$$0 = \frac{\pi}{2} - \sum_{n=1,3,5\ldots}^{\infty} \frac{4}{\pi n^2}$$

$$\frac{\pi}{2} = \frac{4}{\pi} \sum_{n=1,3,5\ldots}^{\infty} \frac{1}{n^2}$$

$$\sum_{n=1,3,5\ldots}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{8}$$

$$\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right] = \frac{\pi^2}{8}$$

$$i.e \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

Q.1 Obtain the Fourier Series for the function $f(x) = \begin{cases} 1 - \frac{2x}{\pi}, for\ 0 \le x \le \pi \\ 1 + \frac{2x}{\pi}, for - \pi \le x \le 0 \end{cases}$

Hence deduce that

$$\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right] = \frac{\pi^2}{8}$$

Q.2 Express $f(x) = x$ as Half range cosine series in $0 \le x \le 2$.

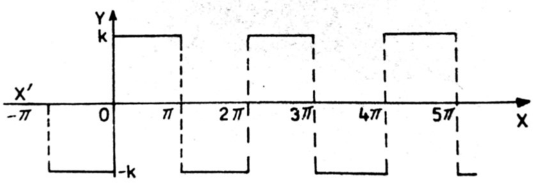
Q.3 Obtain the Fourier Series for the function $f(x) = \begin{cases} x, for\ 0 \le x \le \frac{\pi}{2} \\ \pi - x, for \frac{\pi}{2} \le x \le \pi \end{cases}$

L3

Applications:

1. **Square waveform:**

Find the Fourier series of the periodic function $f(x) = \begin{cases} k, for\ 0 \le x \le \pi \\ -k, for - \pi \le x \le 0 \end{cases}$ and $f(x + 2\pi) = f(x)$. Sketch the graph of f(x) and deduce that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} = \frac{\pi}{4}$.



Function of this kind occur as external forces acting on mechanical systems, electromotive forces in electric circuits etc.

Here, $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \left\{ \int_{-\pi}^{0} -k\ dx + \int_{0}^{\pi} k\ dx \right\} = \frac{1}{\pi} \{-k\pi + k\pi\} = 0$

Note: This can also be seen without integration, since the area under the curve of f(x) between -π and π (taken with a minus sign where f(x) is negative) is zero.

$$\text{Again } a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\ dx = \frac{1}{\pi}\left\{\int_{-\pi}^{0} -k\cos nx\ dx + \int_{0}^{\pi} k\cos nx\ dx\right\}$$
$$= \frac{1}{\pi}\left\{\left|-k\frac{\sin nx}{n}\right|_{-\pi}^{0} + \left|k\frac{\sin nx}{n}\right|_{0}^{\pi}\right\} = 0$$
$$\textit{since } \sin nx = 0 \textit{ at } x = -\pi, 0, \pi$$

Note: Since f(x) is odd function in (-π, π), hence we can write $a_n = 0$.

Now, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\, dx \quad \textit{as } f(x) \textit{is odd function.}$

Therefore, $b_n = \frac{2}{\pi}\int_{0}^{\pi} k\sin nx\, dx = \frac{2k}{\pi}\left|\frac{-\text{co}}{n}\right|_{0}^{\pi} = \frac{2k}{n\pi}\{1-(-1)^n\}$

$$\therefore b_n = \begin{cases} 0, & \textit{for } n \textit{ is even} \\ \frac{4k}{n\pi}, & \textit{for } n \textit{ is odd} \end{cases}$$

Thus, $f(x) = \sum_{1,3,5,\ldots}^{\infty} \frac{4k}{n\pi}\sin nx$

Putting $x = \frac{\pi}{2}$ in the above equation we get,

$$f\left(\frac{\pi}{2}\right) = k = \sum_{n=1,3,5,\ldots}^{\infty} \frac{4k}{n\pi}\sin\frac{n\pi}{2} = \frac{4k}{\pi}\left\{1-\frac{1}{3}+\frac{1}{5}-\cdots\right\}$$

