Ex: Let

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$$

Prove that $\lim_{x \to a} f(x)$ does not exist for any real number a.

Soln: Let $\{x_n\}$ be a sequence of rational numbers such that $x_n \to a$ and $x_n \neq a$, $n = 1, 2, 3, \ldots$).

Then $f(x_n) = 1$.

i.e., $\{f(x_n)\} \to 1$ as $n \to \infty$.

Let $\{y_n\}$ be a sequence of irrational numbers such that $y_n \to a$ and $y_n \neq a$, $n = 1, 2, 3, \ldots$

Then $f(y_n) = 0$

i.e., $\{f(y_n)\} \to 0$ as $n \to \infty$.

So, there are two sequences $\{x_n\}$ & $\{y_n\}$ which converge to the same limit a.

But the $\{f(x_n)\} \to 1$ & $\{f(y_n)\} \to 0$.

$\Rightarrow \lim_{x \to a} f(x)$ does not exist.

Ex: Let $f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$

Prove that $\lim_{x \to a} f(x)$ exists only if $a = 0$.

Soln: Let $a \neq 0$.

Let $\{x_n\}$ be a sequence of rational numbers s.t. $x_n \to a$ and $x_n \neq a$, $n \geq 1$,

$\{y_n\}$ be a sequence of irrational numbers such that $y_n \to a$ and $y_n \neq a$ for ~~n=1,2,3,...~~ $n \geq 1$.

Then $f(x_n) = x_n \to a$ and
$f(y_n) = 0 \to 0$

Since $a \neq 0$, ~~therefore~~
$\{f(x_n)\}$ & $\{f(y_n)\}$ converges to two different points where $\{x_n\}$ & $\{y_n\}$ converge to the same pt. a.

$\Rightarrow$ $\lim_{x \to a} f(x)$ does not exist when $a \neq 0$.

When $a = 0$, both $\{f(x_n)\}$ & $\{f(y_n)\}$ converges to the same limit 0.

Therefore $\lim_{x \to a} f(x)$ exists only if $a=0$.

Theorem :- Let $A \subseteq \mathbb{R}$, let $f: A \to \mathbb{R}$ and let $c \in \mathbb{R}$ be a cluster pt. of A.
If $a \leq f(x) \leq b \ \forall x \in A, x \neq c$ and if $\lim_{x \to c} f(x)$ exists, then $a \leq \lim_{x \to c} f(x) \leq b$.

Proof :- Let $l = \lim_{x \to c} f(x)$ (given).

Equivalently, for every sequence $\{x_n\}$ in A that converges to c such that $x_n \neq c \ \forall n \in \mathbb{N}$, then $\{f(x_n)\}$ converges to l.

But it is known that $a \leq f(x) \leq b \ \forall x \in A, x \neq c$

$$\Rightarrow a \leq f(x_n) \leq b$$

As $n \to \infty$, $a \leq l \leq b$

$$\Rightarrow a \leq \lim_{x \to c} f(x) \leq b.$$

Theorem (Squeeze Thm) :-

Let $A \subseteq \mathbb{R}$, let $f, g, h: A \to \mathbb{R}$, and $c \in \mathbb{R}$ be the cluster pt. of A. If $f(x) \leq g(x) \leq h(x)$ $\forall x \in A, x \neq c$ and $\lim_{x \to c} f = l = \lim_{x \to c} h$

$$\Rightarrow \text{then } \lim_{x \to c} g(x) = l.$$

Ex1 :- Show that $\lim_{x \to 0} x^{3/2} = 0$. $(x > 0)$

If power is natural number then it is obvious that the limit is zero. But when it's fraction we can't say directly.

