(1) Show that $\lim_{x\to 0} \frac{1}{1+e^{1/x}}$ does not exist.

(2) Show that $\lim_{x\to 0} \cos(1/x)$ does not exist.

(3) $\lim_{x\to 0} e^{1/x}$ does not exist

(4) $\lim_{x\to 0} [x]$ does not exist,

(5) $\lim_{x\to 0} \frac{1}{x^2}$ $(x>0)$ does not exist.

(6) $\lim_{x\to 0} \frac{1}{\sqrt{x}}$ $(x>0)$

(7) $\lim_{x\to 0} \sin(1/x^2)$

(8) $\lim_{x\to 0} (x+\operatorname{sgn}(x))$

Find the following limits —

(i) $\lim_{x\to 2} \frac{x^2-3x+7}{4x^3+x+1}$

(iii) $\lim_{x\to 0} \frac{\frac{1}{x-2}-\frac{1}{x}}{\frac{1}{x-3}+\frac{1}{x}}$

(ii) $\lim_{x\to 3} \frac{x^2+x-12}{x-3}$

(iv) $\lim_{x\to 3} \frac{x-3}{\sqrt{x-2}-\sqrt{4-x}}$

(v) $\lim_{x\to 1} \frac{x^7-2x^5+1}{x^3-3x^2+2}$.

(vi) $\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x}$

(vii) $\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1+x^2}}{\sqrt{1-x^2}-\sqrt{1-x}}$

(viii) $\lim_{x\to 1} (x+1)(2x+3)$

(ix) $\lim_{x\to 0} \frac{(x+1)^2-1}{x}$

(x) $\lim_{x\to 1} \frac{\sqrt{x}-1}{x-1}$

(xi) $\lim_{x\to 0} \frac{\sqrt{1+2x}-\sqrt{1+3x}}{x+2x^2}$ where $x>0$.

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Ex: Let
$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$
Prove that $\lim_{x \to a} f(x)$ does not exist for any real number a.

Sol^n: Let $\{x_n\}$ be a sequence of rational numbers such that $x_n \to a$ and $x_n \neq a$, $n = 1, 2, 3, \ldots$;

Then $f(x_n) = 1$.

i.e., $\{f(x_n)\} \to 1$ as $n \to \infty$.

Let $\{y_n\}$ be a sequence of irrational numbers such that $y_n \to a$ and $y_n \neq a$, $n = 1, 2, 3, \ldots$

Then $f(y_n) = 0$

i.e., $\{f(y_n)\} \to 0$ as $n \to \infty$.

So, there are two sequences $\{x_n\}$ & $\{y_n\}$ which converge to the same limit a.

But the $\{f(x_n)\} \to 1$ & $\{f(y_n)\} \to 0$.

$\Rightarrow \lim_{x \to a} f(x)$ does not exist.

Ex: Let
$$f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$$
Prove that $\lim_{x \to a} f(x)$ exists only if $a = 0$.

Sol^n: Let $a \neq 0$.

Let $\{x_n\}$ be a sequence of rational numbers s.t. $x_n \to a$ and $x_n \neq a$.
$n = 1, 2, \ldots$

$\{y_n\}$ be a sequence of irrational numbers such that $y_n \to a$ and $y_n \neq a$ for ~~n=1,2,3,...~~ $n \geq 1$.

Then $f(x_n) = x_n \to a$ and
$f(y_n) = 0 \to 0$

Since $a \neq 0$, ~~therefore~~
$\{f(x_n)\}$ & $\{f(y_n)\}$ converges to two different points where $\{x_n\}$ & $\{y_n\}$ converge to the same pt. a.

$\Rightarrow \lim_{x \to a} f(x)$ does not exist when $a \neq 0$.

When $a = 0$, both $\{f(x_n)\}$ & $\{f(y_n)\}$ converges to the same limit 0.

Therefore $\lim_{x \to a} f(x)$ exists only if $a=0$.

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Theorem:- Let $A \subseteq \mathbb{R}$, let $f: A \to \mathbb{R}$ and let $c \in \mathbb{R}$ be a cluster pt. of A. If $a \leq f(x) \leq b$ $\forall x \in A$, $x \neq c$ and if $\lim\limits_{x \to c} f(x)$ exists, then $a \leq \lim\limits_{x \to c} f(x) \leq b$.

Proof:- Let $l = \lim\limits_{x \to c} f(x)$ (given).

Equivalently, for every sequence $\{x_n\}$ in A that converges to c such that $x_n \neq c$ $\forall n \in \mathbb{N}$, then $\{f(x_n)\}$ converges to l.

But it is known that $a \leq f(x) \leq b$ $\forall x \in A$, $x \neq c$

$$\Rightarrow a \leq f(x_n) \leq b$$

As $n \to \infty$, $a \leq l \leq b$

$$\Rightarrow a \leq \lim_{x \to c} f(x) \leq b.$$

Theorem (Squeeze Thm):-

Let $A \subseteq \mathbb{R}$, let $f, g, h: A \to \mathbb{R}$, and $c \in \mathbb{R}$ be the cluster pt. of A. If $f(x) \leq g(x) \leq h(x)$ $\forall x \in A$, $x \neq c$ and $\lim\limits_{x \to c} f = l = \lim\limits_{x \to c} h$

$\Rightarrow$ then $\lim\limits_{x \to c} g(x) = l$.