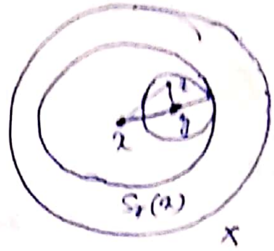
2017-18

Th: Let (X, d) be a metric space, then each open sphere in X is an open set.

Proof :- Let $S_r(x)$ be any open sphere in X.
Also, let $y \in S_r(x)$ be arbitrary point.
Then $d(x, y) < r$



Let $r_1 = r - d(x, y)$ ——— (1)
Then $r_1 > 0$.

Let $z \in S_{r_1}(y)$ be any point.
Then $d(y, z) < r_1$ ——— (2)

By triangle inequality, we have,

$d(x, z) \leq d(x, y) + d(y, z)$
$< d(x, y) + r_1$ [by (2)]
$= r$
or $d(x, z) < r$ [by (1)]
$\Rightarrow z \in S_r(x)$.

$\therefore z \in S_{r_1}(y) \Rightarrow z \in S_r(x)$

$\therefore S_{r_1}(y) \subset S_r(x)$

or, $y \in S_{r_1}(y) \subset S_r(x)$

Hence $S_r(x)$ is a nbd of y.
But y be arbitrary point of $S_r(x)$.
Then $S_r(x)$ is a nbd of each of its point.
$\therefore S_r(x)$ is an open set.
So, every open sphere in X is an open set.

* **Note :-** Converse of the above theorem need not be true. For example $(-\infty, \infty)$ is an open set, but it is not an open sphere in the usual metric space $\mathbb{R}_u$.

Let (X,d) be a metric space then prove that arbitrary union of open sets in X is open set. [VH-11, 13]

Proof: Let $\{G_i : i \in \Lambda\}$, be arbitrary collection of open sets in X.

Let $G = \bigcup_{i \in \Lambda} G_i$

Let $x \in G$ be arbitrary point.

then x belongs to at least one open set of the collection, say. G_λ $(\lambda \in \Lambda)$.

Since G_λ is open set and $x \in G_\lambda$, then $\exists$ an $r>0$ such that $x \in S_r(x) \subset G_\lambda$

$\Rightarrow x \in S_r(x) \subset G$ $[\because G_\lambda \subset G]$

$\Rightarrow G$ is a nbd of x

But x is arbitrary point of G.

Then G is nbd of each of its points.

$\therefore$ G is an open set.

Hence arbitrary collection of open sets in X is open.

Th: Let (X,d) be a metric space, prove that finite intersection of open sets in X is open.

Proof: Let $\{G_1, G_2, \cdots, G_n\}$ be finite collection of open sets in X.

Let $G = G_1 \cap G_2 \cap \cdots \cap G_n = \bigcap_{i=1}^{n} G_i$

Case-I: If $G = \phi$, the G is open set, as empty set ϕ is open set.

Case-II: If $G \neq \phi$

Let $x \in G$ be arbitrary point.

then $x \in G_i$, $\forall i = 1, 2, \cdots, n$.

since G_1 is open set and $x \in G_1$, then $\exists$ an $r_1 > 0$, such that $x \in S_{r_1}(x) \subset G_1$

since G_2 is open set and $x \in G_2$, then $\exists$ an $r_2 > 0$, such that $x \in S_{r_2}(x) \subset G_2$

$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$

since G_n is open set and $x \in G_n$, then $\exists$ an $r_n > 0$, such that $x \in S_{r_n}(x) \subset G_n$.

Let $r = \min\{r_1, r_2, \cdots, r_n\}$. then $r > 0$

$\therefore$ $S_r(x) \subset S_{r_1}(x) \subset G_1$

$S_r(x) \subset S_{r_2}(x) \subset G_2$

$\cdots\cdots\cdots\cdots$

$S_r(x) \subset S_{r_n}(x) \subset G_n$

$\therefore$ $S_r(x) \subset G_1 \cap G_2 \cap \cdots \cap G_n = G$

$\therefore \Rightarrow x \in S_r(x) \subset G$

$\Rightarrow$ G is a nbd of x

But x is arbitoary point of G.

$\therefore$ G is a nbd of each of its points.

Then, G is an open set.

Hence finite intersection of open sets in X is open set.

Note: Arbitoary intersection of open sets need not be open.

Ex: In the usual metric space $\mathbb{R}_u$, we consider the family $\{(-\frac{1}{n}, \frac{1}{n}) : n \in \mathbb{N}\}$ of open sets.

Then $\bigcap_1^{\infty} (-\frac{1}{n}, \frac{1}{n}) = \{0\}$, which is not open set.

Again, if we consider the family $\{(-n, n) : n \in \mathbb{N}\}$ of open sets.

Then $\bigcap_1^{\infty} (-n, n) = (-1, 1)$, which is an open set.

