VU CBCS Semester-IV 2019: Planck's quantum, Planck's constant and light as a collection of photons; Blackbody Radiation: Quantum theory of Light; Photo-electric effect and Compton scattering. De Broglie wavelength and matter waves; Davisson-Germer experiment. Wave description of particles by wave packets. Group and Phase velocities and relation between them. Two-Slit experiment with electrons. Probability. Wave amplitude and wave functions.

Historical Note

1. Situation towards the end of the 19th century and the beginning of the 20th century

1.1 Advancement in Physics:

- *Classical mechanics:*

  Newtonian Mechanics (Principia 1687-1713-1726; Sir Isaac Newton, English, 1643-1727) > Lagrangian Formulation (1750s) (Joseph-Louis Lagrange, Italian-French, 1736-1813) > Hamiltonian Formulation (1833) (William Rowan Hamilton, Irish, 1805-1865)

- *Electrodynamics:*

  Maxwell's (James Clerk Maxwell, Scottish, 1831-1879) Equations of Electromagnetic waves [1861]. [In present form by Oliver Heaviside (English), Josiah W Gibbs (American), Heinrich Hertz (German, 1857-1894) in 1884]

  Lorentz (Dutch, 1853-1928) Force Equation [1861 Maxwell > 1881J. J. Thomson[1] (English) > 1884 Heaviside (English) > 1895 Lorentz]

- *Thermodynamics:*

  Carnot Theorem (1824) [Nicolas Leonard Carnot, French 1796-1832], Maxwell-Boltzmann (Ludwig Eduard Boltzmann, German, 1844-1906) Statistics (1868).

1.2. Major unsolved Questions:

- Energy Distribution $[u(\nu)d\nu \;\; or \;\; u(\lambda)d\lambda]$ of Blackbody Radiation.
- Photo electric effect: Experiment by German Physicist Hertz in 1887.
- Stability of Rutherford's (New Zealand-born British) atom (1911 Gold Foil Expt.).
- Existence of aether: Michelson (American) –Morley (American) Experiment (1887, at Western Reserve University, Ohio).
- Atomic Spectra: Balmer (Swiss Mathematician) series:

  Balmer formula (1885) $\lambda = B\left(\frac{n^2}{n^2-m^2}\right) = B\left(\frac{n^2}{n^2-2^2}\right)$

  Rydberg (Swedish Physicist) Formula (1888) $\bar{\nu} = \frac{1}{\lambda} = \frac{4}{B}\left(\frac{1}{2^2} - \frac{1}{n^2}\right) = R_H\left(\frac{1}{2^2} - \frac{1}{n^2}\right)$

  With $R_H = 1.09737309 \times 10^7 \; m^{-1}$.

  Anomalous Zeeman (Dutch) effect, Fine structures and other observation in atomic spectroscopy.

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[1] William Thomson is a different scientist having other name Lord Kelvin, Scots-Irish, 1824-1907.

1.3 End of the era of triumph of classical physics.

- 1900 Planck (German) distribution formula of Blackbody Radiation > Assumption of **radiation quanta of energy $h\nu$**, where ν is the frequency of radiation and h is a constant determined by Planck to fit the experimental distribution curve and is called Planck constant.

- 1905 Einstein (German Jewish)> Photo electric effect > Particle nature of light/radiation > Photon.

- 1905 Einstein Special Theory of Relativity > Non existence of aether; dependence of mass, length and time on velocity.

- 1913 Niels Bohr (Danish) > Model of Hydrogen Atom> quantisation of angular momentum of atomic electron > explanation of atomic stability, Balmer formula, atomic spectroscopy.

- 1923 New observations: Compton (American) Effect > recoil of electron which scatters X-ray. X-ray photon has momentum $h\nu/c$ > Radiation has particle nature.

- 1923 de Broglie (French) hypothesis: Electron and all matter have wave nature.

- 1925 Heisenberg (German): Matrix Formulation.

- 1926 Schrodinger (German): Schrodinger Equation > Wave mechanics.

- 1927 Heisenberg: Uncertainty Relation (Earle Hesse Kennard in late 1927 & Hermann Wey in 1928 gave the formal relation involving standard deviations as uncertainties: $\sigma_x \sigma_p \geq \hbar/2$).

- 1923-27 Davisson (American) and Germer (American) experiment and explanation > Diffraction of electrons > Confirmation of wave nature of electrons i.e. de Broglie hypothesis.

- 1927 Max Born (German Jewish) probabilistic interpretation of wave mechanics > $P(x,t)dx = \int_{x_1}^{x_1} |\psi(x,t)|^2 dx$, where x_1 and x_2 are the limits within which the particle exists. In 3D $P(\vec{r},t)dx = \iiint |\psi(\vec{r},t)|^2 d\tau$, where the integration is over the region of space in which the particle exists.

