Exercise

(1) Using ε-δ definition show the following limits :-

(1) $\lim\limits_{x \to c} \frac{1}{x} = \frac{1}{c}$ if $c > 0$

(2) $\lim\limits_{x \to 2} \frac{x^3 - 4}{x^2 + 1} = \frac{4}{5}$

(3) $\lim\limits_{x \to c} x^3 = c^3$ for any $c \in \mathbb{R}$.

(4) $\lim\limits_{x \to c} \sqrt{x} = \sqrt{c}$ for any $c > 0$.

(5) $\lim\limits_{x \to 2} \frac{1}{1 - x} = -1$

(6) $\lim\limits_{x \to 1} \frac{x}{1 + x} = \frac{1}{2}$

(7) $\lim\limits_{x \to 0} \frac{x^2}{|x|} = 0$

(8) $\lim\limits_{x \to 1} \frac{x^2 - x + 1}{x + 1} = \frac{1}{2}$

(9) $\lim\limits_{x \to 2} (x^2 + 4x) = 12$

(10) $\lim\limits_{x \to -1} \frac{x + 4}{2x + 3} = 4$

(11) $\lim\limits_{x \to 3} \frac{2x + 3}{4x - 9} = 3$

(12) $\lim\limits_{x \to 6} \frac{x^2 - 3x}{x + 3} = 2$

(13) $\lim\limits_{x \to 7} (x^2 + 1) = 50$

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Sequential Criteria for limit of functions:-

Theorem:- Let $f: A \to \mathbb{R}$ and let c be a cluster point of A. Then the following are equivalent.

(1) $\lim_{x \to c} f(x) = l$

(2) For ~~on~~ every sequence $\{x_n\}$ in A that converges to c s.t. $x_n \neq c$ $\forall n \in \mathbb{N}$, then the sequence $\{f(x_n)\}$ converges to l.

Proof:- (i) $\Rightarrow$ (ii)

Given $\lim_{x \to c} f(x) = l$.

By def^n, for given $\varepsilon > 0$, $\delta(\varepsilon) > 0$ such that if $x \in A$, $x \neq c$ satisfies $0 < |x - c| < \delta$, then $f(x)$ satisfies $|f(x) - l| < \varepsilon$.

(ii) tells $\forall \{x_n\}$ in A that converges to c s.t. $x_n \neq c$.

So, for given $\delta > 0$, $\exists$ $K(\delta)$ s.t. if $n > K(\delta)$ then $|x_n - c| < \delta$.

$\Rightarrow |f(x_n) - l| < \varepsilon \quad \forall n \geq K$

$\Rightarrow \{f(x_n)\}$ converges to l.

Hence (i) $\Rightarrow$ (ii)

(ii) $\Rightarrow$ (i)

This we prove by contradiction.

Suppose (ii) holds but (i) does not hold.

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i.e., $\lim_{x \to c} f(x)$ does not exist or is not equal to l.

So, there exists ε_0-nbd of l i.e., $N_{\varepsilon_0}(l)$ s.t. no matter what δ-nbd of c, we may get one number x_δ in the set $A \cap N_\delta(c)$ with $x_\delta \neq c$ s.t. $f(x_\delta) \notin N_{\varepsilon_0}(l)$.

Hence for every n, the $\frac{1}{n}$ nbd of c $N_{1/n}(c)$ contains a number x_n s.t. $0 < |x_n - c| < 1/n$ and $x_n \in A$ but $|f(x_n) - l| \geq \varepsilon_0$

~~$\Rightarrow |f(x_n) - l| >$~~

$\Rightarrow \{f(x_n)\} \not\to l$.

This contradicts the assumption (ii). Therefore our assumption is not true.

$\therefore$ (ii) $\Rightarrow$ (i).

Divergence Criteria :- Let $A \subseteq \mathbb{R}$, let $f: A \to \mathbb{R}$ and let c be a cluster point of A. If $l \in \mathbb{R}$, then f does not have limit l at c if and only if there exists a sequence $\{x_n\}$ in A with $x_n \neq c$ $\forall n \in \mathbb{N}$. s.t. the sequence $\{x_n\}$ converges to c but the sequence $\{f(x_n)\}$ ~~conv~~ does not converge to l.