Therefore, $\left\{1-\frac{1}{3}+\frac{1}{5}-\cdots\right\} = \frac{\pi}{4}$

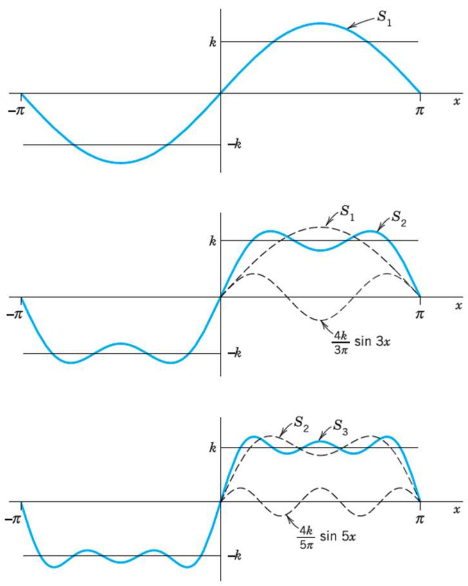
or, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} = \frac{\pi}{4}$

Note: Here the Fourier series expansion is

$$f(x) = \sum_{1,3,5,\ldots}^{\infty} \frac{4k}{n\pi}\sin nx = \frac{4k}{\pi}\sin x + \frac{4k}{3\pi}\sin 3x + \frac{4k}{5\pi}\sin 5x + \cdots$$

The first three partial sums are as follows

$$S_1 = \frac{4k}{\pi}\sin x,\ \ S_2 = \frac{4k}{\pi}\left(\sin x + \frac{\sin 3x}{3}\right), \textit{and } S_3 = \frac{4k}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5}\right) \text{ etc.}$$



Their graphs (shown above) seem to indicate that the series is convergent and has the sum $f(x)$, the given function. We notice that at $x = 0$ and $x = \pi$, the points of discontinuity of $f(x)$, all partial sums have the value zero, the arithmetic mean of the limits -k and k of our function, at these points.

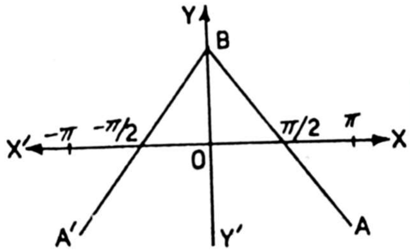
These illustrate the manner in which the successive approximation to the series $f(x) = \sum_{1,3,5,\ldots}^{\infty} \frac{4k}{n\pi} \sin nx$ approach more and more closely to $f(x) = \begin{cases} k, for\ 0 \le x \le \pi \\ -k, for -\pi \le x \le 0 \end{cases}$ for all the values of x in the interval (-π, π), but not for $x = 0, \pm\pi$

2. Triangular waveform:

Obtain the Fourier Series for the function $f(x) = \begin{cases} 1 - \frac{2x}{\pi}, for\ 0 \le x \le \pi \\ 1 + \frac{2x}{\pi}, for -\pi \le x \le 0 \end{cases}$

Hence deduce that

$$\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right] = \frac{\pi^2}{8}$$



Since $f(-x) = 1 - \frac{2x}{\pi}\ in\ (-\pi, 0) = f(x)\ in\ (0, \pi)$ and

$$f(-x) = 1 + \frac{2x}{\pi}\ in\ (0, \pi) = f(x)\ in\ (-\pi, 0)$$

Here the function is an even function in $(-\pi, \pi)$, therefore $b_n = 0$.

Now $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

Where, $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx = \frac{2}{\pi}\int_0^{\pi}\left(1 - \frac{2x}{\pi}\right)dx = \frac{2}{\pi}\left[x - \frac{x^2}{\pi}\right]_0^{\pi} = 0$

And $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx = \frac{2}{\pi}\int_0^{\pi}(1 - \frac{2x}{\pi})\cos nx\, dx$

$$= \frac{2}{\pi}\left[\left(1 - \frac{2x}{\pi}\right)\frac{\sin nx}{n} - \left(-\frac{2}{\pi}\right)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left(-\frac{2\cos n\pi}{\pi n^2} + \frac{2}{\pi n^2}\right) = \frac{4}{n^2\pi^2}[1 - (-1)^n]$$

Therefore $a_n = \begin{cases} 0,\ for\ n\ is\ even \\ \frac{8}{n^2\pi^2},\ for\ n\ is\ odd \end{cases}$

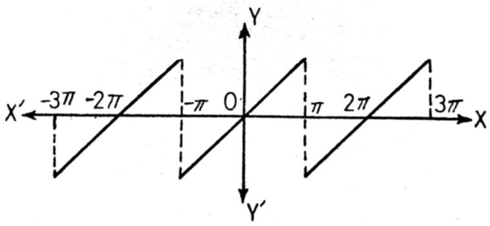
Thus, $f(x) = \frac{8}{\pi^2}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right]$

Now, putting x=0 in the above equation we get

$$1 = f(0) = \frac{8}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

Hence follows the desired result.

