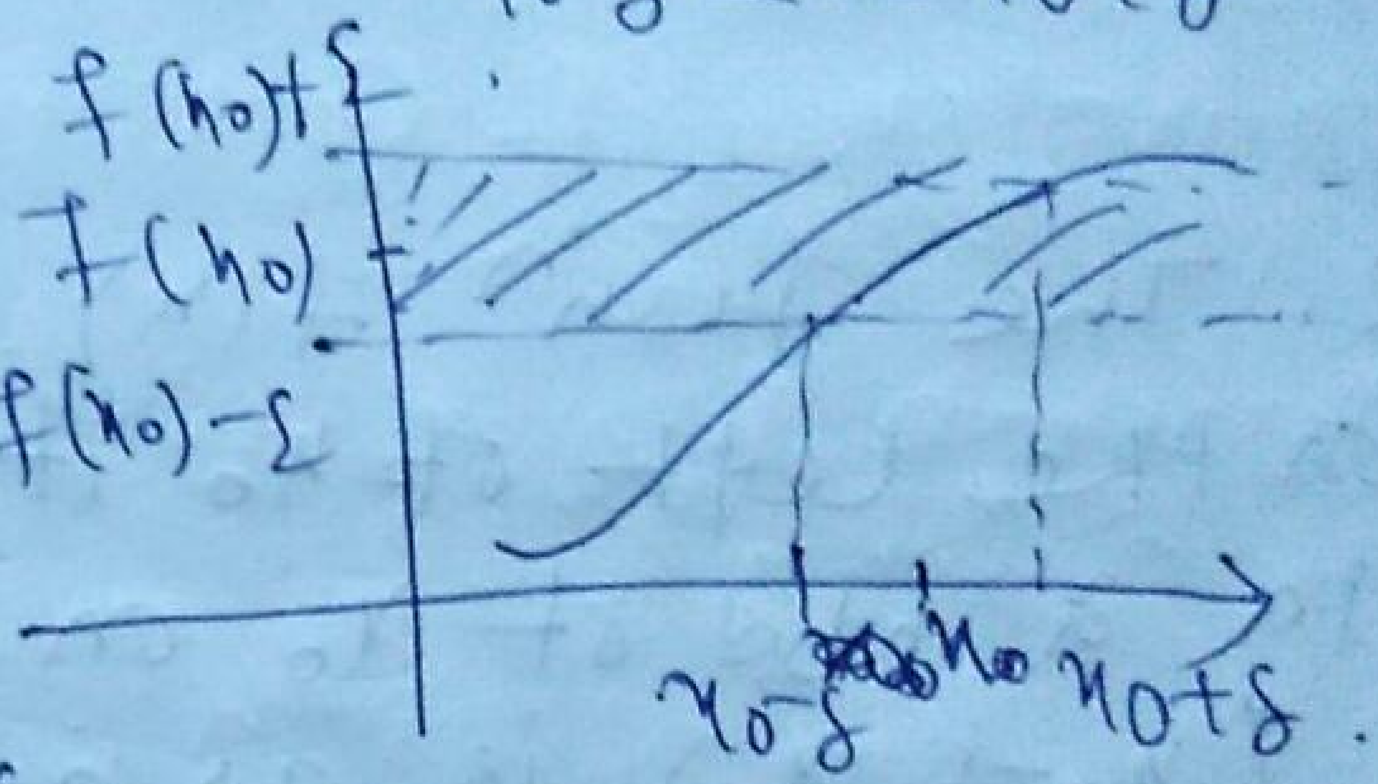
# Continuity of Function :-

**Defn (Cauchy)** Let $f(x)$ be a real valued funn defined over a continuous interval $(a,b)$.
The funn $f(x)$ is said to be continuous at a point $x_0$ in $(a,b)$, if corresponding to an arbitrarily chosen positive number $\epsilon$ however small, there exists a positive number $\delta(\epsilon, x_0)$ such that