Let $\{x_n\}$ and $\{y_n\}$ be two sequences in a metric space (X,d). If $x_n \to x$ and $y_n \to y$, then the sequence $\{d(x_n, y_n)\}$ of real numbers converges to $d(x,y)$ in the real line $\mathbb{R}$. [VH-10] ry

<u>Proof!</u> Since $x_n \to x$ and $y_n \to y$, then for any $\epsilon > 0$, $\exists$ natural numbers m_1 and m_2 such that $d(x_n, x) < \epsilon/2$, $\forall\, n \geq m_1$

and $d(y_n, y) < \epsilon/2$, $\forall\, n \geq m_2$

Let $m = \max\{m_1, m_2\}$

Then for $n \geq m$, we have

$$d(x_n, y_n) \leq d(x_n, x) + d(x,y) + d(y, y_n)$$
$$< \epsilon/2 + d(x,y) + \epsilon/2$$

$$\text{or. } d(x_n, y_n) - d(x,y) < \epsilon \quad \text{——— (1)}$$

Again, $$d(x,y) \leq d(x, x_n) + d(x_n, y_n) + d(y_n, y)$$
$$< \epsilon/2 + d(x_n, y_n) + \epsilon/2$$

$$\text{or } d(x,y) - d(x_n, y_n) < \epsilon \quad \text{——— (2)}$$

From (1) and (2) we get

$$|d(x_n, y_n) - d(x,y)| < \epsilon, \quad \forall\, n \geq m.$$

$$\Rightarrow d(x_n, y_n) \to d(x,y) \text{ as } n \to \infty$$

$\therefore$ the sequence $\{d(x_n, y_n)\}$ converges to $d(x,y)$ in $\mathbb{R}$.

<u>Ex!</u> For any points x, y, a, b in (X, d) prove that

$$|d(x,y) - d(a,b)| \leq d(x,a) + d(y,b)$$ 2011-11.

<u>Sol?</u>

Th: Let $\{x_n\}$, $\{y_n\}$ be sequences in a metric space (X, d) such that $x_n \to x$ in X. Then $y_n \to x$ in X iff $d(x_n, y_n) \to 0$ in $\mathbb{R}$.

Proof: Let $x_n \to x$ and $y_n \to x$ as $n \to \infty$.

then for any $\epsilon > 0$, $\exists$ natural numbers m_1 and m_2 such that $d(x_n, x) < \epsilon/2$, $\forall n \geq m_1$

and $d(y_n, x) < \epsilon/2$, $\forall n \geq m_2$

Let $m = \max\{m_1, m_2\}$.

Then for $n \geq m$, we have by triangle inequality

$$d(x_n, y_n) \leq d(x_n, x) + d(x, y_n)$$
$$< \epsilon/2 + \epsilon/2 = \epsilon,$$

$\Rightarrow$ $d(x_n, y_n) \to 0$ in $\mathbb{R}$.

Conversely:- Let $x_n \to x$ in X and $d(x_n, y_n) \to 0$ in $\mathbb{R}$.

then for any $\epsilon > 0$, $\exists$ a natural numbers k_1, k_2 such that $d(x_n, x) < \epsilon/2$, $\forall n \geq k_1$

and $d(x_n, y_n) < \epsilon/2$, $\forall n \geq k_2$.

Let $k = \max\{k_1, k_2\}$

For $n \geq k$, by triangle inequality we have,

$$d(y_n, x) \leq d(y_n, x_n) + d(x_n, x)$$
$$< \epsilon/2 + \epsilon/2 = \epsilon.$$

$\Rightarrow$ $d(y_n, x) < \epsilon$, $\forall n \geq k$

$\Rightarrow$ $y_n \to x$ as $n \to \infty$.

Th: If $d(x_n, y_n) \to 0$ in $\mathbb{R}$ and $\{x_n\}$ is cauchy or is convergent to a point x of X. Then prove that $\{y_n\}$ is respectively so.

Proof: Since $d(x_n, y_n) \to 0$ in $\mathbb{R}$ and $\{x_n\}$ is a cauchy sequence in X, then for any $\epsilon > 0$, $\exists$ natural numbers k_1 and k_2 such that $d(x_n, y_n) < \epsilon/3$ $\forall$ $n \geq k_1$

and $d(x_n, x_m) < \epsilon/3$, $\forall$ $n, m \geq k_2$

Let $k = \max\{k_1, k_2\}$

Then for $m, n \geq k$, we have

$$d(y_n, y_m) < d(y_n, x_n) + d(x_n, x_m) + d(x_m, y_m)$$
$$< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon$$

$\Rightarrow$ $\{y_n\}$ is a cauchy sequence.

