L-5

Complex representation of Fourier series:

Fourier series of a function $f(x)$ of period $2l$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$= \frac{a_0}{2} + (a_1 \cos x + a_2 \cos 2x + \cdots) + (b_1 \sin x + b_2 \sin 2x + \cdots)$$

We know that $\cos x = \frac{e^{ix}+e^{-ix}}{2}$ *and* $\sin x = \frac{e^{ix}-e^{-ix}}{2i}$

Therefore,

$$f(x) = \frac{a_0}{2} + \left(a_1 \frac{e^{i\pi x/} + e^{-i\pi x/l}}{2} + a_2 \frac{e^{2i\pi x/l} + e^{-2i\pi x/}}{2} + \cdots\right)$$
$$+ \left(b_1 \frac{e^{i\pi x/l} - e^{-i\pi x/l}}{2i} + b_2 \frac{e^{2i\pi x/l} - e^{-2i\pi x/l}}{2i} + \cdots\right)$$

$$= \frac{a_0}{2} + \frac{1}{2}(a_1 - ib_1)e^{i\pi x/l} + \frac{1}{2}(a_2 - ib_2)e^{2i\pi x/l} + \cdots + \frac{1}{2}(a_1 + ib_1)e^{-i\pi x/l}$$
$$+ \frac{1}{2}(a_2 + ib_2)e^{-2i\pi x/l} + \cdots$$

$$= C_0 + C_1 e^{i\pi x/l} + C_2 e^{2i\pi x/l} + \cdots + C_{-1} e^{i\pi x/l} + C_{-2} e^{-2i\pi x/l} + \cdots$$

$$= C_0 + \sum_{n=1}^{\infty} C_n e^{in\pi x/} + \sum_{n=1}^{\infty} C_{-n} e^{-in\pi x/l}$$

This is complex representation of Fourier series expansion of a function, where $C_0 = \frac{a_0}{2}$, $C_n = \frac{1}{2}(a_n - ib_n)$, *and* $C_{-n} = \frac{1}{2}(a_n + ib_n)$

In integral form we have $C_0 = \frac{1}{2l}\int_0^{2l} f(x)dx$

$$C_n = \frac{1}{2}\left[\frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx - \frac{i}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{1}{2l}\int_0^{2l} f(x)\left[\cos\frac{n\pi x}{l} - i\sin\frac{n\pi x}{l}\right]dx = \frac{1}{2l}\int_0^{2l} f(x)\,e^{-in\pi x/l}dx$$

And

$$C_{-n} = \frac{1}{2}\left[\frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx + \frac{i}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{1}{2l}\int_0^{2l} f(x)\left[\cos\frac{n\pi x}{l} + i\sin\frac{n\pi x}{l}\right]dx = \frac{1}{2l}\int_0^{2l} f(x)\,e^{in\pi x/l}dx$$

Example 1: Obtain complex form of Fourier series of the function

$$f(x)=\begin{cases}0, & for -\pi \le x \le 0\\ 1, for\ 0 \le x \le \pi\end{cases}$$

Here, $C_0=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx=\frac{1}{2\pi}\int_0^{\pi} 1\ \ dx=\frac{1}{2\pi}\times\pi=\frac{1}{2}$

And

$$C_n=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,e^{-inx}dx=\frac{1}{2\pi}\int_0^{\pi} 1\,e^{-inx}dx=\frac{1}{2\pi}\left|\frac{e^{-inx}}{-in}\right|_0^{\pi}=-\frac{1}{2i\pi n}\left[e^{-in\pi}-1\right]$$

$$=-\frac{1}{2i\pi n}[\cos n\pi-i\sin n\pi-1]==-\frac{1}{2i\pi n}[(-1)^n-1]$$

Therefore, $C_n=\begin{cases}\frac{1}{in\pi}, for\ n\ is\ odd\\ 0, for\ n\ is\ even\end{cases}$

Similarly, $C_{-n}=\begin{cases}\frac{-1}{in\pi}, for\ n\ is\ odd\\ 0, for\ n\ is\ even\end{cases}$