- 1928 Paul Dirac (English): Relativistic Quantum Mechanics > Prediction of Positron > Proof in 1932;

- 1939 bra ket notation by Dirac: Both Heisenberg's matrix formulation and Schrodinger's wave mechanics formulation can be handled with this.

1.3.1 Blackbody Radiation

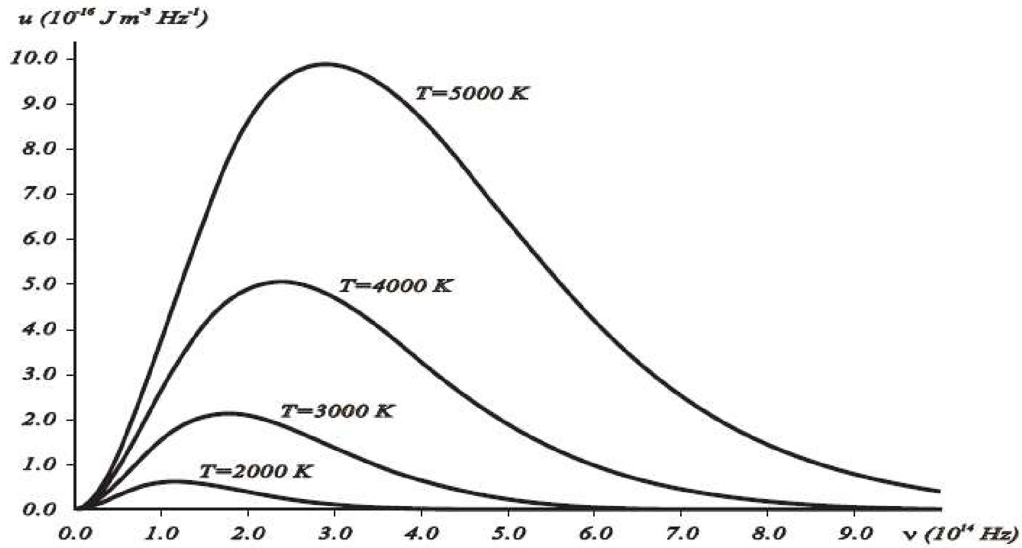

Figure 1.1 Spectral energy density $u(\nu, T)$ of blackbody radiation at different temperatures as a function of the frequency ν.

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1879 J. Stefan (Carinthian Slovene) established from Tyndall's experimental results of IR emissions by platinum filament and its colour:

Per unit area of the surface of a radiating solid at absolute temperature T radiates normally (perpendicularly) a power (or energy per second)-

$P = a\sigma T^4$ ……….. (1)

where $\sigma = 5.670367 \times 10^{-8}$ Wm^2 K^{-4} is called Stefan's constant; a is a coefficient ≤ 1. For ideal blackbody $a = 1$. Equation (1) is called Stefan's law or Stefan-Boltzmann Law.

In 1884 a theoretical derivation of the law was done by Boltzmann (German).

Up to a temperature 1535 K this law accurately matches experimental observations. But at higher temperature deviation from experimental results are observed.

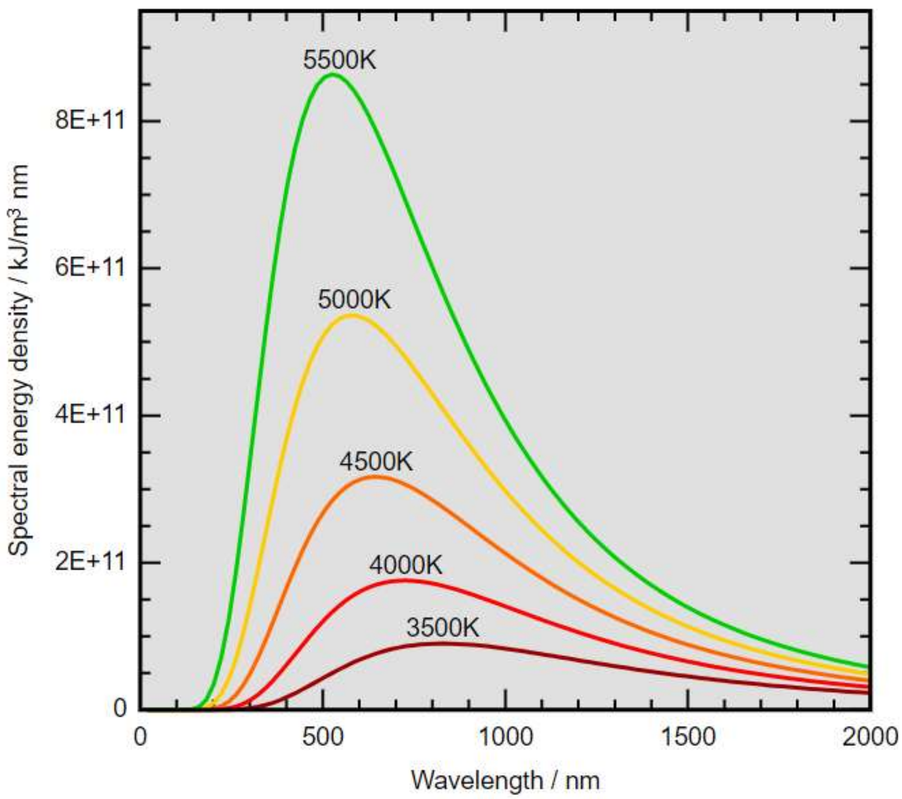
1893 Wien (Wilhelm Wien, 1864-1928, German Physicist) displacement law:

$$\lambda_{max} T = constant \approx 2900\ \mu m.K$$

1894 Wien energy density distribution:

Wien proposed (from thermodynamic consideration) that, Stefan-Boltzmann law and Wien displacement law can be derived if the energy density of blackbody radiation at temperature T per unit wavelength at λ i.e. $u(\lambda, T)$ must be given by a relation:

$u(\lambda, T)d\lambda = \frac{a}{\lambda^5} f(\lambda T)d\lambda$, where $f(\lambda T)$ is any function of λT.



From some arbitrary assumptions regarding mechanisms of emission he proposed that $f(\lambda T) = ae^{-b/\lambda T}$ and so

$u(\lambda, T)d\lambda = \frac{a}{\lambda^5} e^{-b/\lambda T} d\lambda.$

In terms of frequency $u(\nu, T)d\nu = A\nu^3 e^{-\beta\nu/T} d\nu.$

Unit of $u(\nu, T)$ is $Jm^{-3}Hz^{-1}$ or $Jm^{-3}s$ and unit of $u(\lambda, T)$ is Jm^{-4}.

Constants a, b or A, β were determined to fit these equations to experimental curves.

Failure of Wien distribution: Wien's distribution satisfies experimental curve at lower wavelengths or higher frequencies but fails to explain them at higher wavelengths or lower frequencies. ***[In those days producing radiation of higher frequencies or lower wavelengths was not easy.]*** Thus Wien's distribution was insufficient to satisfy observations.

1900 Rayleigh's (Lord Rayleigh, 1842-1919, British physicist) energy density distribution:

Rayleigh assumed that in the cavity of a blackbody radiation exists in the form of electromagnetic standing waves with their nodes at the walls of the cavity. Density of states (or vibrational modes) of these standing waves i.e. number of states (or modes) per unit volume per frequency range of such standing waves is equal to $\frac{8\pi\nu^2}{c^3}$.

The electromagnetic standing waves are excited by the linear oscillation of the tiny electric dipoles of atomic or molecular dimension in the walls of the cavity. The energy of an oscillating dipole can have any value between 0 & ∞ i.e. the energy spectrum of an oscillator is continuous. At temperature T, the number of electric dipoles having energy E is given by M-B statistics, i.e. $N(E) = N_0 e^{-E/kT}$, where N_0 is the number of oscillators with zero energy and $k\ (= 1.38\, JK^{-1})$ is Boltzmann constant. Then it can be shown that at temperature T, the average energy of the oscillators in the walls is $\langle E \rangle = kT$.

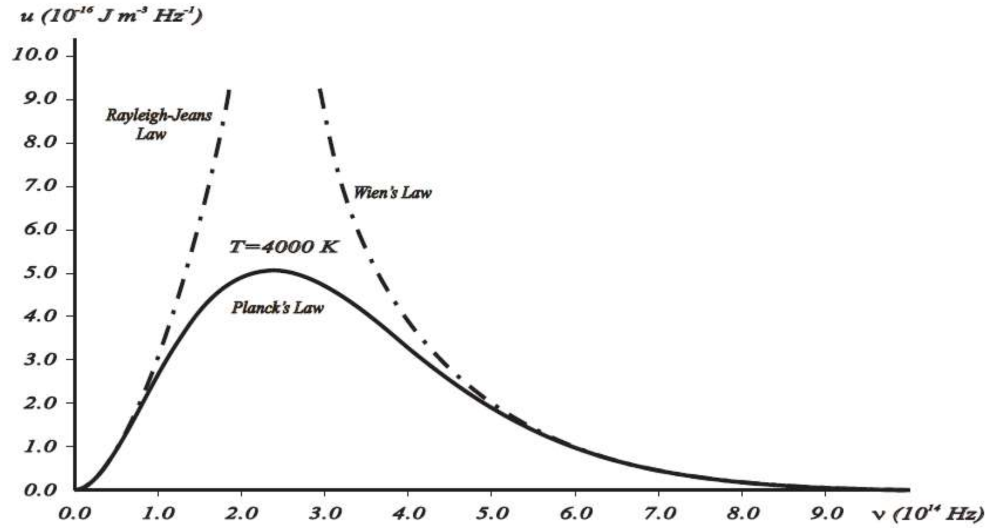


Figure 1.2 Comparison of various spectral densities: while the Planck and experimental distributions match perfectly (solid curve), the Rayleigh–Jeans and the Wien distributions (dotted curves) agree only partially with the experimental distribution.