$$|f(x) - f(x_0)| < \epsilon, \text{ provided } |x - x_0| < \delta$$

$$x_0 - \delta < x < x_0 + \delta$$

$$f(x_0) - \epsilon < f(x) < f(x_0) + \epsilon$$

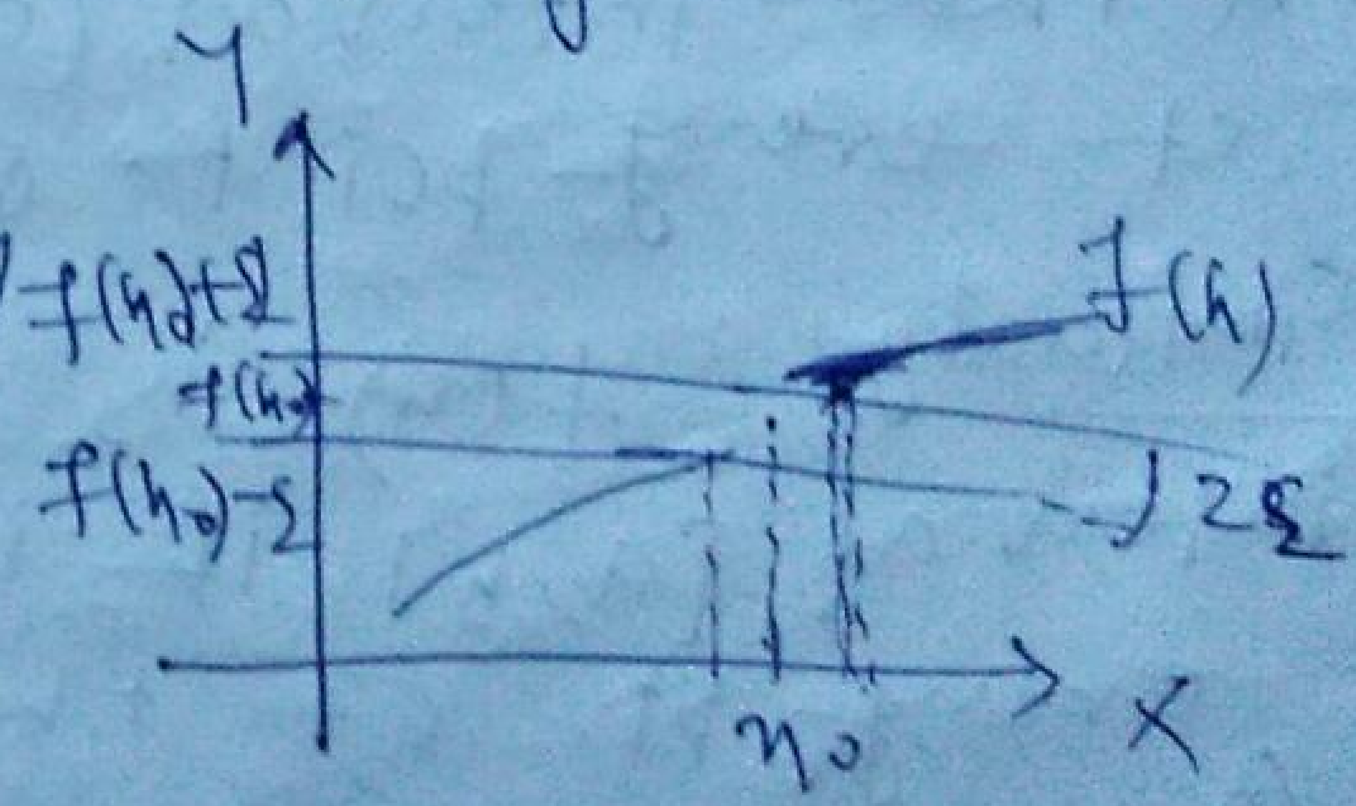



fluction of $f(x)$ is $(x_0 - \delta, x_0 + \delta)$ is

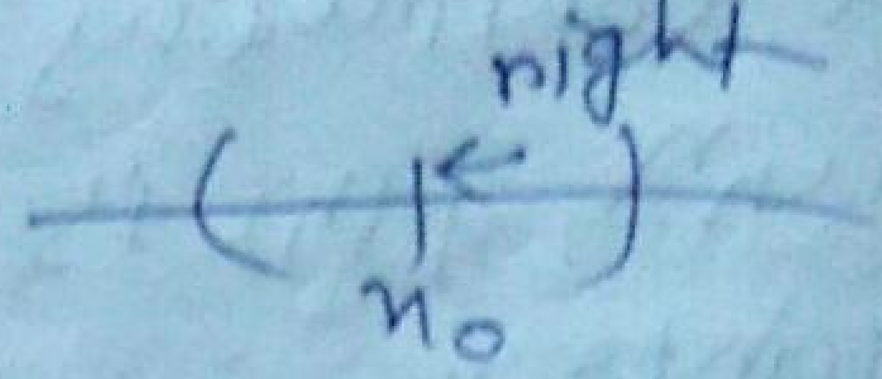
$$= f(x_0) + \epsilon - (f(x_0) - \epsilon)$$
$$= 2\epsilon$$

**Note:-** The funn $f(x)$ is continuous at a point if a nbd of the point can be determined in which the fluction is as small as we ~~please~~ wish. → Necessary condn.

Case of Discontinuity →



Def$^n$: 1. The fun$^n$ $f(x)$ is said to continuous at on the right at $x_0$ if the fluctuation of the fun$^n$ in a nbd of $x_0$ on the right




i.e., $(x_0, x_0+\delta)$ is as small as we wish.

i.e., for a given $\varepsilon>0$ $\exists$ $\delta>0$

s.t. $|f(x)-f(x_0)|<\varepsilon$ whenever $0<x-x_0<\delta$.

2. The fun$^n$ $f(x)$ is said to be continuous on the left of $x_0$ if the fluction of $f(x)$ in a nbd of $x_0$ on the left i.e., $(x_0-\delta, x_0)$ is as small as we wish.

i.e, $|f(x)-f(x_0)|<\varepsilon$ whenever $0<x_0-x<\delta$

Def$^n$ (continuity in the interval):-

The fun$^n$ $f(x)$ defined for a continuous interval $(a,b)$ is said to be continuous in that interval $(a,b)$ if it is continuous at every point of the interval.