$\Rightarrow f(x) = x^{3/2} > 0$

since $x < x^{1/2} \leq 1$ for $0 < x \leq 1$

multiplying by x we get,

$$x^2 < x^{3/2} \leq x$$

$\Rightarrow$ By squeeze thm. $\lim_{x \to 0} x^{3/2} = 0$

Let $1 < x$

then $1 < x^{1/2} < x$

$\Rightarrow x < x^{3/2} < x^2$

By squeeze theorem

$$\lim_{x \to 0} x^{3/2} = 0$$

Show that

(2) $\lim_{x \to 0} \left(\frac{\cos x - 1}{x} \right) = 0$ using squeeze thm.

Sol^n:- $1 - x^2/2 \leq \cos x \leq 1 \quad \forall x \in \mathbb{R}$

$f(x) = \cos x - 1 + x^2/2$

$f'(x) = -\sin x + x > 0$

$\forall x \in \mathbb{R}$

If $f'(x) > 0$ then $f(x)$ is an increasing fun^n

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[Now if we take
$g(x) = x - \sin x$
$g'(x) = 1 - \cos x \geq 0$
$\Rightarrow g$ is a increasing func^n
$\Rightarrow$ $g(x) \geq 0 \; \forall x \in \mathbb{R}$]
$\Rightarrow$

Hence $1 - \frac{x^2}{2} \leq \cos x \leq 1$ holds $\forall x \in \mathbb{R}$.

$-\frac{x}{2} \leq \frac{\cos x - 1}{x} \leq 0$ when $x > 0$

$0 \leq \frac{\cos x - 1}{x} \leq -x/2$ if $x < 0$

$\Rightarrow \lim\limits_{x \to 0} \frac{\cos x - 1}{x} = 0$.

③ $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$

Hint:- $x - \frac{1}{6}x^3 \leq \sin x \leq x$ for $x \geq 0$.
$x \leq \sin x \leq x - \frac{1}{6}x^3$ for $x \leq 0$.

④ $\lim\limits_{x \to 0} x \sin \frac{1}{x} = 0$

$-1 \leq \sin \frac{1}{x} \leq 1$

$\Rightarrow -x \leq x \sin \frac{1}{x} \leq x$

$\Rightarrow \lim\limits_{x \to 0} x \sin \frac{1}{x} = 0$.

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Theorem:- Let $A \subseteq \mathbb{R}$, let $f: A \to \mathbb{R}$ and let $c \in \mathbb{R}$ be a cluster pt. of A. If $\lim_{x \to c} f(x) > 0$ (<0) then there exists a neighbourhood $N_\delta(c)$ s.t. ~~$f(x)$~~

$f(x) > 0$ (<0) $\forall x \in A \cap N_\delta(c)$ & $x \neq c$.

Proof:- Let $l = \lim_{x \to c} f$

suppose $l > 0$.

Choose $\varepsilon = l/2 > 0$

Corresponding to $\varepsilon = l/2$ we can find $\delta(\varepsilon)$ s.t. if $0 < |x - c| < \delta$ & $x \in A$ then

$|f(x) - l| < l/2$

$\Rightarrow -l/2 < f(x) < \frac{3l}{2}$

so, if $x \in A \cap N_\delta(c)$, $x \neq c$ then $f(x) > l/2 > 0$.

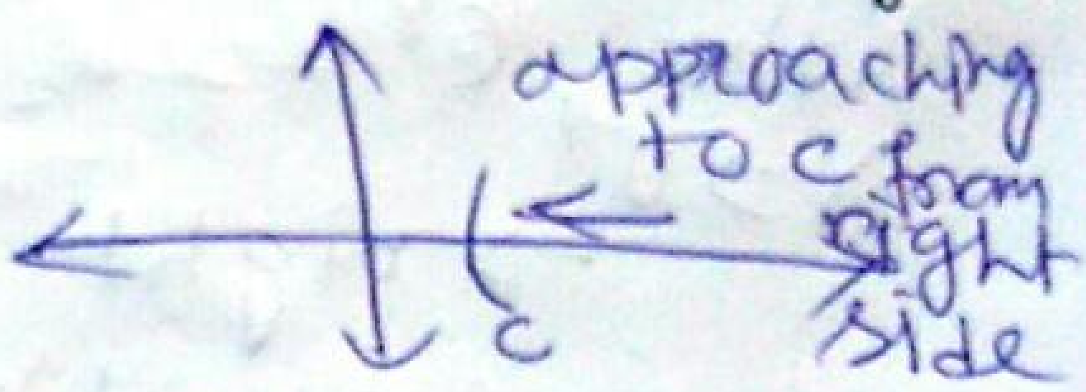
Hence the proof.