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In equilibrium the energy distribution of the standing waves is same as the energy distribution of the oscillators over the frequency range. Therefore the average energy of the vibrational modes of the standing waves will also be $\langle E \rangle = kT$.

So, according to Rayleigh, the energy density distribution is given by:

$u(\nu, T)d\nu = n(\nu, T)\langle E \rangle d\nu = \frac{8\pi\nu^2}{c^3} kTd\nu;$

Or, in terms of wavelength $u(\lambda, T)d\lambda = \frac{8\pi}{\lambda^4} kTd\lambda.$

Density of states of vibrations in a cubical cavity of side L filled with a continuous elastic medium:

3D wave equation: $\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}$ *…………(A)*

Standing wave solution: $\psi(x, y, z, t) = A sin\left(\frac{n_x\pi x}{L}\right) sin\left(\frac{n_y\pi y}{L}\right) sin\left(\frac{n_z\pi z}{L}\right) cos(2\pi\nu t).$ *……..(B)*

n_x, n_y, n_z *are integers* ≥ 1. *A vibrational mode is determined by the set of integers* (n_x, n_y, n_z).

What is the number of such modes within frequency range $\boldsymbol{\nu}$ ***to*** $\boldsymbol{\nu + d\nu}$ ***?***

Consider a coordinate system in which coordinates are the positive integers and zero. This is the first octant of the 3D integer space. Each point in this space will be at unit distance from its nearest neighbours; in other words each point will share unit volume of this space.

Substituting (B) in (A) and simplifying: $n_x^2 + n_y^2 + n_z^2 = \frac{4L^2\nu^2}{c^2} = R^2\ (say)$.

Above equation represents the portion of a sphere of radius $R = \frac{2L}{c}$ *in the first octant of the integer space. In this space a spherical shell between radii* R *and* $R + dR$ *corresponds to the frequency range* ν *to* $\nu + d\nu$. *Volume of such a shell in integer space is:*

$$\frac{1}{8} \times 4\pi R^2 dR = \frac{1}{2}\pi R^2 dR = \frac{1}{2}\pi \times \frac{4L^2\nu^2}{c^2} \times \frac{2L}{c} d\nu = \frac{4\pi L^3\nu^2}{c^3} d\nu$$

The number of coordinate points (n_x, n_y, n_z) *in this shell will also be* $\frac{1}{2}\pi R^2 dR$, *since each point shares unit volume in integer space. But this is equal to the number of vibrational modes in the frequency range* ν *to* $\nu + d\nu$. *Thus the number of modes in the frequency range* ν *to* $\nu + d\nu$ *per unit volume of the cavity will be*

$$\frac{1}{L^3} \times \frac{4\pi L^3\nu^2}{c^3} d\nu = \frac{4\pi\nu^2}{c^3} d\nu$$

Now unpolarised electromagnetic waves contains two types of circularly polarised waves with the plane of polarisation rotating in clockwise and anticlockwise sense. Now two modes of the electromagnetic standing waves with plane of polarisation rotating in opposite sense but identical in all other respect will have same set of (n_x, n_y, n_z), *i.e. each point in the integer space represents two states. Therefore number of states per unit volume of the cavity in the frequency range* ν *to* $\nu + d\nu$ *will be*

$$n(\nu)d\nu = 2 \times \frac{4\pi\nu^2}{c^3} d\nu = \frac{8\pi\nu^2}{c^3} d\nu.$$

$$n(\nu) = \frac{8\pi\nu^2}{c^3} \ldots\ldots\ldots\ldots (C)$$

is called the density of states.

Though for simplicity here we derive this result for a cubical medium, it is applicable to any shape.

Average energy per vibrational mode:

According to M-B statics the number of oscillators (or vibrational states in this case) having energy E *at temperature* T *is* $N_E = N_0 e^{-E/kT}$, *where* N_0 *is the number of oscillators in the state of zero energy (ground state) and* k *is Boltzmann constant (*$k = 1.38 \times 10^{-23} JK^{-1}$*). Therefore for a continuous energy distribution:*

$$\langle E \rangle = \frac{\int_0^\infty E N_0 e^{-E/kT} dE}{\int_0^\infty N_0 e^{-E/kT} dE} = \frac{(kT)^2 \int_0^\infty (E/kT) e^{-E/kT} d(E/kT)}{(kT) \int_0^\infty e^{-E/kT} d(E/kT)} = kT \frac{\Gamma(2)}{\Gamma(1)} = kT \quad \ldots\ldots \ (D).$$

Rayleigh formula satisfies experimental curves at higher wavelengths or lower frequencies but deviates badly from the experimental curves towards lower wavelengths or higher frequencies i.e. towards ultraviolet region of the spectrum. This failure is known as ultraviolet catastrophe.

