Efficiency of Estimators

Let T_1 & T_2 be two unbiased estimators of a parameter $g(\theta)$. Let $E(T_1^2) < \infty$, $E(T_2^2) < \infty$.

We define the efficiency of T_2 relative to T_1 by

$$eff_\theta(T_2|T_1) = \frac{var(T_2)}{var(T_1)}$$

We say that T_2 is more efficient T_1 if $eff(T_2|T_1) < 1$

We can also define the efficiency of an unbiased estimator w.r.to the FRC lower bound.

i.e, $Ef(T) = \frac{var(T)}{FRCLB}$ (If 1 is attained, T is the most efficient)

If $\lim_{n\to\infty} Ef(T) = 1$, we say that T is asymptotically eff.

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Method of Moments :-

Let $X_1, X_2, \ldots, X_n$ be a random sample from a pop^n with dist^n $P_{\underline{\theta}}, \underline{\theta} \in \Theta$. $\underline{\theta} = (\theta_1, \ldots, \theta_k)$

Consider first k non-central moments —

$$\left.\begin{aligned} \mu_1' &= E(X_1) = g_1(\underline{\theta}) \\ \mu_2' &= E(X_1^2) = g_2(\underline{\theta}) \\ &\vdots \\ \mu_k' &= E(X_1^k) = g_k(\underline{\theta}) \end{aligned}\right\} \text{— (1)}$$

Assume the system of eq^{ns} (1) have solution

$$\left.\begin{aligned} \theta_1 &= h_1(\mu_1', \ldots, \mu_k') \\ &\vdots \\ \theta_k &= h_k(\mu_1', \mu_2', \ldots, \mu_k') \end{aligned}\right] \text{— (2)}$$

Define the 1st k non-central sample moments

$$\alpha_1 = \frac{1}{n}\sum_{i=1}^{n} X_i$$

$$\alpha_2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 \quad \ldots \quad \alpha_k = \frac{1}{n}\sum_{i=1}^{n} X_i^k$$

In method of moments we estimate k^{th} pop^n moment by k^{th} sample moment —

i.e, $\hat{\mu}_j' = \alpha_j \quad ; j = 1, 2, \ldots, k$

Thus the Method of Moments estimators of $\theta_1, \ldots, \theta_k$ ~~are~~ defined as

$$\left.\begin{aligned}\hat{\theta}_1 &= h_1(\alpha_1, \alpha_2, \dots, \alpha_k)\\ \hat{\theta}_2 &= h_2(\alpha_1, \alpha_2, \dots, \alpha_k)\\ &\vdots\\ \tilde{\theta}_k &= h_k(\alpha_1, \alpha_2, \dots, \alpha_k)\end{aligned}\right\} \quad (3)$$

Examples:-

1. $X_1, X_2, \dots, X_n \sim P(\lambda)$

$\hat{\lambda} = \bar{X}$ is MME of λ. $\to$ MME is unbiased and consistent for λ.

2. $X_1, X_2, \dots, X_n \overset{\text{i.i.d}}{\sim} N(\mu, \sigma^2)$

$\mu_1' = \mu, \quad \mu_2' = \mu^2 + \sigma^2$

$$\left.\begin{aligned}\mu &= \mu_1'\\ \sigma^2 &= \mu_2' - \mu_1'^2\end{aligned}\right\} \quad \hat{\mu}_{MME} = \bar{X}$$

$$\hat{\sigma}^2_{MME} = \alpha_2 - \alpha_1^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - \bar{X}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

$\boxed{\begin{array}{l}\hat{\mu}_{MME} \text{ is unbiased \& consistent}\\ \hat{\sigma}^2_{MME} \text{ is biased but consistent}\end{array}}$

3. $X_1, X_2, \dots, X_k \sim Bin(n, p)$.

Case I:- n is known $\to$ only one parameter so only one eqn.

$\mu_1' = np \to p = \frac{\mu_1'}{n} \Rightarrow \hat{p} = \frac{\bar{X}}{n}$ is MME of p

$\bar{X} = np$

$\hookrightarrow$ Unbiased and consistent

Case II:- n & p are unknown

$\mu_1' = np, \quad \mu_2' = n^2p^2 + np(1-p)$.