Thus, $f(x)=\frac{1}{2}+\frac{1}{i\pi}\left\{\frac{e^{ix}}{1}+\frac{e^{3ix}}{3}+\frac{e^{5ix}}{5}+\cdots\right\}+\frac{1}{i\pi}\left\{\frac{e^{-i}}{-1}+\frac{e^{-3ix}}{-3}+\frac{e^{-5ix}}{-5}+\cdots\right\}=\frac{1}{2}+\frac{1}{i\pi}\left\{\left(e^{ix}-e^{-ix}\right)+\frac{1}{3}\left(e^{3ix}-e^{-3ix}\right)+\frac{1}{5}\left(e^{5ix}-e^{-5ix}\right)+\cdots\right\}$

Find complex form of Fourier series for the following functions:

1. $f(x)=e^{-x}, for -1\le x\le 1$
2. $f(x)=e^{ax}, for -l\le x\le l$
3. $f(x)=\cos ax\,, for -\pi\le x\le \pi.$

Term-by-Term differentiation of Fourier Series:

Now let us discuss the question of differentiating a Fourier series term by term. First consider a Fourier series in which a_n and b_n are proportional to $1/n$. Since the derivative of $1/n$ sin nx is cos nx (and a similar result for the cosine terms), we see that the differentiated series has no $1/n$ factors to make it converge. Now you might suspect (correctly) that if you can't differentiate the Fourier series, then the function $f(x)$ which it represents can't be differentiated either, at least not at all points. Turn back to examples and problems for which the Fourier series have coefficients proportional to $1/n$ and look at the graphs. Note in every case that $f(x)$ is discontinuous (that is, has jumps) at some points, and so can't be differentiated there.

Example 1: Is it possible to perform term by term differentiation on following function?

$f(x)=x,\ for\ -l\le x\le l.$

We know that periodic expansion of the above function results a sawtooth waveform (which is odd function), and thus Fourier series will give a sine series in the given interval.

Therefore, $f(x)=\sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{l}$, where $b_n=\frac{2l}{n\pi}(-1)^{n+1}$

Now differentiate the above Fourier series expression of the function term by term

$$\frac{d}{dx}[f(x)] = f'(x) = \frac{d}{dx}\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} = \sum_{n=1}^{\infty}\frac{d}{dx}\left[b_n \sin\frac{n\pi x}{l}\right]$$

$$= \sum_{n=1}^{\infty}\frac{2l}{n\pi}(-1)^{n+1}\frac{n\pi}{l}\cos\frac{n\pi x}{l}, \qquad since\ b_n = \frac{2l}{n\pi}(-1)^{n+1}$$

$$= \sum_{n=1}^{\infty} 2(-1)^{n+1}\cos\frac{n\pi x}{l} \dots\dots\dots\dots\dots\dots\dots.(1)$$

Next, we want to differentiate the given function and then expand it in Fourier series.

Since $f(x) = x$, then $\frac{d}{dx}[f(x)] = f'(x) = 1$

Therefore, we can write a cosine series (to match equation 1) as

$$f'(x) = 1 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} \dots\dots\dots\dots\dots\dots(2),$$

$$having\ a_0 = 2\ and\ a_n = 0$$

Certainly, the above equation (2) is not the same cosine series as equation (1).

So, term by term differentiation of this function is not possible as $f(x)\ and\ f'(x)\ are\ both\ piecewise\ continuous.$ Note in every case that *f*(*x*) is discontinuous (that is, has jumps) at some points, and so can't be differentiated there.

Example 2: Let us consider another function for verification

$$f(x) = |x|, for - l \le x \le l$$

Here, $f(x)$ is continuous everywhere in its periodic extension and even function, so we can get cosine series by Fourier series expansion as $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$

Where $a_0 = \frac{2}{l}\int_0^l f(x)\, dx = l$

And $a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l} dx = \frac{2}{l}\int_0^l x\cos\frac{n\pi x}{l} dx = \begin{cases} -\frac{4l}{(n\pi)^2}, for\ n\ is\ odd \\ 0, for\ n\ is\ even \end{cases}$