Particle nature of wave

1905 Planck blackbody radiation formula:

Planck's quantisation rule / Planck's quantum hypothesis / Planck's postulate: According to classical mechanics, a harmonic oscillator of frequency ν can have any amount of energy $E\ [= 4\pi^2 ma^2\nu^2]$, which is proportional to the square of its amplitude a. And it can have any energy between 0 & ∞. But to explain blackbody radiation Planck made the following revolutionary assumptions:

i) An oscillator frequency ν in the wall of the blackbody can have only discrete energies given by $\varepsilon_n = nh\nu$, where $n = 0,1,2 \ldots ..$ and h is a constant, which was determined by him to fit his formula with the experimental distribution curves of blackbody radiation.

ii) When an oscillator of frequency ν absorbs or emits energy in the form of radiation its energy can change only in the steps of $h\nu$. Since the radiation absorbed or emitted by an oscillator have same frequency as that of the oscillator therefore it follows from Planck assumptions that an oscillator of frequency ν can absorb or emit radiation of frequency ν and this emitted or absorbed radiation can have only an amount of energy $h\nu$, no less no more.

Regarding the nature and density of states of the radiation inside the cavity of blackbody, Planck's assumption was same as that of Rayleigh.

Average energy of the oscillators of frequency ν in the walls of the blackbody:

Oscillators of frequency ν have discrete energies $\varepsilon_n = nh\nu, n = 0,1,2\ldots$. According to M-B statics the number of such oscillators at temperature T is $N_n = N_0 e^{-E_n/kT} = N_0 e^{-nh\nu/kT}$. Therefore the average energy of the oscillators of frequency ν will be:

$$\langle\varepsilon_\nu\rangle = \frac{\sum_{n=0}^{\infty} \varepsilon_n N_0 e^{-\varepsilon_n/kT}}{\sum_{n=0}^{\infty} N_0 e^{-\varepsilon_n/kT}} = \frac{\sum_{n=0}^{\infty} \varepsilon_n e^{-\varepsilon_n/kT}}{\sum_{n=0}^{\infty} e^{-\varepsilon_n/kT}}.$$

Note that the integrations of equation (D) have been replaced here by summations since in this case discrete energies are assumed for the oscillators in place of continuous energies of the oscillators in Rayleigh theory.

Now

$$\langle\varepsilon_\nu\rangle = \frac{\sum_{n=0}^{\infty} \varepsilon_n e^{-\varepsilon_n/kT}}{\sum_{n=0}^{\infty} e^{-\varepsilon_n/kT}} = \frac{\sum_{n=0}^{\infty} nh\nu e^{-nh\nu/kT}}{\sum_{n=0}^{\infty} e^{-nh\nu/kT}} = \frac{h\nu e^{-h\nu/kT} + 2h\nu e^{-2h\nu/kT} + 3h\nu e^{-3h\nu/k} + \cdots}{1 + e^{-h\nu/kT} + e^{-2h\nu/kT} + e^{-3h\nu/kT} + \cdots}$$

$$= \frac{h\nu x(1 + 2x + 3x^2 + 4x^3 \ldots)}{1 + x + x^2 + x^3 \ldots} \quad (where\ x = e^{-h\nu/kT})$$

$$= h\nu x \frac{(1-x)^{-2}}{(1-x)^{-1}} = h\nu x \frac{1}{1-x} = \frac{h\nu}{\frac{1}{x} - 1} = \frac{h\nu}{e^{h\nu/kT} - 1}.$$

Since the radiation in the cavity of the blackbody is in equilibrium with the oscillators in the wall so this above expression will also give the average energy of the vibrational modes of the standing waves in the cavity.

The number of vibrational modes or states per unit volume of the cavity in the frequency range ν *to* $\nu + d\nu$ can be determined as before and is given by:

$n(\nu)d\nu = \frac{8\pi\nu^2}{c^3}d\nu.$

Thus the energy distribution of the radiation is given by: $u(\nu)d\nu = \langle\varepsilon_\nu\rangle n(\nu)d\nu = \frac{8\pi\nu^2}{c^3}\cdot\frac{h\nu}{e^{h\nu/kT}-1}d\nu.$

$$u(\nu)d\nu = \frac{8\pi h\nu^3}{c^3}\cdot\frac{1}{e^{h\nu/kT}-1}d\nu \dots\dots\dots\dots (1)$$

In terms of wavelength:

$$u(\lambda)d\lambda = \frac{8\pi hc}{\lambda^5}\cdot\frac{1}{e^{hc/\lambda kT}-1}d\lambda \dots\dots\dots\dots\dots (2)$$

Note that the discreteness of vibrational modes indexed by (n_x, n_y, n_z) arises here from purely classical considerations. But the discreteness of possible energies of an oscillator is due to the assumptions of a new type, called Planck's quantum conditions.