Def^n:- Let $A \subseteq \mathbb{R}$ and $f: A \to \mathbb{R}$ -

(i) If $c \in \mathbb{R}$ is a cluster point of the set $A \cap (c, \infty) = \{x \in A \mid c < x < \infty\}$

then we say that $l \in \mathbb{R}$ is a right hand limit of f at c and is defined by $\lim_{x \to c^+} f(x) = l$. If for

$$\lim_{x \to c^+} f(x) = l$$



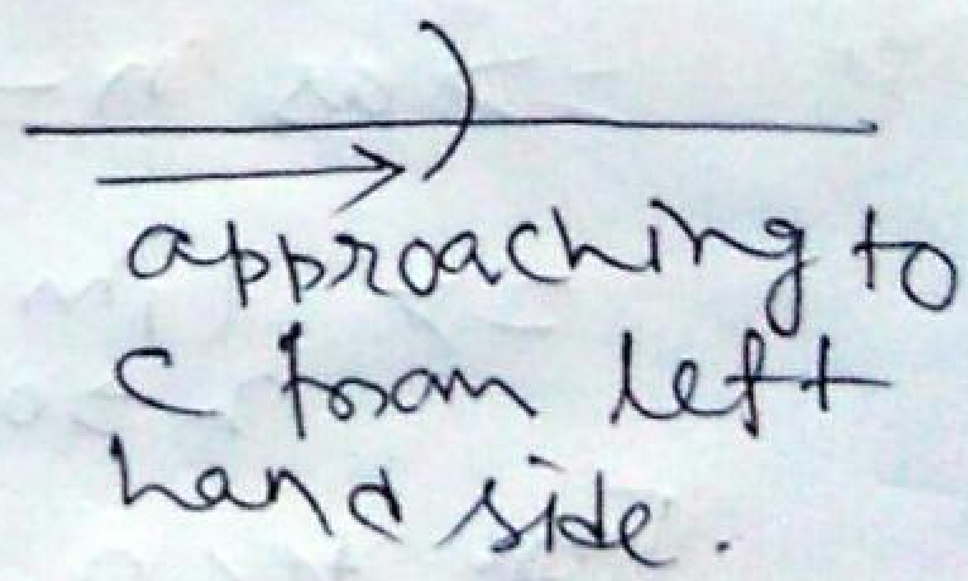
If for any $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that ~~for~~ $\forall x \in A$ and with $0 < x - c < \delta$ then $|f(x) - l| < \varepsilon$.

(ii) If $c \in \mathbb{R}$ is a cluster pt. of the set

$A \cap (-\infty, c) = \{x \in A \mid x < c\}$, then we say $l \in \mathbb{R}$ is a left hand limit of f at c, denoted by $\lim_{x \to c^-} f = l$,

If given any $\varepsilon > 0$ there exists a $\delta > 0$ s.t. for all $x \in A$ with $0 < c - x < \delta$ then

~~$|f(x) - l| < \varepsilon$~~. $|f(x) - l| < \varepsilon$



Ex:- $f(x) = \operatorname{sign} x$, $c = 0$

$$\operatorname{sign}(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$

$\lim_{x \to 0^-} \operatorname{sgn}(x) = -1$, $\lim_{x \to 0^+} \operatorname{sgn} x = 1$

② $f(x) = \frac{1}{x}$, $c = 0$

$\lim_{x\to 0^-} \frac{1}{x} \to -\infty$ $\lim_{x\to 0^+} \frac{1}{x} \to \infty$

Both limit does not exist.