Similarly: $f(x)$ is continuous in the closed interval $[a,b]$ if it is continuous at every pt. of the open interval $(a,b)$, and continuous on the right at $x=a$ and continuous on the left at $x=b$

**Ex:-** Prove that $f(x)=\sin x$ is continuous in $(0,\pi/2)$. Infact it is continuous over $\mathbb{R}$.

**Solⁿ:-** $\varepsilon>0$ be given
Let $x_0\in(0,\pi/2)$
Consider

$$|f(x)-f(x_0)| = |\sin x-\sin x_0|$$
$$= \left|2\sin\left(\frac{x+x_0}{2}\right)\sin\left(\frac{x-x_0}{2}\right)\right|$$
$$\le |x-x_0|$$

$\because \sin x\le x$ for $0<x<\pi/2$

choose $\delta=\varepsilon$

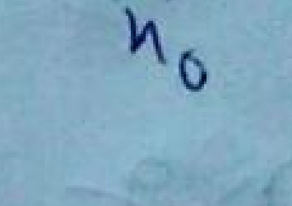
$\Rightarrow \sin x$ is continuous at $x_0\in(0,\pi/2)$

since $x_0$ is arbitrary, so $\sin x$ is continuous over $(a,b)$ on $\mathbb{R}$.

**Corollary:-** If $f(x)$ is continuous at a point $x_0$ and $f(x_0)>0$, then $f(x)>0$ over $(x_0-\delta, x_0+\delta)$

**Proof:-** For any given $\varepsilon>0$ $\exists$ $\delta>0$

s.t. $|f(x)-f(x_0)|<\varepsilon$ provided $|x-x_0|<\delta$.



$\Rightarrow f(x_0)-\varepsilon < f(x) < f(x_0)+\varepsilon$

choose $\varepsilon=\frac{1}{2}f(x_0)$ $\Rightarrow f(x)>\frac{1}{2}f(x_0)>0$
$\forall x\in N(x_0,\delta)$.