Example :-

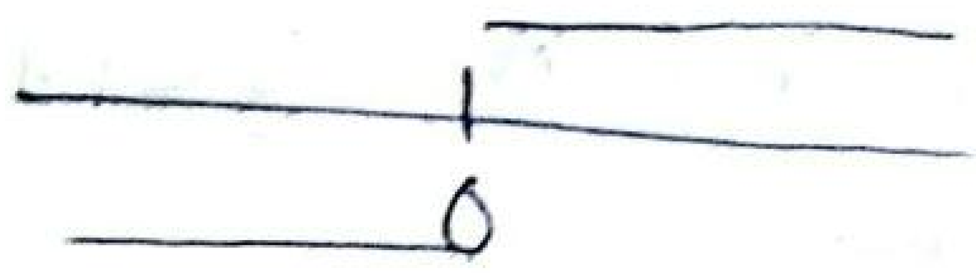
(1) $\lim_{x \to 0} \text{sgn}(x)$ does not exist.

signum funⁿ

$$\text{sgn}(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x = 0 \end{cases}$$

sign of the funⁿ

$\lim_{x \to 0} \text{sgn}(x)$ does not exist.



It means if we take a sequence $\{x_n\}$ which goes to 0, but the corresponding funⁿ $\{\text{sgn}(x_n)\}$ does not go to $\{\text{sgn}(0)\}$.

So, let us choose the sequence $\{x_n\}$,

$x_n = \frac{(-1)^n}{n} \to 0$ as $n \to \infty$

~~$\text{sgn}(x_n) =$~~ $(-1)^n$

$\text{sgn}(x_n) = (-1)^n \quad \forall n \in \mathbb{N}$

$\to 1$ when n is even

-1 when n is odd.

But $\text{sgn}(0) = 0$.

Therefore $\text{sgn}(x_n) \not\to \text{sgn}(0)$.

$\Rightarrow$ ~~sgn~~ $\lim_{x \to 0} \text{sgn}(x)$ does not exist.

(2) $\lim_{x \to 0} (1/x)$ does not exist in $\mathbb{R}$.

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Theorem: If $f: A \to \mathbb{R}$ and if c is the cluster point of A, then f can have only one limit at c. (We can't have two limits)

Proof: Let $\lim_{x \to c} f(x)$ be l and l'.

$\lim_{x \to c} f(x) = l$	$\lim_{x \to c} f(x) = l'$
Given $\varepsilon > 0$, $\exists\ \delta(\varepsilon)$ s.t. If $x \in A$, $0 < \lvert x - c \rvert < \delta$, then $\lvert f(x) - l \rvert < \varepsilon/2$	For $\varepsilon > 0$, $\exists\ \delta_2(\varepsilon)$ s.t. If $x \in A$, $0 < \lvert x - c \rvert < \delta_2$ then $\lvert f(x) - l' \rvert < \varepsilon/2$.

consider $\delta = \inf(\delta_1, \delta_2)$

consider

~~$\lvert l - l' \rvert = \lvert f(x) - l + f(x) - l' \rvert$~~

$$\lvert l - l' \rvert = \lvert -f(x) + l + f(x) - l' \rvert$$

$$\leq \lvert f(x) - l \rvert + \lvert f(x) - l' \rvert$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon$$ whenever $0 < \lvert x - c \rvert < \delta$ $\forall\ x \in A$

ε is arbitrary small

$\Rightarrow l = l'$.

Ex:- $\lim_{x\to 0} \sin(1/x)$ does not exist in $\mathbb{R}$.

~~Choose $x_n = \frac{1}{n\pi}$, $y_n = \frac{1}{\pi/2 + 2n\pi}$~~

We are claiming limit does not exist. It means along two different paths, if the limit of the sequence has different value and both path tending to 0. Then obviously, the limit will not exist.

So, choose

$x_n = \frac{1}{n\pi}$, $y_n = \frac{1}{\pi/2 + 2n\pi}$

$\downarrow$ 0 $\quad$ $\downarrow$ 0 as $n\to\infty$.

~~But~~

Both the sequence $\{x_n\}$ and $\{y_n\}$ converging to 0.

$f(x) = \sin(1/x)$

$f(x_n) = 0$ while $f(y_n) = \sin(\pi/2 + 2n\pi)$

Along two paths limits are different.

So, the limit does not exist.

~~$\therefore$ lt~~

$\therefore \lim_{x\to 0} \sin(1/x)$ does not exist.