Theorem :- Let $g : D \to \mathbb{R}$ be a funⁿ on D and $(c, \infty) \subset D$ for ~~for~~ some $c \in \mathbb{R}$. Then $\lim_{n \to \infty} g(n) = l$ $(l \in \mathbb{R})$ if and only if $\lim_{n \to 0^+} g(1/n) = l$.

Note :- And $\lim_{n \to -\infty} g(x) = l$ iff $\lim_{n \to 0^-} g(1/n) = l$.

Some Important Limits

(1) $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$.

Let $[n] = k$, $k \in \mathbb{N}$

Then $k \le n < k+1$

$\Rightarrow \frac{1}{k+1} < \frac{1}{n} \le \frac{1}{k}$

$\Rightarrow 1 + \frac{1}{k+1} < 1 + \frac{1}{n} \le 1 + \frac{1}{k}$

$\Rightarrow \left(1 + \frac{1}{1+k}\right)^k < \left(1 + \frac{1}{n}\right)^n \le \left(1 + \frac{1}{k}\right)^{k+1}$

- Taking limit $n \to \infty$, $n \to \infty \Rightarrow k \to \infty$

So, we have

$$\lim_{k \to \infty} \left(1 + \frac{1}{1+k}\right)^k \le \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n \le \lim_{k \to \infty} \left(1 + \frac{1}{k}\right)^{k+1}$$

$$\Rightarrow \lim_{n \to \infty} \left(1 + \frac{1}{1+k}\right)^{k+1} \frac{1}{\left(1 + \frac{1}{1+k}\right)} \le \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n \le \lim_{n \to \infty} \frac{\left(1 + \frac{1}{k}\right)^k}{\left(1 + 1/k\right)}$$

$$\Rightarrow e \le \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n \le e$$

$$\Rightarrow \lim_{n \to \infty} \left(1 + \frac{1}{n}\right) = e$$

② $\lim_{n \to -\infty} \left(1+\frac{1}{n}\right)^{n} = e$

$= \lim_{y \to \infty} \left(1-\frac{1}{y}\right)^{-y}$ $\quad y = -n$

$= \lim_{y \to \infty} \left(\frac{y-1}{y}\right)^{-y}$

$= \lim_{y \to \infty} \left(\frac{y}{y-1}\right)^{y}$ $\quad$ let $y-1 = t$

$= \lim_{t \to \infty} \left(\frac{t+1}{t}\right)^{t+1}$ $\quad y = t+1$

$= \lim_{t \to \infty} \left(1+1/t\right)^{t} \left(1+1/t\right)$

$= e \cdot 1 = e.$

③ $\lim_{n \to 0} \left(1+\cancel{\infty} n\right)^{1/n} = e$

let $y = 1/n$, $n \to 0^{+} \Rightarrow y \to \infty^{+}$.

$\lim_{y \to \infty^{+}} \left(1+1/y\right)^{y}$

$= e.$

$n \to 0^{-} \Rightarrow y \to \infty^{-}$

$\lim_{y \to \infty^{-}} \left(1+1/y\right)^{y} = e.$

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① $\lim_{x\to\infty} \frac{x-\sin x}{x+\cos x} = \lim_{x\to\infty} \frac{1-\frac{\sin x}{x}}{1+\frac{\cos x}{x}} = 1.$

$\cancel{x} \Rightarrow$ let $y = 1/x$, $x \to \infty \Rightarrow y \to 0$

$\lim_{y\to 0} \frac{1/y - \sin 1/y}{1/y + \cos 1/y}$

$= \lim_{y\to 0} \frac{1 - y\sin 1/y}{1+y\cos 1/y} = \frac{1}{1} = 1.$

Find the following limits—

① $\lim_{x\to 0} \frac{\sqrt{x^2+4}-2}{x^2}$

$= \lim_{x\to 0} \frac{(\sqrt{x^2+4}-2)(\sqrt{x^2+4}+2)}{x^2(\sqrt{x^2+4}+2)}$

② $\lim_{x\to 0} \frac{1-\cos x}{x}$

$= \lim_{x\to 0} \frac{1-\cos^2 x}{x(1+\cos x)} = \lim_{x\to 0} \frac{\sin^2 x}{x(1+\cos x)}$

$= \lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{x\to 0} \frac{\sin x}{1+\cos x}$

$= 1 \cdot 0 = 0$

③ $\lim_{x\to 0} \frac{\tan x}{x}$ (4) $\lim_{x\to 0} \frac{\sin 3x}{x}$

(5) $\lim_{x\to 0} \frac{1-\cos x}{\sin x}$ (6) $\lim_{x\to\infty} (1+\frac{3}{x})^x$ $\frac{3}{x} = \frac{1}{y}$

$= \lim_{y\to\infty} (1+\frac{1}{y})^{3y}$ 9 € 10

$= \lim_{y\to\infty} (1+\frac{1}{y}) \cdot (1+\frac{1}{y}) \cdot (1+1/y)$

$= e^3$

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