## Def^n of Continuity (Heine):—

Let $x_1, x_2, \dots, x_n, \dots$ be a convergent sequence of real numbers in $(a, b)$ whose limit is say $a$.

Then $f(x)$ is said to be continuous at $a$ if for every such convergent sequences $\{x_n\}$, the sequence of numbers $\{f(x_n)\}$ converges to $f(a)$.

i.e., $f(x)$ is continuous at $x = a$ iff

$$\lim_{x \to a} f(x) = f(a).$$

## Equivalence of Cauchy & Heine Def^n of Continuity:—

Assume $f(x)$ is continuous at $x_0$ according to Cauchy.

i.e., for a given $\varepsilon > 0$, $\exists\ \delta(\varepsilon)$ s.t.

$$|f(x) - f(x_0)| < \varepsilon, \text{ provided } |x - x_0| < \delta \quad \text{—(1)}$$

Consider a convergent sequence $\{x_n\}$ having limit $x_0$ so

$$|x_n - x_0| < \delta \text{ for } n \geq m, \text{ say}$$

$$\Rightarrow x_0 - \delta < x_n < x_0 + \delta \text{ for all } n \geq m.$$

Using (1), $|f(x_n) - f(x_0)| < \varepsilon \quad \forall\, n \geq m$

$$\Rightarrow \lim_{n \to \infty} f(x_n) = f(x_0)$$

Hence Heine def^n.

Conversely,

Suppose $f$ is continuous at $x_0$ in accordance to Heine def$^n$ but Cauchy's def$^n$ is not satisfied i.e., for given $\epsilon>0$, there exists point in $(x_0-\delta, x_0+\delta)$ for which

$|f(x)-f(x_0)| \geq \delta$ , howevery small $\delta$ may be $-(2)$

consider a sequence $\{\delta_n\}$ & converges to 0 and $x_n \in (x_0-\delta_n, x_0+\delta_n)$

$(2) \Rightarrow |f(x_n)-f(x_0)| \geq \epsilon$

$\Rightarrow \cancel{\lim_{x \to x_0} f(x_n)} \Rightarrow \lim_{n \to \infty} f(x_n) \neq f(x_0)$

$\Rightarrow f(x)$ is not continuous.

which is a contradiction.

Def$^n$:- Let $A \subseteq \mathbb{R}$, let $f: A \to \mathbb{R}$ and let $c \in A$. $f$ is continuous at $c$, if for given $\epsilon>0$, $\exists\ \delta(\epsilon, c)$ s.t.