3. **Sawtooth waveform:**
Express $f(x) = x$ as a Fourier series in the interval $-\pi \leq x \leq \pi$.



Since, $f(-x) = -x \; in \; (-\pi, 0) = -f(x) \;\; in \; (0, \pi)$
And $f(-x) = -x \; in \; (0, \pi) = -f(x) \;\; in \; (-\pi, 0)$
Therefore, $f(x)$ is odd function in (-π, π) and thus $a_n = 0$.
Since area under the curve of $f(x)$ in (-π, π) is zero, therefore, $a_0 = 0$.
Now, $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx \, dx = \frac{2}{\pi}\int_0^{\pi} x \; \sin nx \, dx = \frac{2}{\pi}\left\{\left|x\frac{-\cos nx}{n}\right|_0^{\pi} + \left|\frac{\sin nx}{n^2}\right|_0^{\pi}\right\}$

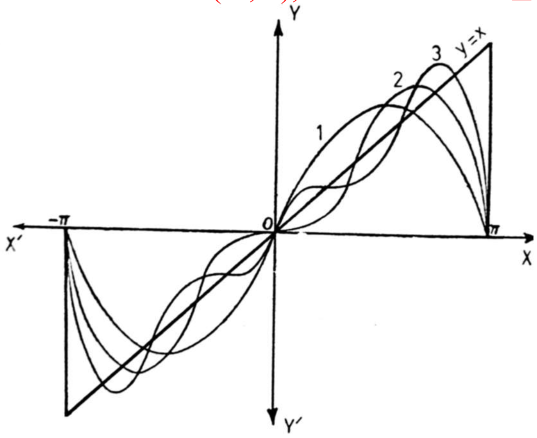
$$= \frac{2}{\pi}\left\{-\pi\frac{\cos n\pi}{n}\right\} = \frac{2}{n}(-1)^{n+1}$$

Therefore, $f(x) = \sum_{n=1}^{\infty}\frac{2}{n}(-1)^{n+1}\sin nx$.

Note: Here the Fourier series expansion is

$$f(x) = \sum_{n=1}^{\infty}\frac{2}{n}(-1)^{n+1}\sin nx = 2\left[\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots\right]$$

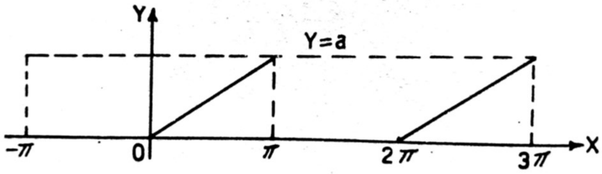
The first three partial sums are as follows
$S_1 = 2\sin x, \;\; S_2 = 2\left(\sin x - \frac{\sin}{2}\right), and \; S_3 = 2\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3}\right)$ etc.
Their graphs (shown below) seem to indicate that the series is convergent and has the sum $f(x)$, the given function. We notice that at $x = \pm\pi$, the points of discontinuity of $f(x)$, all partial sums have the value zero. These illustrate the manner in which the successive approximation to the series $f(x) = \sum_{n=1}^{\infty}\frac{2}{n}(-1)^{n+1}\sin nx$ approach more and more closely to $f(x) = x$ for all the values of x in the interval (-π, π), but not for $x = \pm\pi$.



4. **Modified Sawtooth waveform:**
Find the Fourier series of the periodic function

$$f(x) = \begin{cases} 0, for -\pi \leq x \leq 0 \\ x, for \; 0 \leq x \leq \pi \end{cases}$$

Hence show that $\left\{\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right\} = \frac{\pi^2}{8}$.