③ $f(x) = e^{1/x}$, $c = 0$

$\lim_{x\to 0^+} e^{1/x} = \infty$ $\lim_{x\to 0^-} e^{1/x} = 0$

→ right hand limit does not exist.

→ left hand limit exist

For any $0<t$, $0 < t < e^t$ $\forall\, t > 0$

$e^t = 1 + t + t^2/2! + \cdots$

Replace t by $1/x$

Then $0 < 1/x < e^{1/x}$ for $x>0$

Take a sequence $x_n = 1/n \to 0$

$0 < 1/x_n < e^{1/x_n}$

$\Rightarrow \cancel{0 < e^{n}} < e^{1}$

$\Rightarrow 0 < n < e^n < \infty$

$\Rightarrow$ as $n \to \infty$ $e^n \to \infty$

$\Rightarrow \lim_{x\to 0} e^{1/x} = \infty$

For $x<0$, $t = -1/x$ if $x<0$

$0 < -1/x < e^{-1/x}$ for $x<0$

when $x \to 0^-$ $\lim_{x\to 0^-} e^{1/x} = \frac{1}{\infty} = 0$.

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Equivalent Defn of left/right hand limit of funn f at c

Theorem:- Let $A \subseteq \mathbb{R}$, let $f: A \to \mathbb{R}$ and $c \in \mathbb{R}$ be cluster point of $A \cap (c, \infty)$. Then the following statements are equivalent:

(i) $\lim_{x \to c^+} f = l$

(ii) For every sequence $\{x_n\}$ that converges to c such that $x_n \in A$ and $x_n > c$ $\forall n \in \mathbb{N}$, then the sequence $\{f(x_n)\}$ converges to l.

Theorem:- Let $A \subseteq \mathbb{R}$, let $f: A \to \mathbb{R}$ and let c be a cluster pt. of both the sets $A \cap (-\infty, c)$ & $A \cap (c, \infty)$. Then $\lim_{x \to c} f = l$ iff $\lim_{x \to c^+} f = l = \lim_{x \to c^-} f$.

Determine the give limit, if it exists.

① $f(x) = \begin{cases} 2x-1, & x \leq -2 \\ -x+2, & x > -2 \end{cases}$ find $\lim_{x \to 2^-} f(x)$

② $f(x) = \begin{cases} -x^2+4x-3, & x < 1 \\ x-7, & x \geq 1 \end{cases}$, $\lim_{x \to 1^-} f(x)$

③ $f(x) = \begin{cases} x^2-2x+1, & x < -1 \\ -\frac{x}{2}+\frac{7}{2}, & x \geq -1 \end{cases}$, find $\lim_{x \to 1} f(x)$

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④ $f(x) = \begin{cases} x+3 & , x \in (-\infty, 0) \\ -x+2 & , x \in [0, 2) \\ (x-2)^2 & , x \in [2, \infty) \end{cases}$

⑤ $\lim_{x \to 0} f(x)$ & $\lim_{x \to 2} f(x)$

⑤ $f(x) = \begin{cases} (x+1)^2 - 1 & , -2 \le x < 0 \\ \frac{5}{4} \sin\left(\frac{\pi x}{2}\right) & , 0 \le x < 2 \\ (x-3)^2 - 1 & , 2 \le x \le 4 \end{cases}$

$\lim_{x \to 2} f(x)$

⑥ $f(x) = \begin{cases} 2x-1 & , x \le -1 \\ x^2+1 & , -1 < x \le 1 \\ -x+3 & , x > 1 \end{cases}$

$\lim_{x \to 1^-} f(x)$ & $\lim_{x \to 1} f(x)$

⑦ $f(x) = \begin{cases} -x^2 - 4x - 2 & , x \le 0 \\ (x-1)^2 - 1 & , x > 0 \end{cases}$

⑧ $f(x) = \begin{cases} x^2 + 6x + 8 & x \le -1 \\ -x + 4 & x > -1 \end{cases}$