Therefore, $f(x) = \frac{l}{2} + \sum_{n=1,3,5\dots}^{\infty} -\frac{4l}{(n\pi)^2}\cos\frac{n\pi x}{l}$

Differentiate it term by term we get

$$\frac{d}{dx}[f(x)] = f'(x) = \frac{d}{dx}\left[\frac{l}{2} + \sum_{n=1,3,5,..}^{\infty} -\frac{4l}{(n\pi)^2}\cos\frac{n\pi x}{l}\right]$$

$$= \frac{d}{dx}\left[\frac{l}{2}\right] + \frac{d}{dx}\left[\sum_{n=1,3,5,..}^{\infty} -\frac{4l}{(n\pi)^2}\cos\frac{n\pi x}{l}\right]$$

$$= 0 + \sum_{n=1,3,5,\dots}^{\infty}\frac{d}{dx}\left[-\frac{4l}{(n\pi)^2}\cos\frac{n\pi x}{l}\right]$$

$$= 0 + \sum_{n=1,3,5,..}^{\infty}\frac{4l}{(n\pi)^2}\times\frac{n\pi}{l}\sin\frac{n\pi x}{l}$$

$$\therefore \qquad f'(x) = \sum_{n=1,3,5,..}^{\infty}\frac{4}{n\pi}\sin\frac{n\pi x}{l} ------------ (1)$$

Again, differentiating the given function, we get

$$f'(x) = \frac{d}{dx}[|x|] = \begin{cases} +1, for\ x > 0 \\ -1, for\ x < 0 \\ does\ not\ exist\ at\ x = 0 \end{cases}$$

So $f'(x)$ has discontinuity at x=0. If we expand $f'(x)$ in Fourier series we get sine series as

$$f'(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \text{ , } where\ b_n = \begin{cases} \frac{4}{n\pi}, for\ n\ is\ odd \\ 0, for\ n\ is\ even \end{cases}$$

$$\therefore \quad f'(x) = \sum_{n=1,3,5,..}^{\infty} \frac{4}{n\pi} \sin\frac{n\pi x}{l} - - - - - - - - - - - - (2)$$

Here, equation (1) and (2) represent same Fourier series. As $f(x)$ is continuous everywhere in its periodic extension and $f'(x)$ is piecewise continuous in given interval so term by term differentiation is possible for this function.

Theorem:

If each term of an infinite series has a derivative and the series of derivatives is uniformly convergent, then the series can be differentiated term-by-term.

$$i.e. \quad \frac{d}{dx}\sum_{n=1}^{\infty} u_n(x) = \sum_{n=1}^{\infty} \frac{d}{dx} u_n(x)$$

Next consider Fourier series with a_n and b_n proportional to $1/n^2$. If we differentiate such a series once, there are still $1/n$ factors left but we can't differentiate it twice. In that case we would (correctly) expect the function to be continuous with a discontinuous first derivative. (Look for examples.) Continuing, if a_n and b_n are proportional to $1/n^3$, we can find two derivatives, but the second derivative is discontinuous, and so on for Fourier coefficients proportional to higher powers of $1/n$.

It is interesting to plot (by computer) a given function together with enough terms of its Fourier series to give a reasonable fit. You will find that the more continuous derivatives a function has, the fewer terms of its Fourier series are required to approximate it. We can understand this: The higher order terms oscillate more rapidly (compare sin x, sin2x, sin10x), and this rapid oscillation is what is needed to fit a curve which is changing rapidly (for example, a jump). But if $f(x)$ has several continuous derivatives, then it is quite "smooth" and does not require so much of the rapid oscillation of the higher order terms. This is reflected in the dependence of the Fourier coefficients on a power of $1/n$.

Term-by-Term integration of Fourier Series:

Theorem:

If each term of an infinite series is continuous in an interval (a, b) and the series is uniformly convergent to the sum $f(x)$in this interval, then

1. $f(x)$ is also continuous in this interval.
2. The series can be integrated term-by-term

$$i.e. \quad \int_a^b \left\{\sum_{n=1}^{\infty} u_n(x)\right\} dx = \sum_{n=1}^{\infty} \int_a^b u_n(x)\, dx$$