$\mu_2' - \mu_1'^2 = np(1-p)$

$\Rightarrow 1 - p = \frac{\mu_2' - \mu_1'^2}{\mu_1'}$

$$\boxed{\begin{aligned}\Rightarrow p &= 1 - \frac{\mu_2' - \mu_1'^2}{\mu_1'}\\ \Rightarrow n &= \frac{\mu_1'}{p}.\end{aligned}}$$

$$\hat{p}_{MME} = 1 - \frac{\alpha_2 - \alpha_1^2}{\alpha_1} = \frac{\bar{X} - \frac{1}{k}\sum_{i=1}^{k}(X_i - \bar{X})^2}{\bar{X}}$$

$$\hat{n}_{MME} = \frac{\bar{X}^2}{\bar{X} - \frac{1}{k}\sum_{i=1}^{k}(X_i - \bar{X})^2}$$

~~MME is unbiased and~~

$\overline{X} \longrightarrow np$

$\frac{1}{n}\Sigma x_i^2 \longrightarrow n^2p^2 + np(1-p)$

~~However~~ $\hat{n} \xrightarrow{P} n$

However $\hat{n} \xrightarrow{P}\!\!\!\!\!/\;\; n$

eg, $k=2$, $x_1 = 2$, $x_2 = 6$, $\overline{x} = 4$

$\frac{1}{k}\Sigma(x_i - \overline{x})^2 = 4$

so, $\overline{X} - \frac{1}{k}\sum_{i=1}^{k}(x_i - \overline{x})^2$

So, MME need not be consistent always

Remark

1. The Method of moment estimators need not be unbiased.
2. If the fun^n g_i's are continuous & one-to-one $\Rightarrow$ h_i's are continuous, then MME's will be consistent.

Examples:-

1. $x_1, x_2, \ldots, x_n \sim U[a, b]$

$\mu_1' = \frac{a+b}{2}$, $\mu_2' = \frac{a^2+b^2+ab}{3}$

$a = \mu_1' - \sqrt{3(\mu_2' - \mu_1'^2)}$

$b = \mu_1' + \sqrt{3(\mu_2' - \mu_1'^2)}$ since $a < b$.

$\hat{a}_{MME} = \overline{X} - \sqrt{\frac{3}{n}\Sigma(X_i - \overline{X})^2}$

$\hat{b}_{MME} = \overline{X} + \sqrt{\frac{3}{n}\Sigma(X_i - \overline{X})^2}$

Inverse fun^{n}s are continuous so they will be consistent not unbiased.

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Sometimes for a k-dimensional parameter we may have to consider more than k-eqn's.

Example :- 1. $X_1, X_2, \ldots, X_n \sim U[-\theta, \theta], \theta > 0$

$\mu_1' = 0$

$\mu_2' = \frac{\theta^2}{3}$

$\theta = \sqrt{3\mu_2'}$

$\hat{\theta}_{MME} = \sqrt{3\alpha_2} = \sqrt{\frac{3}{n}\sum_{i=1}^{n} X_i^2}$

2. $X_1, X_2, \ldots, X_n \sim \text{Gamma}(p, \lambda)$

$f(x) = \frac{\lambda^p}{\Gamma(p)} e^{-\lambda x} x^{p-1}, x > 0$

$\mu_1' = \frac{p}{\lambda}, \quad \mu_2' = \frac{p(p+1)}{\lambda^2}$

$p = \frac{\mu_1'^2}{\mu_2' - \mu_1'^2}, \quad \lambda = \frac{\mu_1'}{\mu_2' - \mu_1'^2}$

$\hat{p}_{MME} = \frac{\bar{X}^2}{\frac{1}{n}\Sigma(X_i - \bar{X})^2}, \quad \hat{\lambda}_{MME} = \frac{\bar{X}}{\frac{1}{n}\Sigma(X_i - \bar{X})^2}$

These are consistent but not unbiased.

MME's are generally consistent but not unbiased.

3. $X_1, \ldots, X_n \sim \text{Beta}(\alpha, \beta)$

$f(x) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1}(1-x)^{\beta-1}, \quad 0 < x < 1, \quad \alpha, \beta > 0$

$\left\{\begin{array}{l} \alpha_1 \xrightarrow{P} \mu_1' \\ \alpha_2 \xrightarrow{P} \mu_2' \end{array}\right.$ by weak law of large

$\mu_1' = \frac{\alpha}{\alpha+\beta}, \quad \mu_2' = \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}$

$\alpha = \frac{\mu_1'(\mu_1' - \mu_2')}{(\mu_2' - \mu_1'^2)} \rightarrow$ cont. $\rightarrow$ continuity my

$\beta = \frac{(1-\mu_1')(\mu_1' - \mu_2')}{(\mu_2' - \mu_1'^2)}$

$\hat{\alpha}_{MME} = \frac{\bar{X}(\bar{X} - \frac{1}{n}\Sigma X_i^2)}{\frac{1}{n}\Sigma(X_i - \bar{X})^2}, \quad \hat{\beta}_{MME} = \frac{(1-\bar{X})(\bar{X} - \frac{1}{n}\Sigma X_i^2)}{\frac{1}{n}\Sigma(X_i - \bar{X})^2}$

$\hat{\alpha}_{MME}$ & $\hat{\beta}$ are consistent but biased.