Fitting his equation with experimental curves Planck determined the value of h. Its value is $6.626 \times 10^{-34}\ J.s$ and it is a universal constant of immense importance as was revealed later years with the advancement of quantum mechanics.

Derivations from Planck's law:

Stefan-Boltzmann law: Total energy (in all wavelength range) per unit volume of the cavity of a black body is

$$u = \int_0^\infty u(\nu)d\nu = \frac{8\pi h}{c^3}\int_0^\infty \frac{\nu^3}{e^{h\nu/kT}-1}d\nu = \frac{8\pi h}{c^3}\left(\frac{kT}{h}\right)^4\int_0^\infty \frac{x^3}{e^x-1}dx \quad [where\ x = h\nu/kT]$$

$$= \frac{8\pi h}{c^3}\left(\frac{kT}{h}\right)^4\int_0^\infty x^3e^{-x}(1-e^{-x})^{-1}dx = \frac{8\pi h}{c^3}\left(\frac{kT}{h}\right)^4\int_0^\infty x^3e^{-x}(1+e^{-x}+e^{-2x}+e^{-3x}+\cdots)dx$$

$$= \frac{8\pi h}{c^3}\left(\frac{kT}{h}\right)^4\int_0^\infty x^3(e^{-x}+e^{-2x}+e^{-3x}+e^{-4x}+\cdots)dx = \frac{8\pi h}{c^3}\left(\frac{kT}{h}\right)^4\sum_{p=1}^\infty\int_0^\infty x^3e^{-px}dx$$

$$= \frac{8\pi h}{c^3}\left(\frac{kT}{h}\right)^4\sum_{p=1}^\infty\frac{1}{p^4}\int_0^\infty (px)^{4-1}e^{-px}d(px) = \frac{8\pi h}{c^3}\left(\frac{kT}{h}\right)^4\sum_{p=1}^\infty\frac{1}{p^4}\Gamma(4) = \frac{8\pi h}{c^3}\left(\frac{kT}{h}\right)^4\sum_{p=1}^\infty\frac{3!}{p^4}$$

$$= \frac{48\pi h}{c^3}\left(\frac{kT}{h}\right)^4\frac{\pi^4}{90} = \frac{8\pi^5k^4}{15c^3h^3}T^4$$

It can be shown that the energy radiated normally per unit area from a blackbody is $E = u\frac{c}{4}$. Thus:

$$E = \frac{8\pi^5 k^4}{15c^3h^3} \cdot \frac{c}{4} T^4 = \left(\frac{2\pi^5 k^4}{15c^2h^3}\right) T^4 \ \ or, \ \ E \propto T^4.$$

Wien displacement law:

$$u(\lambda) = \frac{8\pi hc}{\lambda^5} \cdot \frac{1}{e^{hc/\lambda kT} - 1} = \frac{8\pi hc}{z(\lambda)}, say. \quad Where \quad z(\lambda) = \lambda^5\left(e^{hc/\lambda kT} - 1\right).$$

The value of λ for which $u(\lambda)$ is maximum is obtained from the condition:

$$\left.\frac{\partial u(\lambda)}{\partial \lambda}\right|_{\lambda=\lambda_{max}} = 0. \quad \text{This is equivalent to:} \quad \left.\frac{\partial z(\lambda)}{\partial \lambda}\right|_{\lambda=\lambda_{max}} = 0$$

$$Or, \quad 5{\lambda_{max}}^4 \cdot \left(e^{hc/\lambda_{max}kT} - 1\right) - {\lambda_{max}}^5 \cdot \frac{hc}{{\lambda_{max}}^2 kT} e^{hc/\lambda_{max}kT} = 0;$$

$$Or, 1 - e^{-hc/\lambda_{max}kT} = \frac{hc}{5\lambda_{max}kT};$$

Or, with $\frac{hc}{\lambda_{max}kT} = x$, this equation can be written as: $1 - e^{-x} = \frac{x}{5}$

This equation can not be solved analytically, but can be solved numerically. Or by writing as a pair of equation: $\boldsymbol{y = 1 - e^{-x}}$; and $\boldsymbol{y = \frac{x}{5}}$ it can also be solved graphically. The curves represented by these equations intersect for $x \approx 4.965$.

$$Thus \ \frac{hc}{\lambda_{max}kT} \approx 4.965;$$

$$\Rightarrow \lambda_{max}T = \frac{hc}{4.965k} = \frac{6.626 \times 10^{-34} \times 3 \times 10^8}{4.965 \times 1.38 \times 10^{-23}} \approx 0.0029\, m.K = constant$$

Wien and Rayleigh-Jeans distribution law:

Planck's law: $u(\nu)d\nu = \frac{8\pi h\nu^3}{c^3} \cdot \frac{1}{e^{h\nu/kT} - 1} d\nu$