Here, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_0^{\pi} x\, dx = \frac{1}{\pi}\times\frac{\pi^2}{2} = \frac{\pi}{2}$

And $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = \frac{1}{\pi}\int_0^{\pi} x\cos nx\, dx = \frac{1}{\pi}\left\{\left|x\frac{\sin nx}{n}\right|_0^{\pi} + \left|\frac{\cos nx}{n^2}\right|_0^{\pi}\right\}$

$$= \frac{1}{\pi n^2}\{\cos n\pi - \cos 0\} = \frac{1}{\pi n^2}\{(-1)^n - 1\}$$

$$\therefore a_n = \begin{cases} 0, for\ n\ is\ even \\ -\frac{2}{\pi n^2}, for\ n\ is\ odd \end{cases}$$

Again $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx = \frac{1}{\pi}\int_0^{\pi} x\ \sin nx\, dx = \frac{1}{\pi}\left\{\left|-x\frac{\cos nx}{n}\right|_0^{\pi} + \left|\frac{\sin nx}{n^2}\right|_0^{\pi}\right\}$

$$= \frac{1}{\pi}\left\{-\pi\frac{\cos n\pi}{n}\right\} = -\frac{(-1)^n}{n}$$

$$\therefore b_n = \begin{cases} \frac{1}{n}, for\ n\ is\ odd \\ -\frac{1}{n}, for\ n\ is\ even \end{cases}$$

Therefore, $f(x) = \frac{\pi}{4} - \frac{2}{\pi}\sum_{n=1,3,5,\ldots}^{\infty}\frac{\cos nx}{n^2} - \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx$

Putting x=0 in the above expression we get,

$$f(0) = 0 = \frac{\pi}{4} - \frac{2}{\pi}\sum_{n=1,3,5,..,}^{\infty}\frac{1}{n^2}$$

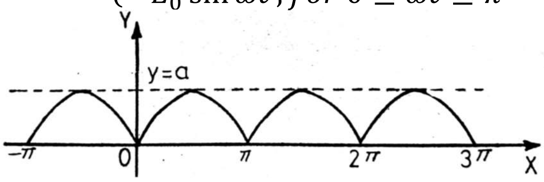
Therefore, $\left\{\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right\} = \frac{\pi^2}{8}$.

5. **Half-wave rectifier:**
   Find the Fourier series of the periodic function

$$f(t) = \begin{cases} 0, for -\pi \le \omega t \le 0 \\ E_0 \sin\omega t\,, for\ 0 \le \omega t \le \pi \end{cases}$$

6. **Full-wave rectifier:**
   Find the Fourier series of the periodic function

$$f(t) = \begin{cases} -E_0 \sin\omega t\,, for -\pi \le \omega t \le 0 \\ E_0 \sin\omega t\,, for\ 0 \le \omega t \le \pi \end{cases}$$



Here the function is even function, therefore $b_n = 0$.

Now $f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega t$

Where, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)d(\omega t) = \frac{1}{\pi}\left\{\int_{-\pi}^{0} -E_0\sin\omega t\, d(\omega t) + \int_0^{\pi} E_0 \sin\omega t\, d(\omega t)\right\}$

$$= \frac{E_0}{\pi}\{[\cos\omega t]_{-\pi}^{0} + [\cos\omega t]_{\pi}^{0}\} = \frac{E_0}{\pi}\{(1+1)+(1+1)\} = \frac{4E_0}{\pi}.$$

And $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos n\omega t\, d(\omega t)$

$$= \frac{1}{\pi}\left\{\int_{-\pi}^{0} -E_0 \sin\omega t\ \cos n\omega t\, d(\omega t) + \int_{0}^{\pi} E_0 \sin\omega t\ \cos n\omega t\, d(\omega t)\right\}$$

$$= \frac{E_0}{2\pi}\left\{\int_{-\pi}^{0} -[\sin(1+n)\omega t + \sin(1-n)\,\omega t]\, d(\omega t)\right.$$

$$\left. + \int_{0}^{\pi} [\sin(1+n)\,\omega t + \sin(1-n)\,\omega t]\ d(\omega t)\right\}$$

$$= \frac{E_0}{2\pi}\left\{\left[\frac{\cos(1+n)\omega t}{(1+n)} + \frac{\cos(1-n)\omega t}{(1-n)}\right]_{-\pi}^{0} + \left[\frac{\cos(1+n)\omega t}{(1+n)} + \frac{\cos(1-n)\omega t}{(1-n)}\right]_{\pi}^{0}\right\}$$

$$= \frac{E_0}{2\pi}\left\{\left[\frac{1}{1+n} + \frac{1}{1-n} - \frac{(-1)^{n+1}}{1+n} - \frac{(-1)^{1-n}}{1-n}\right]\right.$$