Conversely:

Let $d(x_n, y_n) \to 0$ in $\mathbb{R}$ and $x_n \to x$ as $n \to \infty$, then for any $\epsilon > 0$, $\exists$ natural numbers p_1 and p_2 such that $d(x_n, y_n) < \epsilon/2$, $\forall$ $n \geq p_1$

and $d(x_n, x) < \epsilon/2$, $\forall$ $n \geq p_2$

Let $p = \max\{p_1, p_2\}$.

Then for $n \geq p$, we have

$$d(y_n, x) \leq d(y_n, x_n) + d(x_n, x)$$
$$< \epsilon/2 + \epsilon/2 = \epsilon$$

$\Rightarrow$ $y_n \to x$ as $n \to \infty$.

Th: Let $\{x_n\}, \{y_n\}$ be sequences in a metric space (X, d).

(a) $\{x_n\}$ and $\{y_n\}$ are cauchy sequence $\Rightarrow \{d(x_n, y_n)\}$ converges in the real line $\mathbb{R}$.

(b) $d(x_n, y_n) \to 0$ in $\mathbb{R} \Rightarrow \{x_n\}$ is cauchy or is convergent to a point x of X then $\{y_n\}$ is respectively so.

Proof: (a) Since $\{x_n\}$ and $\{y_n\}$ are cauchy sequence, then for any $\epsilon > 0$, $\exists$ natural numbers m_1, m_2 such that

$$d(x_i, x_j) < \epsilon/2 \quad \forall i, j \geq m_1$$

and $$d(y_i, y_j) < \epsilon/2 \quad \forall i, j \geq m_2$$

Let $m = \max\{m_1, m_2\}$.

Then for $i, j \geq m$, we have

$$d(x_i, y_i) \leq d(x_i, x_j) + d(x_j, y_j) + d(y_j, y_i).$$
$$< \epsilon/2 + d(x_j, y_j) + \epsilon/2$$
$$= \epsilon + d(x_j, y_j)$$

$$d(x_i, y_i) - d(x_j, y_j) < \epsilon. \quad \text{——(1)}$$

Again $$d(x_j, y_j) \leq d(x_j, x_i) + d(x_i, y_i) + d(y_i, y_j)$$
$$< \epsilon/2 + d(x_i, y_i) + \epsilon/2$$

$$\therefore d(x_j, y_j) - d(x_i, y_i) < \epsilon \quad \text{——(2)}.$$

From (1) & (2) we write

$$|d(x_i, y_i) - d(x_j, y_j)| < \epsilon. \quad \forall i, j \geq m.$$

$\Rightarrow$ the sequence $\{d(x_n, y_n)\}$ is a cauchy sequence of real numbers and hence it convergent in $\mathbb{R}$.

Complete spaces :-

A metric space (X,d) is said to be complete if every Cauchy sequence in X converges to a point of X.

Th: A metric space (X,d) is complete iff every Cauchy sequence in X has a convergent subsequence.

Proof:- Let, the metric space (X,d) is complete and $\{x_n\}$ be a Cauchy sequence in X that converges to a point x in X.

Then for any $\epsilon > 0$, $\exists$ natural numbers m_1 and m_2 such that $d(x_n, x) < \epsilon/2$, $\forall\ n \geq m_1$

and $d(x_{n_k}, x_n) < \epsilon/2$, $\forall\ n_k, n \geq m_2$

Let $m = \max\{m_1, m_2\}$

Also let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$.

Now for $n_k \geq m$, we have

$$d(x_{n_k}, x) < d(x_{n_k}, x_n) + d(x_n, x)$$
$$< \epsilon/2 + \epsilon/2 = \epsilon.$$

It follows that $\{x_{n_k}\}$ converges to x as $k \to \infty$.

Hence every Cauchy sequence in X has a convergent subsequence.

Conversly :- Let $\{x_n\}$ be a Cauchy sequence in X has a convergent subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to x$ as $k \to \infty$.

Then for any pre-assigned $\epsilon > 0$, $\exists$ natural numbers P_1 and P_2 such that $d(x_{n_k}, x) < \epsilon/2$, $\forall\ n_k \geq P_1$

and $d(x_{n_k}, x_n) < \epsilon/2$ $\forall\ n_k, n \geq P_2$

Let $P = \max\{P_1, P_2\}$.

Now for $n \geq P$, by triangle inequality,

$$d(x_n, x) < d(x_n, x_{n_k}) + d(x_{n_k}, x)$$
$$< \epsilon/2 + \epsilon/2 = \epsilon$$

$\Rightarrow x_n \to x$ as $n \to \infty$.

Thus every Cauchy sequence in X conv. to a point of X.

Hence metric space (X,d) is complete.

Ex! The metric space (X,d), where $X=(0,1]$ and d the usual metric on X is not Complete.