At high frequencies $e^{h\nu/kT} - 1 \approx e^{h\nu/kT}$

$$So \ u(\nu)d\nu = \frac{8\pi h\nu^3}{c^3} \cdot e^{-h\nu/kT} d\nu = A\nu^3 e^{-\beta\nu/T} d\nu \quad \text{[Wien's distribution law.]}$$

$$where \ A = \frac{8\pi h}{c^3} \quad \text{and} \quad \beta = h/k \ are \ constants.$$

At low frequencies $e^{h\nu/kT} - 1 = \left(1 + h\nu/kT + \frac{(h\nu/kT)^2}{2!} + \frac{(h\nu/kT)^3}{3!} \ldots\right) - 1 \approx h\nu/kT$

So $u(\nu)d\nu = \frac{8\pi h\nu^3}{c^3} \cdot \frac{1}{h\nu/kT} d\nu = \frac{8\pi \nu^2 kT}{c^3} d\nu$ [Rayleigh-Jeans law].

Problems:

JAM 2020

Q.46 Consider two spherical perfect blackbodies with radii R_1 and R_2 at temperatures $T_1 = 1000$ K and $T_2 = 2000$ K, respectively. They both emit radiation of power 1 kW. The ratio of their radii, R_1/R_2 is given by _____________.

Ans.: Solve yourself.

JAM 2017

Q.23 In the radiation emitted by a black body, the ratio of the spectral densities at frequencies 2ν and ν will vary with ν as:

(A) $\left[e^{h\nu/k_BT} - 1\right]^{-1}$ (C) $\left[e^{h\nu/k_BT} - 1\right]$

(B) $\left[e^{h\nu/k_BT} + 1\right]^{-1}$ (D) $\left[e^{h\nu/k_BT} + 1\right]$

Ans.: $u(\nu) = \frac{8\pi h\nu^3}{c^3} \cdot \frac{1}{e^{h\nu/kT}-1}$

$$\frac{u(2\nu)}{u(\nu)} = \frac{8\pi h(2\nu)^3}{c^3} \cdot \frac{1}{e^{2h\nu/kT}-1} \cdot \frac{c^3}{8\pi h\nu^3} \cdot \frac{e^{h\nu/kT}-1}{1} = 8 \cdot \frac{e^{h\nu/kT}-1}{e^{2h\nu/kT}-1} = 8 \cdot \frac{e^{h\nu/kT}-1}{(e^{h\nu/kT}-1)(e^{h\nu/kT}+1)}$$

$$= 8 \cdot \frac{1}{e^{h\nu/kT}+1} = 8 \cdot \left(e^{h\nu/kT} + 1\right)^{-1}$$

$$\frac{u(2\nu)}{u(\nu)} \infty \left(e^{h\nu/kT} + 1\right)^{-1} \quad \Rightarrow \quad (B).$$

JAM 2014

Q.42 According to Wien's theory of black body radiation, the spectral energy density in a blackbody cavity at temperature T is given as

$$u_T(\lambda)\,d\lambda = \frac{\alpha}{c^3\lambda^5} e^{-\beta/\lambda T}\,d\lambda$$

where α and β are constants and c is the speed of light. Further, the intensity of radiation coming out of the cavity is $\frac{u_T c}{4}$, where $u_T = \int_0^\infty u_T(\lambda)\,d\lambda$ is the total energy density of radiation. Given that Stefan-Boltzmann constant $\sigma = 5.67\times10^{-8}$ Wm^{-2}K^{-4} and $\lambda_{max}T = 2.90\times10^{-3}$ m.K, find the values of α and β. The value of integral $\int_0^\infty x^3 e^{-x}\,dx = 6$.

Ans.: $\left[\frac{\partial}{\partial\lambda}\left(u_T(\lambda)\right)\right]_{\lambda_{max}} = 0$

So, from $u_T(\lambda) = \frac{\alpha}{c^3\lambda^5} e^{-\beta/\lambda T}$, we have $\left[-\frac{5\alpha}{c^3\lambda^6} e^{-\beta/\lambda T} + \frac{\alpha\beta}{c^3\lambda^7 T} e^{-\beta/\lambda T}\right]_{\lambda_{max}} = 0$

$\Rightarrow -5 + \frac{\beta}{\lambda_{max}T} = 0$ $\Rightarrow \boldsymbol{\beta} = 5\lambda_{max}T = 5 \times 2.9 \times 10^{-3} = \mathbf{0.0145\ m.K}$

$\sigma T^4 = E$

$$= \frac{u_T c}{4} = \frac{c}{4}\int_0^\infty u_T(\lambda)\,d\lambda = \frac{c}{4}\frac{\alpha}{c^3}\int_0^\infty \frac{1}{\lambda^5} e^{-\beta/\lambda T}\,d\lambda$$