$$\left. + \left[\frac{1}{1+n} + \frac{1}{1-n} - \frac{(-1)^{n+1}}{1+n} - \frac{(-1)^{1-n}}{1-n}\right]\right\}$$

$$= \frac{E_0}{\pi}\left\{\frac{1}{1+n} + \frac{1}{1-n} + \frac{(-1)^n}{1+n} + \frac{(-1)^n}{1-n}\right\}$$

$$= \frac{E_0}{\pi}\left\{\frac{1-n+n+1}{1-n^2} + \frac{(-1)^n - n(-1)^n + n(-1)^n + (-1)^n}{1-n^2}\right\}$$

$$= \frac{E_0}{\pi}\left\{\frac{2}{1-n^2} + \frac{2(-1)^n}{1-n^2}\right\} = \frac{2E_0}{\pi}\left\{\frac{1+(-1)^n}{1-n^2}\right\}$$

Therefore, $a_n = \begin{cases} 0\ , for\ n\ is\ odd \\ \frac{4E_0}{\pi}\left\{\frac{1}{1-n^2}\right\}, for\ n\ is\ even \end{cases}$

Hence, $f(t) = \frac{1}{2} \times \frac{4E_0}{\pi} + \frac{4E_0}{\pi}\sum_{n=2,4,6,..}^{\infty} \frac{1}{1-n^2} \cos n\omega t$

$$f(t) = \frac{4E_0}{\pi}\left[\frac{1}{2} - \frac{\cos 2\omega t}{1\times 3} - \frac{\cos 4\omega t}{3\times 5} - \frac{\cos 6\omega t}{5\times 7} - \cdots\right]$$

L-4

Parseval's identity:

This theorem states that if a function *f*(*x*) has the Fourier series expansion within limit -π to +π as $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

then the average of the square of the function over the period will be equal to sum of the squares of the coefficients a_0, a_n, and b_n.

$$i.e. \qquad \frac{1}{\pi}\int_{-\pi}^{\pi}[f(x)]^2\, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

This result is known as Parseval's theorem.

Proof: Let function $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

$$= \frac{a_0}{2} + (a_1 \cos x + a_2 \cos 2x + \cdots) + (b_1 \sin x + b_2 \sin 2x + \cdots)$$

Therefore,

$$\frac{1}{\pi}\int_{-\pi}^{\pi}[f(x)]^2\,dx$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{a_0^2}{4}\,dx+a_1^2\frac{1}{\pi}\int_{-\pi}^{\pi}\cos^2 x\,dx+a_2^2\frac{1}{\pi}\int_{-\pi}^{\pi}\cos^2 2x\,dx+\cdots$$

$$+b_1^2\frac{1}{\pi}\int_{-\pi}^{\pi}\sin^2 x\,dx+b_2^2\frac{1}{\pi}\int_{-\pi}^{\pi}\sin^2 2x\,dx+\cdots$$

Since other terms are zero due to orthogonality relations

$\int_{-\pi}^{\pi}\sin mx\sin nx\,dx=0, for\ m\neq n\ and\ \int_{-\pi}^{\pi}\cos mx\cos nx\,dx=0\,, for\ m\neq n$

Therefore, $\frac{1}{\pi}\int_{-\pi}^{\pi}[f(x)]^2\,dx=\frac{a_0^2}{2}+(a_1^2+a_2^2+\cdots)+(b_1^2+b_2^2+\cdots)=\frac{a_0^2}{2}+\sum_{n=1}^{\infty}(a_n^2+b_n^2)$

Note: Depending on the given intervals Parseval's identity can be represented as follows,

1. If $0\leq x\leq 2l$ then $\frac{1}{l}\int_0^{2l}[f(x)]^2\,dx=\frac{a_0^2}{2}+\sum_{n=1}^{\infty}(a_n^2+b_n^2)$
2. If $0\leq x\leq l$, and the function is expanded as Half range cosine series, then
$$\frac{2}{l}\int_0^{l}[f(x)]^2\,dx=\frac{a_0^2}{2}+\sum_{n=1}^{\infty}a_n^2$$
3. If $0\leq x\leq l$, and the function is expanded as Half range sine series, then
$$\frac{2}{l}\int_0^{l}[f(x)]^2\,dx=\frac{a_0^2}{2}+\sum_{n=1}^{\infty}b_n^2$$

Example 1: **Show that** $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\frac{\pi^2}{8}$

Ans: Let us consider the half range sine series for the function $f(x)=1, for\ 0\leq x\leq\pi$, such that $f(x)=\sum_{n=1}^{\infty}b_n\sin nx$.