$|f(x)-f(c)|<\epsilon$ provided $|x-c|<\delta$

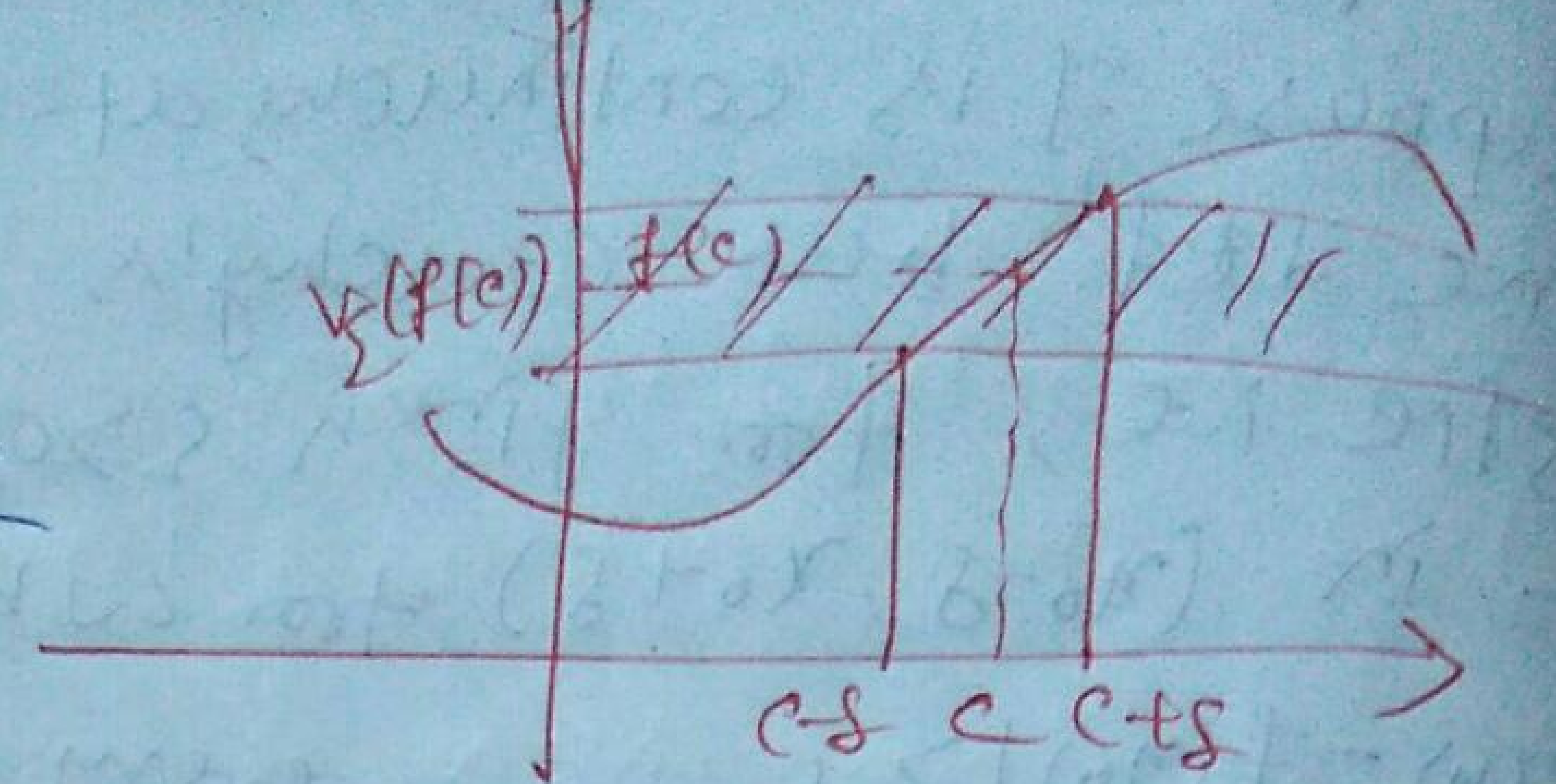
If $f$ fails to be continuous at $c$, then $f$ is discontinuous at $c$.

Thm:- A fun$^n$ $f: A \to \mathbb{R}$ is continuous at $x=c \in A$ if and only if given any $\epsilon$-nbd $V_\epsilon(f(c))$ of $f(c)$ $\exists$ a $\delta$-nbd $V_\delta(c)$ of $c$ such that if $x$ is any point of the $A \cap V_\delta(c)$, then $f(x)$ belongs to $V_\epsilon(f(c))$ i.e.,

$f(A \cap V_\delta(c)) \subseteq V_\epsilon(f(c))$.

**Remark 1:-**

If $c \in A$ is a cluster point (limit pt.) of $A$, then




$f$ is continuous if and only if

$$f(c) = \lim_{x \to c} f(x)$$

So, for the continuity of $f(x)$ at $c$, we need the following conditions must hold.

(i) $f(x)$ must be defined at $x = c$

(ii) $\lim_{x \to c} f(x)$ must exists.

(iii) $\lim_{x \to c} f(x) = f(c)$.

**Ex:-** 1. $f(x) = x^2$ is continuous on $\mathbb{R}$.

Let $c \in \mathbb{R}$

$$\lim_{x \to c} f(x) = c^2 = f(c)$$

2. $f(x) =$ polynomial, $\sin x$, $\cos x$, $\tan x$, $e^x$ ... are continuous.

2. $f(x) = \frac{1}{x}$ is continuous on $A = \{x \in \mathbb{R} / x > 0\}$.

Let $c \in A$, $c > 0$ - $\lim_{x \to c} f(x) = \frac{1}{c} = f(c)$

But $f(x) = \frac{1}{x}$ is not continuous at $x = 0$ because it is not defined at $x = 0$.

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