Let $\beta/\lambda T = x$. Then $dx = -(\beta/\lambda^2 T)\,d\lambda$ $\Rightarrow d\lambda = -\frac{\lambda^2 T}{\beta}\,dx$.

$$\sigma T^4 = E = -\frac{T}{\beta}\frac{c}{4}\frac{\alpha}{c^3}\int_\infty^0 \frac{1}{\lambda^3}e^{-x}dx = \left(\frac{T}{\beta}\right)^4 \frac{c}{4}\frac{\alpha}{c^3}\int_0^\infty \left(\frac{\beta}{\lambda T}\right)^3 e^{-x}dx = \left(\frac{T}{\beta}\right)^4 \frac{c}{4}\frac{\alpha}{c^3}\int_0^\infty x^3 e^{-x}dx$$

$$= \left(\frac{T}{\beta}\right)^4 \frac{c}{4}\frac{\alpha}{c^3}\cdot 6 = \frac{3}{2}\left(\frac{T}{\beta}\right)^4 \frac{\alpha}{c^2}$$

$$\Rightarrow \frac{3}{2}\left(\frac{T}{\beta}\right)^4 \frac{\alpha}{c^2} = \sigma T^4 \quad \Rightarrow \frac{3}{2}\frac{\alpha}{c^2\beta^4} = \sigma \quad \Rightarrow \alpha = \frac{2c^2\sigma\beta^4}{3}$$

$$\Rightarrow \alpha = \frac{2\times(3\times10^8)^2\times5.67\times10^{-8}\times(\mathbf{0.0145})^4}{3} = \frac{2\times9\times5.67\times(\mathbf{1.45})^4}{3} = 150.3856.$$

JAM 2013

Q.6 A blackbody at temperature T emits radiation at a peak wavelength λ. If the temperature of the blackbody becomes $4T$, the new peak wavelength is

(A) $\frac{1}{256}\lambda$ (B) $\frac{1}{64}\lambda$ (C) $\frac{1}{16}\lambda$ (D) $\frac{1}{4}\lambda$

Ans.: $\lambda_{max}T = constant \Rightarrow (\lambda_{max})_2 T_2 = (\lambda_{max})_1 T_1 \Rightarrow (\lambda_{max})_2 = \frac{(\lambda_{max})_1 T_1}{T_2} = \frac{\lambda T}{4T} = \frac{\lambda}{4} \Rightarrow (D).$

JAM 2012

Q.8 When the temperature of a blackbody is doubled, the maximum value of its spectral energy density, with respect to that at initial temperature, would become

(A) $\frac{1}{16}$ times (B) 8 times (C) 16 times (D) 32 times

Ans.: $u(\lambda) = \frac{8\pi hc}{\lambda^5}\cdot\frac{1}{e^{hc/\lambda kT}-1} = \frac{8\pi h}{\lambda^5}\cdot\frac{1}{e^{hc/k\lambda T}-1}$

For $\lambda = \lambda_{max}$ we have $u(\lambda_{max}) = \frac{8\pi hc}{\lambda_{max}{}^5}\cdot\frac{1}{e^{hc/k\lambda_{max}T}-1}$.

Also $\lambda_{max}T = constant \Rightarrow [\lambda_{max}]_2 T_2 = [\lambda_{max}]_1 T_1 \Rightarrow \frac{[\lambda_{max}]_1}{[\lambda_{max}]_2} = \frac{T_2}{T_1}$

$$\frac{u([\lambda_{max}]_2)}{u([\lambda_{max}]_1)} = \frac{8\pi hc}{[\lambda_{max}]_2{}^5}\cdot\frac{1}{e^{hc/k[\lambda_{max}]_2T_2}-1}\cdot\frac{[\lambda_{max}]_1{}^5}{8\pi hc}\cdot\frac{e^{hc/k[\lambda_{max}]_1T_1}-1}{1} = \left(\frac{[\lambda_{max}]_1}{[\lambda_{max}]_2}\right)^5 = \left(\frac{T_2}{T_1}\right)^5$$

$= 2^5 = 32 \quad \Rightarrow (D).$

JAM 2007:

7. The black body spectrum of an object O_1 is such that its radiant intensity (i.e., intensity per unit wavelength interval) is maximum at a wavelength of 200 nm. Another object O_2 has the maximum radiant intensity at 600 nm. The ratio of power emitted per unit area by O_1 to that of O_2 is

(A) $\frac{1}{81}$ (C) 9

(B) $\frac{1}{9}$ (D) 81

Ans.: Clearly the temperatures of the two blackbody will be different. If T_1 and T_2 are the temperatures then:

$$\frac{T_1}{T_2} = \frac{[\lambda_{max}]_2}{[\lambda_{max}]_1} = \frac{600}{200} = 3.$$

$$\frac{P_1}{P_2} = \frac{\sigma T_1{}^4}{\sigma T_2{}^4} = \left(\frac{T_1}{T_2}\right)^4 = 3^4 = 81.$$