Solⁿ: Let $\{x_n\}$ be a Sequence in (X,d), where $x_n=\frac{1}{n}$, $\forall n\in\mathbb{N}$ then $\{x_n\}$ is Cauchy sequence in X as for each $\epsilon>0$, we have $d(x_m,x_n) = |\frac{1}{m}-\frac{1}{n}| < \epsilon$, $\forall\ m,n > 1/\epsilon$.

But $\lim_{n\to\infty} x_n = 0 \notin X$, thus $\{x_n\}$ is not converges in X. Hence (X,d) is not Complete.

Th! Let $\{x_n\}$ be a sequence in a metric space (X,d) such that $\lim_{n\to\infty} x_n = x$. Let A be the range set of the sequence $\{x_n\}$. Then

(a) If A is a finite set, then $x_n = x$, for infinitely many n.

(b) If A is an infinite set, then x is the limit point of A.

** Th! Let (Y, d_Y) be a subspace of a metric space (X,d) then Y is complete $\Rightarrow$ Y is closed.

Proof: Let $x\in X$ be a limit point of Y.
then every open sphere centred on x contains a point of Y, other than x.
In particular, the open sphere $S_{1/n}(x)$ contains a point x_n of Y, other than x, where n is +ve integer.

$\therefore\ d(x_n,x) < 1/n$,

Thus $\{x_n\}$ is a sequence in Y such that $x_n\to x$ in X.
As every conv. sequence is a Cauchy sequence, then $\{x_n\}$ is a Cauchy sequence in X and hence in Y $[\because Y\subset X]$

But Y is complete, hence $x\in Y$.

It follows that Y is closed.

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Th: Let (X,d) be a complete metric space and (Y, d_Y) be a subspace of (X,d). Then Y is closed $\Rightarrow$ Y is complete.

Proof: Let $\{x_n\}$ be a cauchy sequence in Y. and hence in X as $Y \subset X$. But X being complete, then $\{x_n\}$ converges to a point $x \in X$.

We prove $x \in Y$

Case-I: If $\{x_n\}$ has ~~infinitely~~ finitely many distinct points, then $x_n = x$, for finite values of n.

Thus $x \in Y$, As $\{x_n\}$ is a sequence in Y.

Case-II: If $\{x_n\}$ has infinitely many distinct points, then x is limit point of the range set of the sequence $\{x_n\}$.

Therefor x is also limit point of Y, as $\{x_n\} \subset Y$.

Since Y is closed ~~also~~ so $x \in Y$, ~~then~~

It follows that Y is complete.

Th: Let (X,d) be a matric space and $G \subset X$, Then, G is an open set if and only if it is the union of open sphers.

Proof:- Let G be an open set

Then G is a nbd of each of its points.

$\therefore$ Each point G is the centre of some open spher which contain in G.

clearly, union of all such open sphers is the set G.

Conversly, Let G be the union of open sphers.

Let $x \in G$ be arbitorary point.

Then x belongs to at least one open spher of the union say $S_r(P) \subset G$.

Since $S_r(P)$ is an open set and $x \in S_r(P)$, then $\exists$ an $r_1 > 0$ such that $x \in S_{r_1}(P) \subset S_r(x)$ $\quad [\because S_r(P) \subset G]$

$\Rightarrow x \in S_{r_1}(P) \subset G$

$\Rightarrow G$ is a nbd of x

But x is arbitorary point of G, then G is a nbd of each of its points.

Hence G is an open set.

Th: Let (X,d) be a metric space and $A \subset X$. Then A is closed iff A contains all its limit points, i.e., $A' \subset A$.

Proof: Let A is closed set.
We have to prove $A' \subset A$.
Let $x \in A'$ be any point.
Then $x \in A' \Rightarrow x$ is a limit point of A.
$\Rightarrow$ every open sphere centred on x contains at least one point of A other than x.

If possible, let $x \notin A$
Then $x \in X-A$.
But $(X-A)$ is open set as A is closed set.
$\Rightarrow \exists$ an $r>0$, s.t $x \in S_r(x) \subset X-A$
$\Rightarrow x$ is not a limit point of A.
Which is a contradiction.
$\therefore$ our assumption $x \notin A$ is wrong.
Hence ~~$x \in A$ and~~
$\therefore x \in A' \Rightarrow x \in A$
$\therefore A' \subset A$.

Conversely: Let $A' \subset A$.
We have to prove A is closed set, i.e., to prove $X-A$ is open set.
Let $x \in X-A$ be arbitrary point.
$\Rightarrow x \notin A$, $\quad [\because A' \subset A]$
$\Rightarrow x \notin A'$
$\Rightarrow x$ is not a limit point of A.
$\Rightarrow \exists$ an $r>0$ s.t. $x \in S_r(x) \subset X-A$.
$\Rightarrow X-A$ is a nbd of x.
But x is arbitrary point of $(X-A)$
$\therefore X-A$ is a nbd of each of its points.
Hence $X-A$ is open set and A is closed set.