$\therefore b_n=\frac{2}{\pi}\int_0^{\pi}f(x)\sin nx\,dx=\frac{2}{\pi}\int_0^{\pi}1.\sin nx\,dx=\frac{2}{\pi}\left|\frac{-\cos nx}{n}\right|_0^{\pi}=-\frac{2}{n\pi}[(-1)^n-1]$

$$\therefore b_n=\begin{cases}\frac{4}{n\pi}\,, & if\ n\ is\ odd\\ 0\,, & if\ n\ is\ even\end{cases}$$

Now, using Parseval's identity we get

$$\frac{2}{\pi}\int_0^{\pi}[f(x)]^2\,dx=\sum_{n=1}^{\infty}b_n^2$$

$$\int_0^{\pi}1^2\,dx=\frac{\pi}{2}\left\{\left(\frac{4}{\pi}\right)^2+\left(\frac{4}{3\pi}\right)^2+\left(\frac{4}{5\pi}\right)^2+\cdots\right\}$$

$$\pi=\frac{\pi}{2}\times\frac{16}{\pi^2}\left\{\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right\}$$

Therefore, $\left\{\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right\}=\frac{\pi^2}{8}$

Hence, $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\frac{\pi^2}{8}$

Example 2: **Show that** $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}=\frac{\pi^4}{96}$

Ans: Let us consider the half range cosine series for the function $f(x) = x, for\ 0 \le x \le \pi$, such that $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$.

Here, $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx = \frac{2}{\pi}\int_0^{\pi} x\,dx = \pi$, since $f(x) = x$ is even function in extended interval $-\pi \le x \le \pi$.

And $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx = \frac{2}{\pi}\left\{\left|x\frac{\sin nx}{n}\right|_0^{\pi} + \left|\frac{\cos nx}{n^2}\right|_0^{\pi}\right\}$

$$= \frac{2}{\pi n^2}\{(-1)^n - 1\}$$

$$\therefore a_n = \begin{cases} 0, if\ n\ is\ even \\ \frac{-4}{\pi n^2},\ if\ n\ is\ odd \end{cases}$$

Now, using Parseval's identity we get

$$\frac{2}{\pi}\int_0^{\pi}[f(x)]^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2$$

$$\frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{\pi^2}{2} + \sum_{n=1,3,5,\dots}^{\infty}\frac{16}{\pi^2 n^4}$$

$$\frac{2}{\pi}\times\frac{\pi^3}{3} = \frac{\pi^2}{2} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}$$

Hence, $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4} = \frac{\pi^4}{96}$

Example 3: Show that $\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$

Ans: Let us consider the function $f(x) = x^2, for - \pi \le x \le \pi$.

Since this function is even function, we can expand it as $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

Here, $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx = \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2}{3}\pi^2$

And $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\cos nx\,dx = \frac{2}{\pi}\left\{\left|x^2\frac{\sin nx}{n}\right|_0^{\pi} - \int_0^{\pi} 2x\frac{\sin nx}{n}\,dx\right\}$

$$= \frac{2}{\pi}\left\{\left|2x\frac{\cos nx}{n^2}\right|_0^{\pi} + 2\int_0^{\pi}\frac{\cos nx}{n^2}\,dx\right\} = \frac{2}{\pi}\left\{\frac{2\pi(-1)^n}{n^2} + 0\right\} = \frac{4(-1)^n}{n^2}$$

Now, using Parseval's identity we get

$$\frac{2}{\pi}\int_0^{\pi}[f(x)]^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2$$

$$\frac{2}{\pi}\int_0^{\pi} x^4\,dx = \frac{1}{2}\times\frac{4\pi^4}{9} + \sum_{n=1}^{\infty}\frac{16}{n^4}$$

$$\frac{2}{\pi}\times\frac{\pi^5}{5} = \frac{2\pi^4}{9} + 16\sum_{n=1}^{\infty}\frac{1}{n^4}$$

Hence, $\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$.