible solution of the wave equation for a circular membrane. Some of which may have to be discarded to satisfy the boundary conditions or initial conditions. These solutions can also be written as (see Math Methods … by Boas):

$$u(r,\theta,t) = J_m(kr)\begin{Bmatrix}\cos m\theta\\ \sin m\theta\end{Bmatrix}\begin{Bmatrix}\cos\omega t\\ \cos\omega t\end{Bmatrix} \quad \dots \dots \dots (7.30)$$

In case of drum skin, the circumference is fixed i.e. $u(r,\theta,t) = 0$ for $r = a$ (radius of the membrane).

Thus for a drum skin, we must have:

$$J_m(ka) = 0$$

This condition fixes the values of k, since $J_m(\xi)$ becomes zero only for particular values of ξ or in other words $J_m(ka)$ becomes zero only for particular values of k. We can write these values as k_{mn}, where n denotes the serial number of the zero of the Bessel function. Hence the possible frequencies can also be indexed as $\omega_{mn} = k_{mn}c$. Then we should write:

$$u_{mn}(r,\theta,t) = J_m(k_{mn}r)\begin{Bmatrix}\cos m\theta \\ \sin m\theta\end{Bmatrix}\begin{Bmatrix}\cos \omega_{mn}t \\ \cos \omega_{mn}t\end{Bmatrix} \dots\dots\dots (7.30A)$$

The following part is optional

First few modes of vibration of a circular membrane with the circumference at rest (vibration of a drum skin):

The condition satisfied by a drum skin of radius a is:

$$u_{mn}(r,\theta,t) = 0 \text{ for } r = a$$

Which gives:

$$J_m(k_{mn}a) = 0$$

Now there are standard tables where the values of ξ for which $J_m(\xi)$ are zero, i.e. tables for zeros of $J_m(\xi)$. With the help of such tables we can write:

Sl. No. of zero (n)	Value of $k_{mn}a$ for the zeros of $J_m(k_{mn}a)$ for $m =$			
	0	1	2	3
1	2.4048	3.8317	5.1356	6.3802
2	5.5201	7.0156	8.4172	9.7610
3	8.6537	10.1735	11.6198	13.0152

Let us explore the lowest four values of k_{mn}. From the above table we see that these are:

$$k_{01} = \frac{2.4048}{a} \text{ (for } m = 0), k_{11} = \frac{3.8317}{a} \text{ (for } m = 1),$$

$$k_{21} = \frac{5.1356}{a} \text{ (for } m = 2), k_{02} = \frac{5.5201}{a} \text{ (for } m = 0).$$

Therefore the lowest possible frequencies are:

$$\omega_{01} = k_{01}c = \frac{2.4048\text{c}}{a}, \quad \omega_{11} = \frac{3.8317\text{c}}{a}, \quad \omega_{21} = \frac{5.1356\text{c}}{a}, \omega_{02} = \frac{5.5201\text{c}}{a}$$

Then the space parts (amplitudes) of first four modes of vibration are:

$$\omega_{01} = \frac{2.4048\text{c}}{a}; \; \boldsymbol{u_{01}(r,\theta)} \;\; = \; \boldsymbol{J_0\left(2.4048\frac{r}{a}\right)} = 0 \text{ for } r = a$$

$$\omega_{11} = \frac{3.8317\text{c}}{a}; \; \boldsymbol{u_{11}(r,\theta)} = \begin{cases} \boldsymbol{J_1\left(3.8317\frac{r}{a}\right)\cos\theta} \; = 0 \text{ for } r = a \text{ and/or } \theta = \frac{\pi}{2}, \frac{3\pi}{2} \\ \boldsymbol{J_1\left(3.8317\frac{r}{a}\right)\sin\theta} = 0 \text{ for } r = a \text{ and/or } \theta = 0, \pi \end{cases}$$

$$\omega_{21} = \frac{5.1356\text{c}}{a}; \; \boldsymbol{u_{21}(r,\theta)} = \begin{cases} \boldsymbol{J_2\left(5.1356\frac{r}{a}\right)\cos 2\theta} = 0 \text{ for } r = a \text{ and/or } \theta = \frac{\pi}{4}, \frac{3\pi}{4} \\ \boldsymbol{J_2\left(5.1356\frac{r}{a}\right)\sin 2\theta} = 0 \text{ for } r = a \;\text{ and/or } \theta = 0, \frac{\pi}{2} \end{cases}$$

$$\omega_{02} = \frac{5.5201\text{c}}{a}; \; \boldsymbol{u_{02}(r,\theta)} \; = \boldsymbol{J_0\left(5.5201\frac{r}{a}\right)} \;\; = 0 \text{ for } r = a \; and \; r \; = \frac{2.4048}{5.5201} a$$

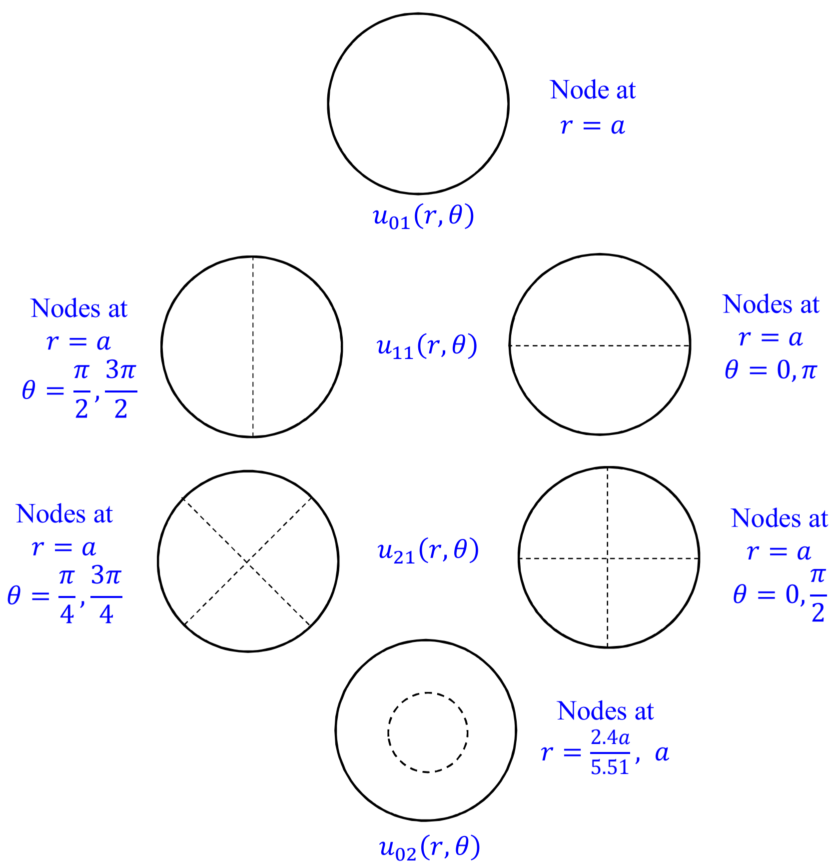

Fig.4: Four lowest frequency vibrational modes of a drum skin. Dotted lines represent nodes where there is no vibration. Rim of the skin is always a node.